Generalizations of Zeckendorf's Theorem to Two-Dimensional Sequences.

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Outline

- How to Combinatorially Interpret Zeckendorf Decompositions

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- Gaussianity for Simple Jump Paths
- Future Research Questions

Definition (Zeckendorf Decompositions)

A **Zeckendorf Decomposition** is a way to write a natural number as the sum of non-adjacent Fibonacci Numbers.

Theorem (Zeckendorf's Theorem)

Every natural number has a unique Zeckendorf Decomposition.

- Example: $335 = 13 + 89 + 233$

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- Example: $335 = 13 + 89 + 233$
- Example: $1033 = 1 + 3 + 8 + 34 + 987$

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Construction of Zeckendorf Diagonal Sequence

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- For each $n \in \mathbb{N}^+$, check if any downward/leftward path sums to the number. If not, add the number to the sequence so that it is added to the shortest unfilled diagonal moving from the bottom right to the top left.

$$\begin{bmatrix}
6992 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
2200 & 6054 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
954 & 2182 & 5328 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
364 & 908 & 2008 & 5100 & \cdots & \cdots & \cdots & \cdots & \cdots \\
138 & 342 & 862 & 1522 & 4966 & \cdots & \cdots & \cdots & \cdots \\
44 & 112 & 296 & 520 & 1146 & 2952 & \cdots & \cdots & \cdots \\
16 & 38 & 94 & 184 & 476 & 1102 & 2630 & \cdots & \cdots \\
4 & 10 & 22 & 56 & 168 & 370 & 1052 & 2592 & \cdots \\
1 & 2 & 6 & 18 & 46 & 140 & 366 & 1042 & 2270
\end{bmatrix}$$

- Our goal is to enumerate how many paths are required for a linear search of a Zeckendorf decomposition.

Definition (Compound Jump Paths)

A **compound jump path** is a path on the lattice grid moving only down and to the left, where movements may be greater than one unit at a time in either direction.

- We count compound jump paths from (a, b) to $(0, 0)$.

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- Let the number of compound jump paths from (a, b) to $(0, 0)$ with k steps be denoted $r_{a,b,k}$.

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- Let the number of simple jump paths from (a, b) to $(0, 0)$ with k steps be denoted $t_{a,b,k}$.

Lemma (Compound Jump Path Partition Lemma)

$\forall a, b \in \mathbb{N}$, $q_{a,b} = \sum_{k=1}^{a+b} r_{a,b,k}$.

Lemma (Simple Jump Path Partition Lemma)

$\forall a, b \in \mathbb{N}$, $s_{a,b} = \sum_{k=1}^{\min\{a,b\}} t_{a,b,k}$.

Lemma (Enumerating Simple Jump Paths)

$\forall a, b \in \mathbb{N}, k \in \min\{a, b\}, t_{a,b,k} = \binom{a-1}{k-1}\binom{b-1}{k-1}$.

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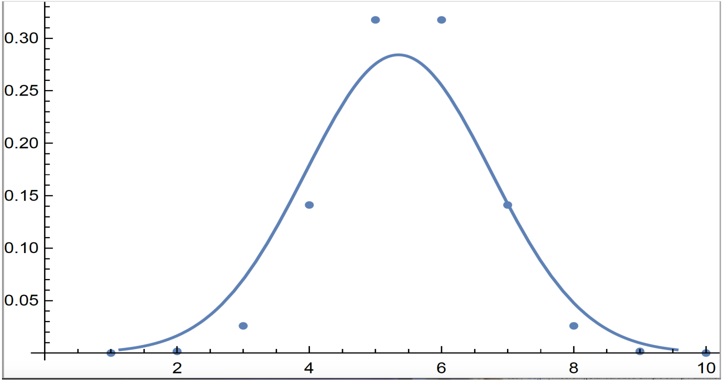
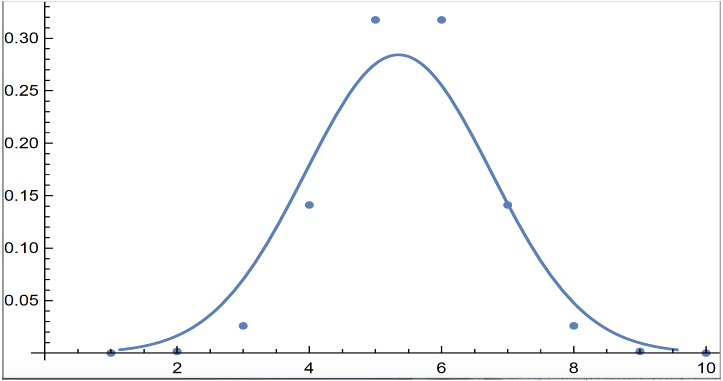
- First factor is number of ways to group *a* objects into *k* nonempty groups
- Second factor is number of ways to group *b* objects into *k* nonempty groups
- Groupings are independently determined



- Represents $\{t_{10,10,k}\}_{k=1}^{10}$



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- Special case: simple jump paths over a square lattice for $n = 10$

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- Simple jump paths: $k \in [n]$

- Densities: number of simple jump paths with a fixed number of steps

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- Study square lattice, i.e. where $a = b$.

Theorem (Mean on Square Lattice)

$\forall n \in \mathbb{N}^+, \mu_{n+1,n+1} \; = \; \frac{1}{2}n + 1 \sim \frac{n}{2}.$

- Derive directly from definition of first moment

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- Use standard techniques for evaluating binomial coefficients

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- Derive directly from definition of second standardized moment
- Use index shift $\sum_{k=1}^{n+1} \left(k - \left(\frac{1}{2}n + 1\right)\right)^2 \binom{n}{k-1}^2 = \sum_{k=0}^{n} (k + 1 - (\frac{1}{2}n + 1))^2 \binom{n}{k}^2$

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- Use standard techniques for evaluating binomial coefficients

- Density function: $f_n(k+1) \;:=\; \frac{t_{n+1,n+1,k+1}}{s_{n+1,n+1}} \;=\; \frac{\binom{n}{k}^2}{\binom{2n}{n}}$

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- Simplifying binomial coefficients gives $\frac{(n!)^4}{(k!)^2((n-k)!)^2(2n)!}$
- Use Stirling's Approximation on each factor: $m! \ \sim \ m^m e^{-m}\sqrt{2\pi m}$

- End result of Stirling expansion is
$f_n(k+1) \;=\; \frac{n^{2n}}{k^{2k}\cdot(n-k)^{2n-2k}\cdot 2^{2n}\cdot\frac{1}{4}\cdot\sqrt{4\pi n}}$

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- Let $P_n(k+1) := \frac{n^n}{k^k (n-k)^{n-k} 2^n}$ and $S_n(k+1) = \frac{1}{\frac{1}{2}\sqrt{\pi n}}$,
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then $f_n(k+1) = P_n(k+1)^2 S_n(k+1)$.

- Let $k := \mu_{n+1,n+1} + x\sigma_{n+1,n+1}$, then
$f_n(k+1)dk = f_n(\mu_n + x\sigma_n + 1)\sigma_n dx \sim f_n(\mu_n + x\sigma_n + 1)\frac{\sqrt{n}}{2}dx$

Apply logarithm to $P_n(k+1) = \frac{n^n}{k^k(n-k)^{n-k}2^n}$:

$$\log P_n(k+1) = n\log(n) - k\log(k) - (n-k)\log(n-k) - n\log(2)$$

Rewrite $k = \frac{n}{2} + \frac{x\sqrt{n}}{2\sqrt{2}} = \frac{n}{2}\left(1 + \frac{x}{\sqrt{2n}}\right)$ to expand $\log(k)$ and $\log(n-k)$:

$$\log(k) = \log\left(\frac{n}{2}\left(1 - \frac{x}{\sqrt{2n}}\right)\right) \approx \log(n) - \log(2) + \log\left(1 - \frac{x}{\sqrt{2n}}\right)$$

$$\log(n-k) = \log\left(\frac{n}{2}\left(1 + \frac{x}{\sqrt{2n}}\right)\right) \approx \log(n) - \log(2) + \log\left(1 + \frac{x}{\sqrt{2n}}\right)$$

Substitute logarithm expansions and approximate $\log\left(1+\frac{x}{\sqrt{2n}}\right)$ and $\log\left(1-\frac{x}{\sqrt{2n}}\right)$ to second order to conclude

$$\log P_n(k+1) \sim -\frac{n}{2}\log\left(1-\frac{x^2}{2n}\right) - \frac{x\sqrt{n}}{2}\left(\frac{x}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right)\right)$$

Approximate $\log\left(1-\frac{x^2}{2n}\right)$ up to second order:

$$-\frac{n}{2}\left(-\frac{x^2}{2n} + O\left(\frac{1}{n^2}\right)\right) - \frac{x\sqrt{n}}{2}\left(\frac{x}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{3}{2}}}\right)\right) \sim -\frac{x^2}{4}$$

It follows that

$$P_n(k+1) \sim e^{-\frac{x^2}{4}} \Rightarrow P_n(k+1)^2 \sim e^{-\frac{x^2}{2}} \Rightarrow$$

$$f_n(k+1) \sim \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

- Normal distribution, mean 0, standard deviation 1.

- Finish generalizations of Gaussian convergence: simple jump paths where $a \neq b$

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- Generalize result to compound jump paths

- H. Alpert, *Differences of Multiple Fibonacci Numbers*, October 20, 2009
- I. Badinskki, C. Huffaker, N. Mccue, C. Miller, K. Miller, S. Miller, M. Stone, *The M&M Game: From Morsels to Modern Mathematics*, September 3, 2015
- O. Beckwith, A. Bower, L. Gaudet, R. Insoft, S. Li, S. Miller, P. Tosteson, *The Average Gap Distribution For Generalized Zeckendorf Decompositions*, Fibonacci Quarterly, December 12, 2012.
- L. Cano, R. Diaz, *Continuous Analogues for the Binomial Coefficients and the Catalan Numbers*, March 22, 2016

- R. Doward, P. Ford, E. Fourakis, P. Harris, S. Miller, E. Palsson, H. Paugh, *New Behavior in Legal Decompositions Arising From Non-Positive Linear Recurrences*, September 10, 2015
- V. Guo, J. Zeng *New Congruences for Sums Involving Apery Numbers or Central Delannoy Numbers*, International Journal of Number Theory, May 25, 2012
- E. Hart, *The Zeckendorf Decomposition of Certain Fibonacci-Lucas Products*, Fibonacci Quarterly, November 1998
- E. Hart, L. Sanchis, *On The Occurrence Of Fn in The Zeckendorf Decomposition of nFn*, February 1997

- M. Kanovich, *Multiset Rewriting Over Fibonacci and Tribonacci Numbers*, Journal of Computer and System Sciences, September 2014
- M. Kologlu, G. Kopp, S. Miller, Y. Wang, *On the Number of Summands in Zeckendorf Decompositons*, Journal of Number Theory, August 19, 2010.
- C. Krattenhaler, *Lattice Path Enumeration*, April 17, 2015
- T. Mansour, A. Munagi, M. Shattuck *Recurrence Relations and Two Dimensional Set Partitions*, Journal of Integer Sequences, March 26, 2011

References

- S. Miller, Y. Wang, *Gaussian Behavior in Generalized Zeckendorf Decompositions*, July 14, 2011
- S. Miller, *The Probability Lifesaver*, 2017
- J. Watkins, *Moments and Generating Functions*, September 29, 2009

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