Distribution of Summands in Generalized Zeckendorf Decompositions

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Joint with: Olivia Beckwith, Amanda Bower, Louis Gaudet, Shiyu Li, and Steven J. Miller
http://www.williams.edu/Mathematics/sjmiller/public_html

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Introduction

Goals of the Talk

- Generalize Zeckendorf decompositions
- Analyze gaps (in the bulk and longest)
- Power of generating functions
- Some open problems (if time permits)

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

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$2012 = 1597 + 377 + 34 + 3 + 1 = F_{16} + F_{13} + F_8 + F_3 + F_1$.

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Results

Central Limit Type Theorem [KKMW]

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

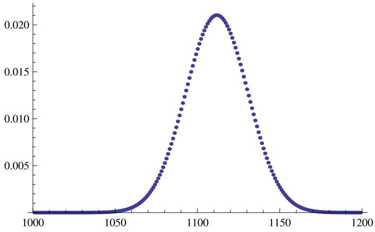

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

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- Zeckendorf
- Lekkerkerker: Average number summands is $C_{\rm Lek} n + d$.
- Central Limit Type Theorem

Gaps Between Summands

Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

$$i_n - i_{n-1}, \; i_{n-1} - i_{n-2}, \; \ldots \;, \; i_2 - i_1.$$

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Let $P_n(m)$ be the probability that a gap for a decomposition in $[H_n, H_{n+1})$ is of length m.

Big Question: What is $P(m) = \lim_{n\to\infty} P_n(m)$?

Main Results

Theorem (Base B Gap Distribution (SMALL 2011))

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$*, and for* $k \geq 1$*,* $P(k) = c_B B^{-k}$*, with* $c_B = \frac{(B-1)(3B-2)}{B^2}$*.*

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(k) = 1/\phi^k$ *for* $k \geq 2$*, with* $\phi = \frac{1+\sqrt{5}}{2}$ *the golden mean.*

Theorem (Zeckendorf Gap Distribution (SMALL 2012))

Gap measures $\nu_{m;n}$ *converge almost surely to average gap measure where* $P(k) = 1/\phi^k$ *for* $k \geq 2$*.*

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Theorem (Zeckendorf Gap Distribution (SMALL 2012))

Gap measures $\nu_{m;n}$ converge almost surely to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

Theorem

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ be a positive linear recurrence of length L where $c_i \geq 1$ for all $1 \leq i \leq L$. Then $P(j) =$

$$\begin{cases} 1 - (\frac{a_1}{C_{Lek}})(\lambda_1^{-n+2} - \lambda_1^{-n+1} + 2\lambda_1^{-1} + a_1^{-1} - 3) & : j = 0 \\ \lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1 - 2a_1) + a_1) & : j = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & : j \geq 2 \end{cases}$$

Proof of Fibonacci Result

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

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$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Calculating $X_{i,i+k}$

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For the indices less than i: F_{i-1} choices. Why? Have F_i, don't have F_{i-1}. Follows by Zeckendorf: like the interval $[F_i, F_{i+1})$ as have F_i, number elements is $F_{i+1} - F_i = F_{i-1}$.

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For the indices greater than $i+k$: $F_{n-k-i-2}$ choices. Why? Have F_n, don't have F_{i+k+1}. Like Zeckendorf with potential summands $F_{i+k+2}, \dots, F_n$. Shifting, like summands $F_1, \dots, F_{n-k-i-1}$, giving $F_{n-k-i-2}$.

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So total choices number of choices is $F_{n-k-2-i}F_{i-1}$.

Determining $P(k)$

$$\sum_{i=1}^{n-k} X_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$

- $\sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$ is the x^{n-k-3} coefficient of $(g(x))^2$, where $g(x)$ is the generating function of the Fibonaccis.
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$P(k) = C/\phi^k$ for some constant C, so $P(k) = 1/\phi^k$.

Proof sketch of almost sure convergence

- $m = \sum_{j=1}^{k(m)} F_{i_j}$,
  $\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right)$.
- $\mu_{m,n}(t) = \int x^t \mathrm{d}\nu_{m;n}(x)$.
- Show $\mathbb{E}_m[\mu_{m;n}(t)]$ equals average gap moments, $\mu(t)$.
- Show $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^2]$ and $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^4]$ tend to zero.

Key ideas: (1) Replace $k(m)$ with average (Gaussianity); (2) use $X_{i,i+g_1,j,j+g_2}$.

Longest Gap

Question

Given a random number m in the interval $[F_n, F_{n+1})$, what is the probability that m has longest gap equal to r?

For the Fibonacci Recurrence

Theorem (Longest Gap Asymptotic CDF)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\mathrm{Prob}\left(L_n(m) \leq f(n)\right) \;\approx\; e^{-e^{\log n - f(n)/\log\phi}}$$

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Immediate Corollary: If $f(n)$ grows ***slower*** or ***faster*** than $\log n/\log\phi$, then $\text{Prob}(L_n(m) \leq f(n))$ goes to **0** or ***1***, respectively.

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From this analysis we get the mean:

$$\mu_n = \frac{\log\left(\frac{\phi^2}{\phi^2+1)}n\right)}{\log \phi} + \frac{\gamma}{\log \phi} - \frac{1}{2} + \text{Error}_{MC} + \epsilon_1(n),$$

where $\epsilon(n) \to 0$ for large n.

Fibonacci Case Generating Function

Let $G(n, k, f)$ be the number of m in $[F_n, F_{n+1})$ that have k nonzero summands in their Zeckendorf Decomposition and all gaps less than $f(n)$.

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$G(n, k, f)$ is the coefficient of x^n for the generating function

$$\frac{1}{1-x}\left[\sum_{j=2}^{f(n)-2} x^j\right]^{k-1}$$

The Combinatorics

Why the n^{th} coefficient of $\frac{1}{1-x}\left[\sum_{j=2}^{f(n)-1} x^j\right]^{k-1}$?

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- The sum of the gaps of x is $\leq n$.
- Each gap is ≥ 2.
- Each gap is $< f(n)$.

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For fixed k, this is surprisingly hard to analyze. We only care about the sum over all k.

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Use **partial fractions** and **Rouché** to find the CDF.

Partial Fractions

Write the roots of $x^f - x^2 - x - 1$ as $\{\alpha_i\}_{i=1}^{f}$. We can write our generating function

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We can take the n^{th} coefficient of this expansion to find the number of y with gaps less than $f(n)$.

Partial Fractions

Divide the **number** of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$, **by** the total number of m, which is $F_{n+1} - F_n \sim \frac{1}{\sqrt{5}}\phi^n$.

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Theorem (Exact CDF)

The proportion of $m \in [F_n, F_{n+1})$ with $L(x) < f(n)$ is exactly

$$\sum_{i=1}^{f(n)} \frac{-\sqrt{5}(\alpha_i)}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \left(\frac{1}{\alpha_i}\right)^{n+1} \frac{1}{(\phi^n - (-1/\phi)^n)}$$

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Now, we find out about the roots of $x^f - x^2 - x + 1$.

Rouché's Theorem

A useful consequence of the argument principle is Rouché's Theorem:

Theorem (Rouché's Theorem)

Suppose we have two functions f and g on a region K and that $|f(x) - g(x)| < |g(x)|$ for all x on the boundary δK. Then f and g have the same number of roots inside K.

Rouché and Roots

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Lemma (Critical Root Behavior)

For $f \in \mathbb{N}$ and $f \geq 4$, the polynomial $p_f(z) = z^f - z^2 - z + 1$ has exactly one root z_f with $|z_f| < .9$. Further, $z_f \in \mathbb{R}$ and $z_f = \frac{1}{\phi} + \left|\frac{z_f^f}{z_f + \phi}\right|$, so as $f \to \infty$, z_f converges to $\frac{1}{\phi}$.

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We only care about the **smallest root**.

Getting the CDF

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$$\lim_{n\to\infty} (\phi z_f)^{-n} = \lim_{n\to\infty} \left(1 + \left|\frac{\phi z_f^{f(n)}}{\phi + z_f}\right|\right)^{-n}.$$

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We can see the double exponential by taking logarithms, Taylor expanding, and re-exponentiating.

Mean/Variance

Note

$$\mu = \sum_{j=1}^{n} j\left(CDF(j) - CDF(j-1) \right)$$

Using **Partial Summation**, Euler-Maclaurin, and evaluating the resulting integrals, we calculate the mean and variance.

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$$\mu = \sum_{j=1}^{n} j(\ CDF(j) - CDF(j-1)\ )$$

Using **Partial Summation**, Euler-Maclaurin, and evaluating the resulting integrals, we calculate the mean and variance.

$$\mu_n = \frac{\log\left(\frac{\phi^2}{\phi^2+1)}n\right)}{\log\phi} + \frac{\gamma}{\log\phi} - \frac{1}{2} + \text{ Error}_{MC} + \epsilon_1(n),$$

and

$$\sigma_n^2 = \frac{\pi^2}{6\log\phi} - \frac{1}{12} + \text{ Error}_{MC}^2 + \epsilon_2(n),$$

where $\epsilon_1(n), \epsilon_2(n)$ go to zero in the limit.

Positive Linear Recurrence Sequences

This method can be greatly generalized to **Positive Linear Recurrence Sequences** ie: linear recurrences with non-negative coefficients. WLOG:

$$H_{n+1} = c_1 H_{n-(j_1=0)} + c_2 H_{n-j_2} + \cdots + c_L H_{n-j_L}.$$

Theorem (Zeckendorf's Theorem for *PLRS* recurrences)

*Any $b \in \mathbb{N}$ has a unique **legal** decomposition into sums of H_n, $b = a_1 H_{i_1} + \cdots + a_{i_k} H_{i_k}$.*

Here **legal** reduces to non-adjacency of summands in the Fibonacci case.

Messier Combinatorics

The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of x^{n-s} in the generating function:

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$$\frac{1}{1-x}\left(c_1 - 1 + c_2x^{t_2} + \cdots + c_Lx^{t_L}\right) \times$$
$$\times \sum_{k\geq 0}\Bigg[\left((c_1-1)x^{t_1} + \cdots + (c_L-1)x^{t_L}\right)\left(\frac{x^{s+1}-x^f}{1-x}\right) +$$
$$+x^{t_1}\left(\frac{x^{s+t_2-t_1+1}-x^f}{1-x}\right) + \cdots + x^{t_{L-1}}\left(\frac{x^{s+t_L-t_{L-1}+1}-x^f}{1-x}\right)\Bigg]^k.$$

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$$+x^{t_1}\left(\frac{x^{s+t_2-t_1+1}-x^f}{1-x}\right) + \cdots + x^{t_{L-1}}\left(\frac{x^{s+t_L-t_{L-1}+1-x^f}}{1-x}\right)\Bigg]^k.$$

A geometric series!

Let $f > j_L$. The number of $x \in [H_n, H_{n+1})$, with longest gap $< f$ is given by **the coefficient of** s^n in the generating function

$$F(s) = \frac{1 - s^{j_L}}{\mathcal{M}(s) + s^f \mathcal{R}(s)},$$

where

$$\mathcal{M}(s) = 1 - c_1 s - c_2 s^{j_2+1} - \cdots - c_L s^{j_L+1},$$

and

$$\mathcal{R}(s) = c_{j_1+1} s^{j_1} + c_{j_2+1} s^{j_2} + \cdots + (c_{j_L+1} - 1) s^{j_L}.$$

and c_i and j_i are defined **as above**.

The **coefficients** in the **partial fraction** expansion might blow up from multiple roots.

The **coefficients** in the **partial fraction** expansion might blow up from multiple roots.

Theorem (Mean and Variance for "Most Recurrences")

For x in the interval $[H_n, H_{n+1})$, the mean longest gap μ_n and the variance of the longest gap σ_n^2 are given by

$$\mu_n = \frac{\log\left(\frac{\mathcal{R}(\frac{1}{\lambda_1})}{\mathcal{G}(\frac{1}{\lambda_1})} n\right)}{\log \lambda_1} + \frac{\gamma}{\log \lambda_1} - \frac{1}{2} + Error_{MC}^1 + \epsilon_1(n),$$

and

$$\sigma_n^2 = \frac{\pi^2}{6 \log \lambda_1} - \frac{1}{12} + Error_{MC}^2 + \epsilon_2(n),$$

where $\epsilon_i(n)$ tends to zero in the limit, and $Error_{MC}$ comes from the Euler-Maclaurin Formula.

Future Research

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- Determine the distribution of the $k(n)^{\text{th}}$ longest gap, with $k(n)$ either constant or slowly growing in n.
- Extend to recurrences with zero coefficients.
- Generalize to signed decompositions

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