

Extending Pythagoras

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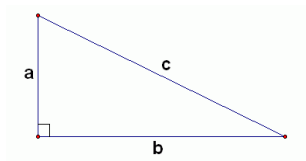
Washington State University, October 12, 2015

Goals of the Talk

- Often multiple proofs: Say **a proof** rather than **the proof**.
- Different proofs highlight different aspects.
- Too often rote algebra: Explore! Generalize! Conjecture!
- General: How to find / check proofs: special cases, 'smell' test.
- Specific: Pythagorean Theorem.

Pythagorean Theorem

Geometry Gem: Pythagorean Theorem



Theorem (Pythagorean Theorem)

Right triangle with sides a , b and hypotenuse c , then $a^2 + b^2 = c^2$.

Most students know the statement, but the proof?

Why are proofs important? Can help see big picture.

Geometric Proofs of Pythagoras

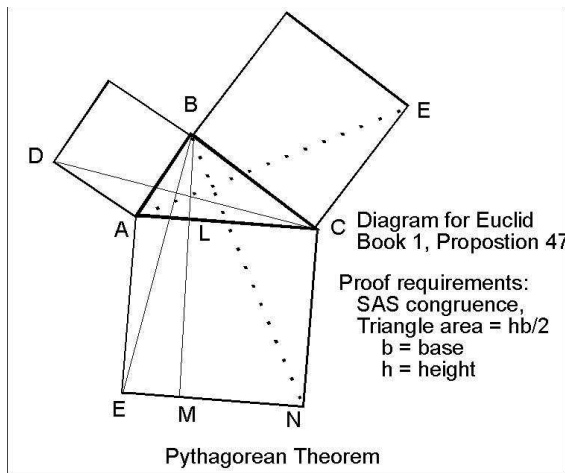


Figure: Euclid's Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?

Geometric Proofs of Pythagoras

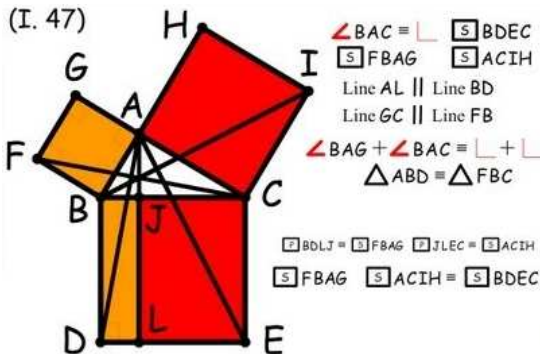


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Geometric Proofs of Pythagoras

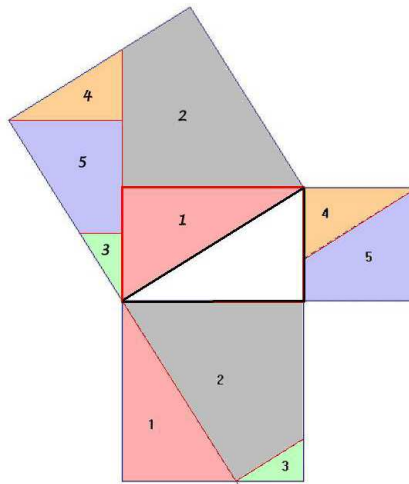
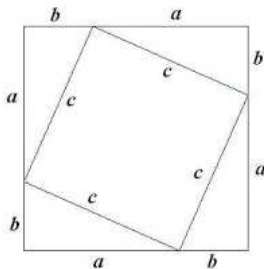


Figure: A nice matching proof, but how to find these slicings!

Geometric Proofs of Pythagoras



$$\begin{aligned}\text{Big square: } (a+b)^2 \\ = a^2 + 2ab + b^2\end{aligned}$$

$$\text{Four triangles} = 2ab$$

$$\text{Little square} = c^2$$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

$$a^2 + b^2 = c^2$$

Figure: Four triangles proof: I

Geometric Proofs of Pythagoras

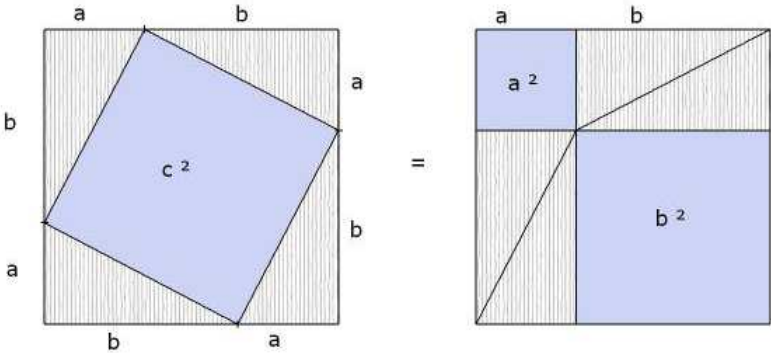


Figure: Four triangles proof: II

Geometric Proofs of Pythagoras

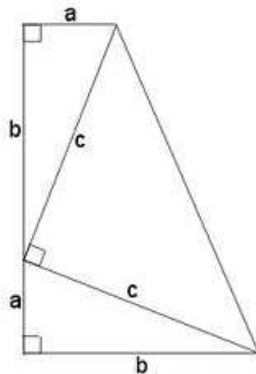
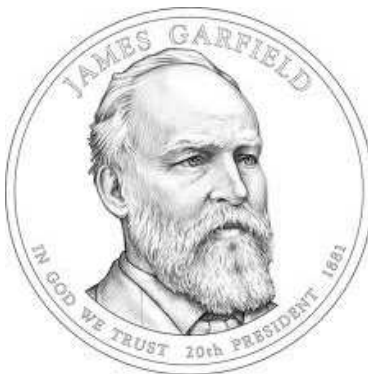


Figure: President James Garfield's (Williams 1856) Proof.

Geometric Proofs of Pythagoras

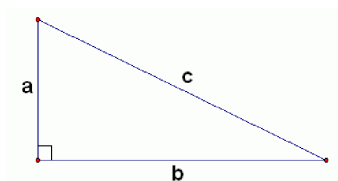
Lots of different proofs.

Difficulty: how to find these combinations?

At the end of the day, do you know *why* it's true?

Dimensional Analysis

Possible Pythagorean Theorems....



$$\diamond c^2 = a^3 + b^3.$$

$$\diamond c^2 = a^2 + 2b^2.$$

$$\diamond c^2 = a^2 - b^2.$$

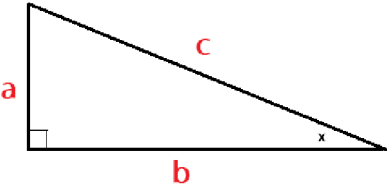
$$\diamond c^2 = a^2 + ab + b^2.$$

$$\diamond c^2 = a^2 + 110ab + b^2.$$

Possible Pythagorean Theorems....

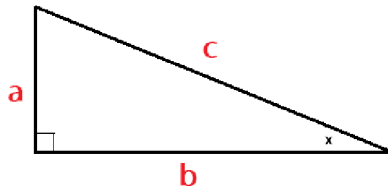
- ◇ $c^2 = a^3 + b^3$. **No**: wrong dimensions.
- ◇ $c^2 = a^2 + 2b^2$. **No**: asymmetric in a, b .
- ◇ $c^2 = a^2 - b^2$. **No**: can be negative.
- ◇ $c^2 = a^2 + ab + b^2$. **Maybe**: passes all tests.
- ◇ $c^2 = a^2 + 110ab + b^2$. **No**: violates $a + b > c$.

Dimensional Analysis Proof of the Pythagorean Theorem



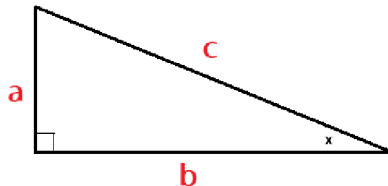
◇ Area is a function of hypotenuse c and angle x .

Dimensional Analysis Proof of the Pythagorean Theorem



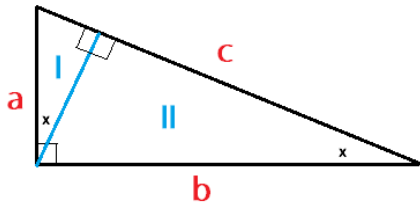
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- ◇ $\text{Area}(c, x) = f(x)c^2$ for some function f (similar triangles).

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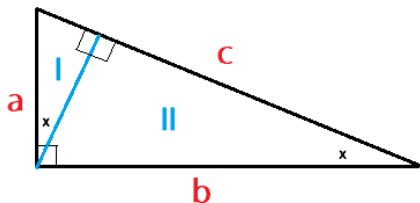
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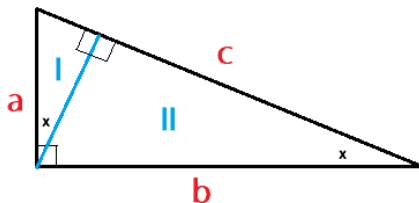
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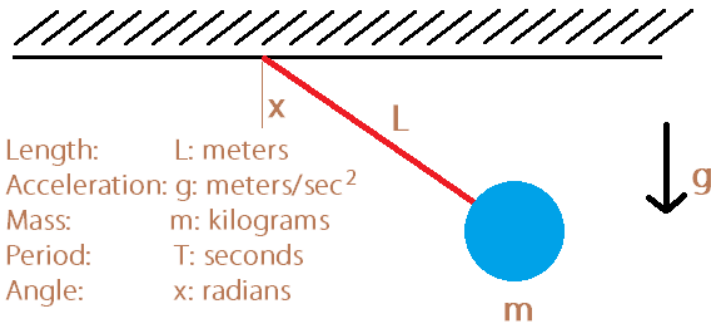
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- ◇ $f(x)a^2 + f(x)b^2 = f(x)c^2$

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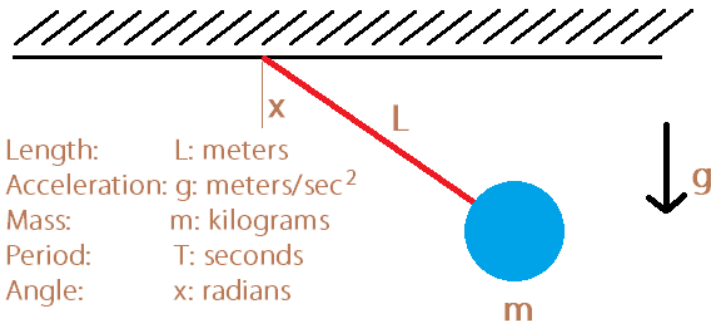


- ◇ Area is a function of hypotenuse c and angle x .
- ◇ $\text{Area}(c, x) = f(x)c^2$ for some function f (CPCTC).
- ◇ Must draw an auxiliary line, but where? Need right angles!
- ◇ $f(x)a^2 + f(x)b^2 = f(x)c^2 \Rightarrow a^2 + b^2 = c^2$.

Dimensional Analysis and the Pendulum

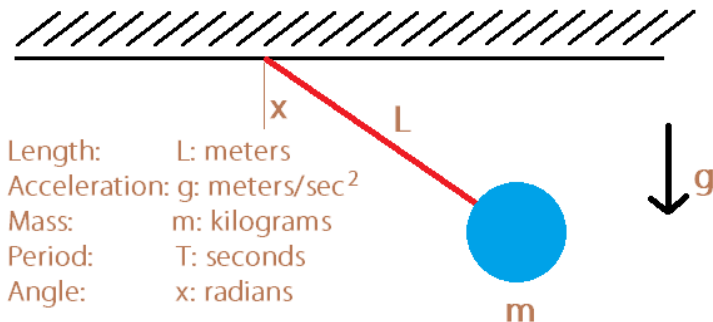


Dimensional Analysis and the Pendulum



Period: Need combination of quantities to get seconds.

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$$T = f(x)\sqrt{L/g}.$$

Guessing Pythagoras:

Finding the Functional Form

Idea is to try and guess the correct functional form for Pythagoras.

Guess will have some free parameters, determine by special cases.

Natural guesses: linear, quadratic,

Linear Attempt

Guess linear relation: $c = \alpha a + \beta b$: what are α, β ?

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- $a \rightarrow 0$ have very thin triangle so $b \rightarrow c$ and thus $\beta = 1$.
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Question: Does $c = a + b$ make sense?

Linear Attempt: Analyzing $c = a + b$ (so $a = b = 1$ implies $c = 2$)

So, if linear, *must be* $c = a + b$. Using:

- Area rectangle x by y is xy .
- Area right triangle of sides x by y is $\frac{1}{2}xy$.

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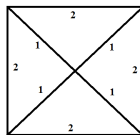


Figure: Four triangles and a square, assuming $c = a + b$ and $a = b = 1$.

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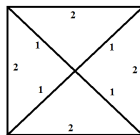


Figure: Four triangles and a square, assuming $c = a + b$ and $a = b = 1$.

Calculate area of big square two ways:

- Four triangles, each area $\frac{1}{2}1 \cdot 1$: total is 2.
- Square of sides 2: area is $2 \cdot 2 = 4$.

Contradiction! Cannot be linear!

Quadratic Attempt:

Guess quadratic: $c^2 = \alpha a^2 + \gamma ab + \beta b^2$: what are α, β, γ ?

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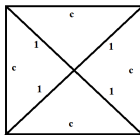


Figure: Four triangles and a square: $a = b = 1$.

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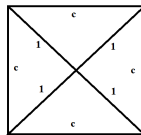


Figure: Four triangles and a square: $a = b = 1$.

Equating areas: $c^2 = 4 \left(\frac{1}{2} 1 \cdot 1 \right)$, so $c^2 = 2$ or $c = \sqrt{2}$.

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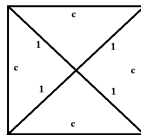


Figure: Four triangles and a square: $a = b = 1$.

Equating areas: $c^2 = 4 \left(\frac{1}{2} 1 \cdot 1 \right)$, so $c^2 = 2$ or $c = \sqrt{2}$.
Thus $2 = 1 + \gamma 1 \cdot 1 + 1$, so $\gamma = 0$ and $c^2 = a^2 + b^2$.

Warnings:

Not a proof: just shows that if quadratic, must be $c^2 = a^2 + b^2$.

In lowest terms:

$$\frac{16}{64} = \frac{1}{4}, \quad \frac{19}{95} = \frac{1}{5}, \quad \frac{49}{98} = \frac{1}{2},$$

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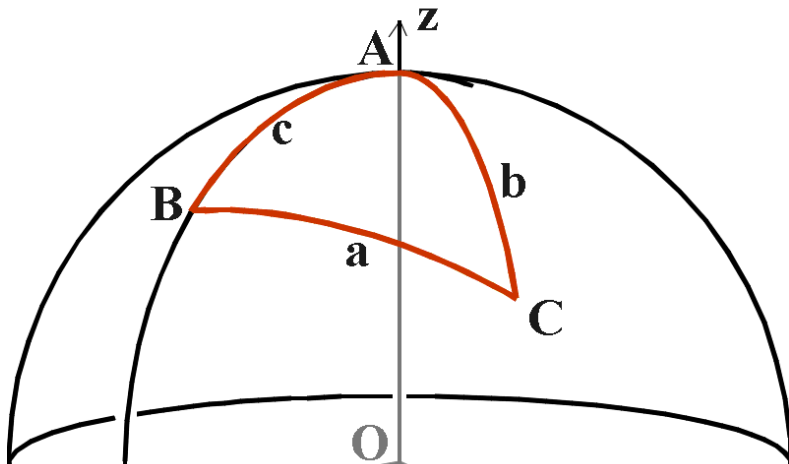
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Extending Pythagoras: The Sphere

Pythagoras on a Sphere

What should the Pythagorean Theorem be on a sphere?



Spherical Coordinates

Spherical Coordinates: $\rho \in [0, R]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$.

- $x = \rho \sin(\theta) \cos(\phi)$.
- $y = \rho \sin(\theta) \sin(\phi)$.
- $z = \rho \cos(\theta)$.

Note $z = \rho \cos(\theta)$, then (x, y) from circle of radius $r = \rho \sin(\theta)$ and angle ϕ .

Special Cases

What could the Pythagorean Formula be on a sphere?

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- If a, b, c small relative to radius R should reduce to planar Pythagoras.
- Can have equilateral right triangle with $a = b = c$.
- Only depends on ratios $a/R, b/R, c/R$.

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- Only depends on ratios $a/R, b/R, c/R$.

Maybe a relation involving cosines of $a/R, b/R, c/R$ as arc length is related to angle!

$$\cos(u) = 1 - u^2/2! + u^4/4! - \dots \approx 1 - u^2/2 \text{ (} u \text{ small).}$$

Ingredients (will consider R large relative to a, b, c):

- $\cos(a/R) \approx 1 - \frac{1}{2} \frac{a^2}{R^2}.$
- $\cos(b/R) \approx 1 - \frac{1}{2} \frac{b^2}{R^2}.$
- $\cos(c/R) \approx 1 - \frac{1}{2} \frac{c^2}{R^2}.$

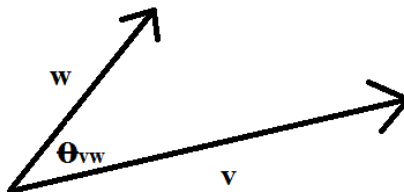
Algebra: $\cos(c/R) \approx \cos(a/R) \cos(b/R)$:

$$1 - \frac{1}{2} \frac{c^2}{R^2} \approx \left(1 - \frac{1}{2} \frac{a^2}{R^2}\right) \left(1 - \frac{1}{2} \frac{b^2}{R^2}\right) = 1 - \frac{a^2 + b^2}{2R^2} + \frac{a^2 b^2}{4R^4}$$

$$c^2 \approx a^2 + b^2 - \frac{2a^2 b^2}{R^2}.$$

Needed Input: Dot Product $\vec{v} \cdot \vec{w}$

$$(v_1, \dots, v_n) \cdot (w_1, \dots, w_n) = v_1 w_1 + \dots + v_n w_n.$$

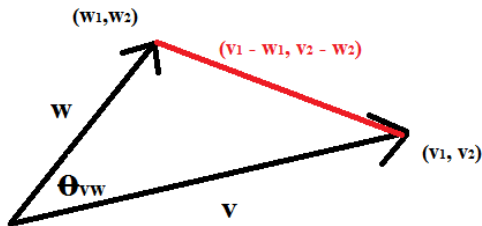


Theorem

If θ_{vw} is the angle between $\vec{v}$ and $\vec{w}$ then

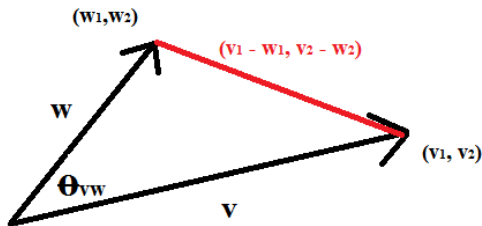
$$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta_{vw}, \text{ where } |\vec{v}| = \sqrt{v_1^2 + \dots + v_n^2}.$$

Proof of Dot Product Formula (Plane)



Use the Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cos \theta_{ab}$.

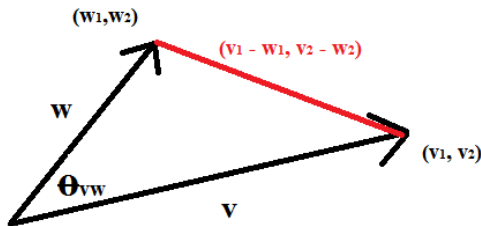
Proof of Dot Product Formula (Plane)



Use the Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cos \theta_{ab}$.

$$(v_1 - w_1)^2 + (v_2 - w_2)^2 = (v_1^2 + v_2^2) + (w_1^2 + w_2^2) - 2|\vec{V}| |\vec{W}| \cos \theta_{vw}.$$

Proof of Dot Product Formula (Plane)



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After some algebra:

$$v_1 w_1 + v_2 w_2 = |\vec{V}| |\vec{W}| \cos \theta_{vw},$$

completing the proof.

Spherical Proof

Three points: P_0, P_A, P_B :

- $P_0 : (R, 0, 0)$.
- $P_A : (R, \theta_A, \phi_A): |\overrightarrow{P_A P_0}| = \theta_A R = a$.
- $P_B : (R, \theta_B, \phi_B): |\overrightarrow{P_B P_0}| = \theta_B R = b$.

Length $\overrightarrow{P_B P_A}$ is $c = \theta_{AB} R$, where θ_{AB} angle between $\overrightarrow{P_A P_0}$ and $\overrightarrow{P_B P_0}$.

Proof follows from dot product:

$$\vec{P} \cdot \vec{Q} = |\vec{P}| |\vec{Q}| \cos(\theta_{PQ}).$$

Spherical Proof: Continued

Cartesian Coordinates for Dot Product:

Remember right triangle: can take $\phi_A = 0$, $\phi_B = \pi/2$.

• $\vec{P_A P_0} : (R \sin \theta_A, 0, R \cos \theta_A)$, length is R .

• $\vec{P_B P_0} : (0, R \sin \theta_B, R \cos \theta_B)$, length is R .

$$\vec{P_B} \cdot \vec{P_A} = 0 + 0 + R^2 \cos \theta_A \cos \theta_B.$$

Dot product now gives

$$\cos(\theta_{AB}) = \frac{\vec{P_B} \cdot \vec{P_A}}{|\vec{P_B P_0}| |\vec{P_A P_0}|} = \frac{\vec{P_B} \cdot \vec{P_A}}{R^2}.$$

Substituting yields

$$\cos(\theta_{AB}) = \frac{R^2 \cos \theta_A \cos \theta_B}{R^2} = \cos \theta_A \cos \theta_B,$$

proving spherical Pythagoras!

Next Step: Generalize

Keep going! Generalize further!

What's the next natural candidate?

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What's the next natural candidate? **Hyperbolic!**

Guess:

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Guess: $\cosh(c) = \cosh(a) \cosh(b)$, where $\cosh$ is the hyperbolic cosine!

$$\cos(x) = \frac{1}{2} (e^{ix} + e^{-ix}), \quad \cosh(x) = \frac{1}{2} (e^x + e^{-x}).$$

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$$\cos(x) = \frac{1}{2} (e^{ix} + e^{-ix}), \quad \cosh(x) = \frac{1}{2} (e^x + e^{-x}).$$

Fun identities:

- $\cosh^2(x) - \sinh^2(x) = 1.$
- $\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y).$
- $\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y)....$

Conclusion

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- ◇ Math is not complete – explore and conjecture!
- ◇ Different proofs highlight different aspects.
- ◇ Get a sense of what to try / what might work.

Feeling Equations

Sabermetrics

Sabermetrics is the art of applying mathematics and statistics to baseball.

Danger: not all students like sports (Red Sox aren't making life easier!).

Lessons: not just for baseball; try to find the **right** statistics that others miss, competitive advantage (business, politics).

Estimating Winning Percentages

Assume team A wins p percent of their games, and team B wins q percent of their games. Which formula do you think does a good job of predicting the probability that team A beats team B? Why?

$$\frac{p + pq}{p + q + 2pq},$$

$$\frac{p + pq}{p + q - 2pq}$$

$$\frac{p - pq}{p + q + 2pq},$$

$$\frac{p - pq}{p + q - 2pq}$$

Estimating Winning Percentages

$$\frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}, \quad \frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq}$$

How can we test these candidates?

Can you think of answers for special choices of p and q ?

Estimating Winning Percentages

$$\frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}, \quad \frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq}$$

Homework: explore the following:

- ◇ $p = 1, q < 1$ (do not want the battle of the undefeated).
- ◇ $p = 0, q > 0$ (do not want the Toilet Bowl).
- ◇ $p = q$.
- ◇ $p > q$ (can do $q < 1/2$ and $q > 1/2$).
- ◇ Anything else where you 'know' the answer?

Estimating Winning Percentages

$$\frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}, \quad \frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq}$$

Homework: explore the following:

- ◇ $p = 1, q < 1$ (do not want the battle of the undefeated).
- ◇ $p = 0, q > 0$ (do not want the Toilet Bowl).
- ◇ $p = q$.
- ◇ $p > q$ (can do $q < 1/2$ and $q > 1/2$).
- ◇ Anything else where you 'know' the answer?

Estimating Winning Percentages

$$\frac{p - pq}{p + q - 2pq} = \frac{p(1 - q)}{p(1 - q) + (1 - p)q}$$

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Estimating Winning Percentages: ‘Proof’



Figure: First see how A does, then B .

Estimating Winning Percentages: 'Proof'

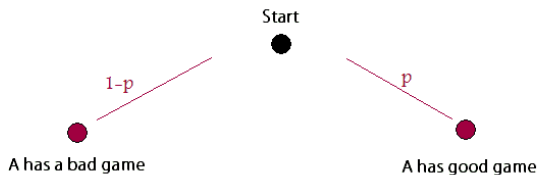


Figure: Two possibilities: A has a good day, or A doesn't.

Estimating Winning Percentages: 'Proof'

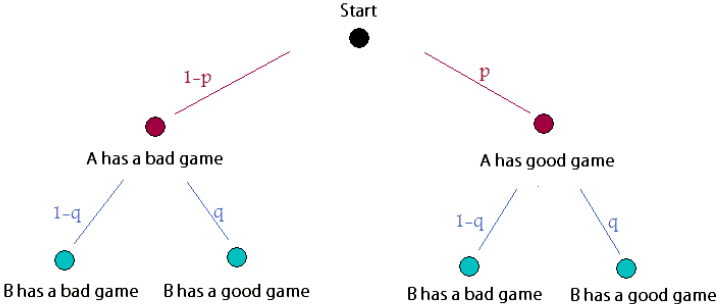


Figure: *B* has a good day, or doesn't.

Estimating Winning Percentages: 'Proof'

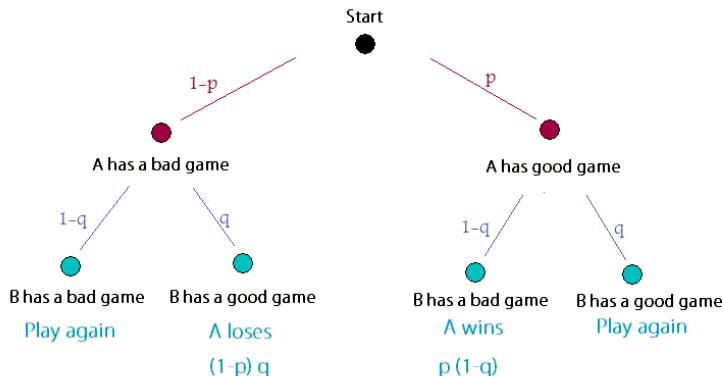
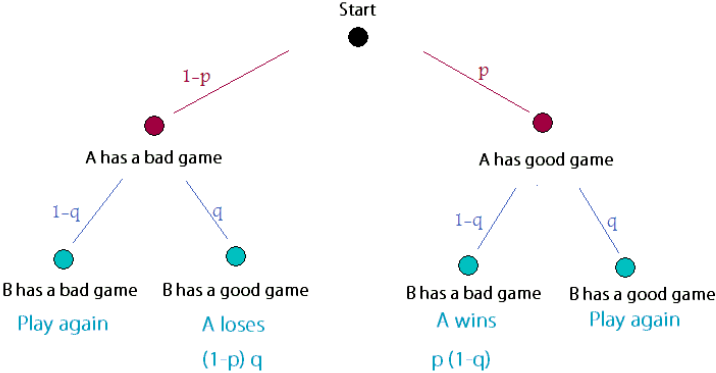


Figure: Two paths terminate, two start again.

Estimating Winning Percentages: ‘Proof’



Probability A wins is

$$\frac{p(1-q)}{p(1-q) + (1-p)q} = \frac{p - pq}{p + q - 2pq}$$

Figure: Probability A beats B.

Lessons

Special cases can give clues.

Algebra can suggests answers.

Better formula: Bill James' Pythagorean Won-Loss formula.

Numerical Observation: Pythagorean Won-Loss Formula

Parameters

- RS_{obs} : average number of runs scored per game;
- RA_{obs} : average number of runs allowed per game;
- γ : some parameter, constant for a sport.

James' Won-Loss Formula (NUMERICAL Observation)

$$\text{Won} - \text{Loss Percentage} = \frac{RS_{\text{obs}}^{\gamma}}{RS_{\text{obs}}^{\gamma} + RA_{\text{obs}}^{\gamma}}$$

γ originally taken as 2, numerical studies show best γ is about 1.82. Used by ESPN, MLB.

See <http://arxiv.org/abs/math/0509698> for a 'derivation'.

Other Gems

Sums of Integers

$$S_n := 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2.$$

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Proof 2: Grouping:

$$2S_n = (\textcolor{red}{1} + \textcolor{blue}{n}) + (\textcolor{red}{2} + \textcolor{blue}{(n-1)}) + \cdots + (\textcolor{red}{n} + \textcolor{blue}{1}).$$

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Can improve: divide and conquer again: lather, rinse, repeat....

$$\frac{n}{4} \frac{n}{4} + \frac{n}{4} \frac{2n}{4} + \frac{n}{4} \frac{3n}{4} \leq S_n, \quad \text{so} \quad \frac{6}{16}n^2 \leq S_n.$$

Geometric Irrationality Proofs: <http://arxiv.org/abs/0909.4913>

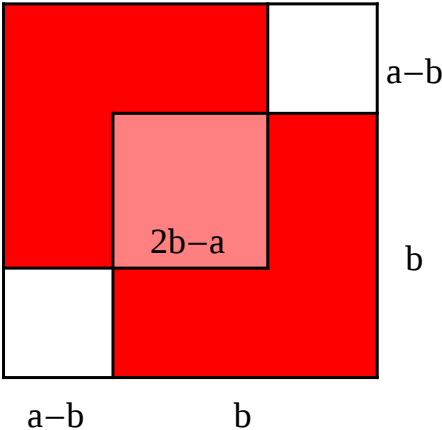


Figure: Geometric proof of the irrationality of $\sqrt{2}$.

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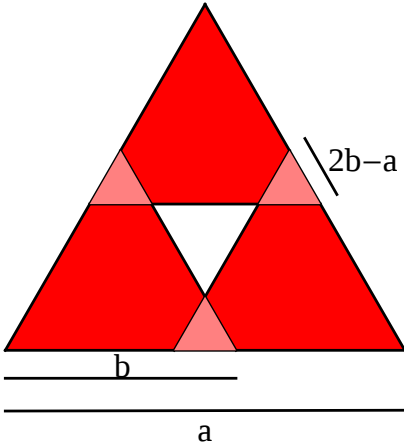


Figure: Geometric proof of the irrationality of $\sqrt{3}$

Geometric Irrationality Proofs: <http://arxiv.org/abs/0909.4913>

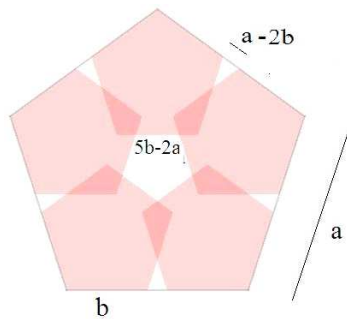


Figure: Geometric proof of the irrationality of $\sqrt{5}$.

Geometric Irrationality Proofs: <http://arxiv.org/abs/0909.4913>

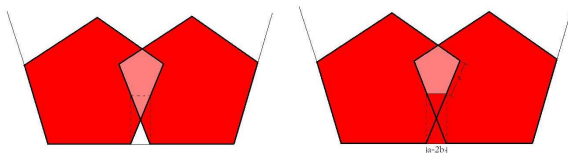


Figure: Geometric proof of the irrationality of $\sqrt{5}$: the kites, triangles and the small pentagons.

Geometric Irrationality Proofs: <http://arxiv.org/abs/0909.4913>

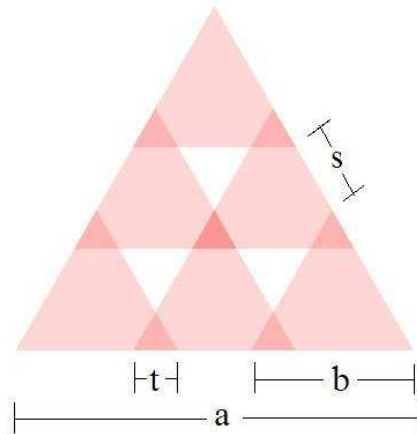


Figure: Geometric proof of the irrationality of $\sqrt{6}$.

Preliminaries: The Cookie Problem

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Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$. Solved $x_1 + \dots + x_P = C, x_i \geq 0$.

Proof: Consider $C + P - 1$ cookies in a line.

Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.

Divides the cookies into P sets.

Example: 8 cookies and 5 people ($C = 8, P = 5$):

