S-Legal Index Difference Sequences

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SMALL REU 2022 at Williams College
Joint work with Astrid Lilly, Steven J. Miller,
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Joint Mathematics Meetings
Boston, January 7th, 2023

AMS-SIAM Special Session on Research in Mathematics by Undergraduates and Students in Post-Baccalaureate Programs III

Zeckendorf's theorem

Let $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Theorem (Zeckendorf)

Every non-negative integer has a unique decomposition as a sum of distinct non-consecutive Fibonacci numbers.

Equivalent Definition (Fibonacci sequence)

Let F_n be the smallest positive integer which cannot be written as as a sum of non-consecutive terms in $\{F_1, \ldots, F_{n-1}\}$.

Interpretation of consecutiveness

We put the Fibonacci sequence in a 1D array of boxes.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	$\cdots$

1	2	3	5	8	13	21	34	$\cdots$

A legal decomposition does not have summands that correspond to boxes sharing an edge.

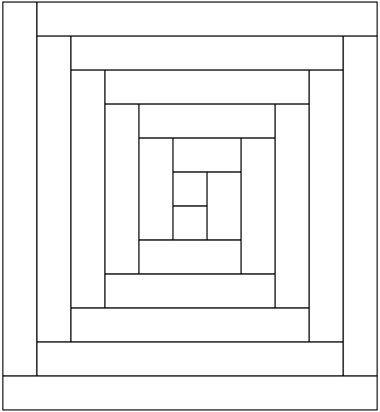
Why stop at 1D?

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	$\cdots$

But, an 1D array of boxes is boring...

Catral, Ford, Harris, Nelson, and Miller decided to use the Fibonacci spiral instead, and defined a new sequence.

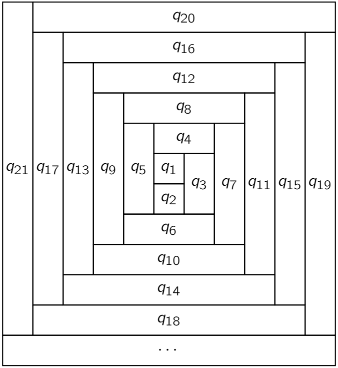
The Fibonacci... spiral?



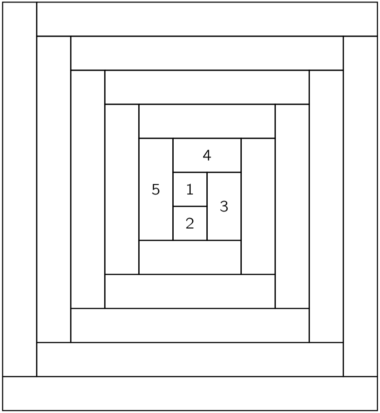
The Fibonacci quilt sequence

Definition (Fibonacci quilt sequence; CFHMN, 2020)

Let q_n be the smallest positive integer which cannot be written as as a sum of non-adjacent terms in $\{q_1, \ldots, q_{n-1}\}$. Two terms are adjacent if their boxes share an edge in the quilt.



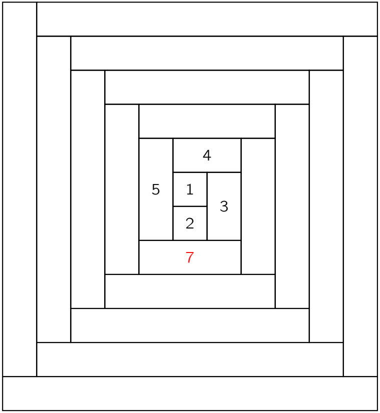
Computing the Fibonacci quilt sequence



All legal sums with at least two elements are at least 6, thus:

$$q_1 = 1,$$
$$q_2 = 2,$$
$$q_3 = 3,$$
$$q_4 = 4,$$
$$q_5 = 5.$$

Computing the Fibonacci quilt sequence

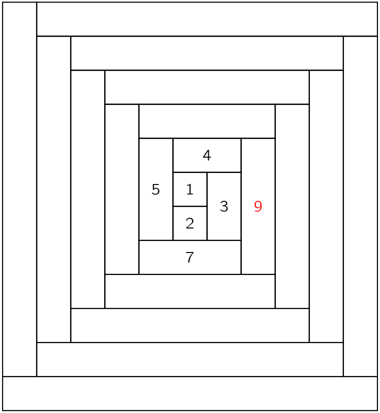


The possible legal sums using $\{1, 2, 3, 4, 5\}$ are:

$$1 = 1,$$
$$2 = 2,$$
$$3 = 3,$$
$$4 = 4,$$
$$5 = 5,$$
$$6 = 2 + 4,$$
$$8 = 3 + 5.$$

Thus, $q_6 = 7$.

Computing the Fibonacci quilt sequence

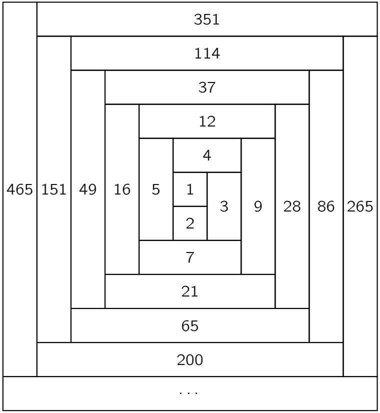


The possible legal sums using $\{1, 2, 3, 4, 5, 7\}$ are:

$1 = 1,$
$2 = 2,$
$3 = 3,$
$4 = 4,$
$5 = 5,$
$6 = 2 + 4,$
$7 = 7,$
$8 = 3 + 5 = 1 + 7,$
$11 = 4 + 7.$

Thus, $q_7 = 9$.

Computing the Fibonacci quilt sequence



And so on...

Recurrence

The behavior of the Fibonacci quilt sequence is well understood:

Proposition (CFHMN, 2020)

Let q_n be the Fibonacci quilt sequence. For $n \geq 5$,

$$q_{n+1} = q_n + q_{n-4}.$$

The triangular quilt sequence

Definition (Triangular quilt sequence; SMALL '22)

Let t_n be the smallest positive integer which cannot be written as as a sum of non-adjacent terms in $\{t_1, \ldots, t_{n-1}\}$. Two terms are adjacent if their boxes share an edge in the Padovan spiral.

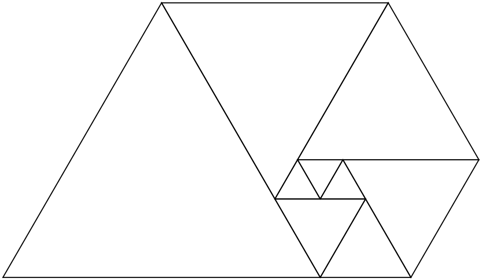


Figure: The Padovan Spiral.

Computing the triangular quilt sequence

We were able to compute the first 50 terms of the triangular quilt sequence.

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	$\cdots$
1	2	3	5	6	11	12	20	23	40	46	80	92	$\cdots$

Question

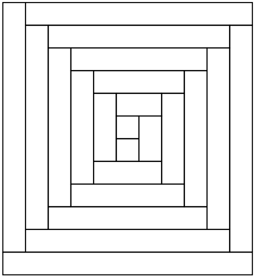
Like the Fibonacci quilt sequence, does the triangular quilt sequence follow a recurrence?

Surprising answer: Based on the first terms, apparently no.

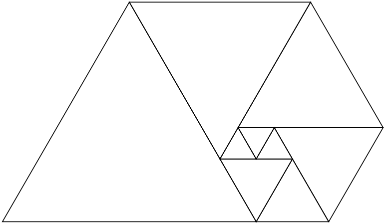
Bonus fact: This sequence is not listed (yet) on the Online Encyclopedia of Integer Sequences.

Simplifying the quilt sequences

Studying 2D constructions is hard. Let's simplify adjacency.



Boxes q_i and q_j are adjacent iff $|i - j| \in \{1, 3, 4\}$ or $\{i, j\} = \{1, 3\}$.



Triangles t_j and t_i are adjacent iff $|i - j| \in \{1, 5\}$ or $\{i, j\} = \{1, 4\}$.

Simplifying the quilt sequences

Fibonacci quilt sequence:

Boxes q_i and q_j are adjacent iff $|i-j| \in \{1,3,4\}$ or $\{i,j\} = \{1,3\}$.

Triangular quilt sequence:

Triangles t_j and t_i are adjacent iff $|i-j| \in \{1,5\}$ or $\{i,j\} = \{1,4\}$.

Fibonacci quilt-like sequence:

a_i and a_j are adjacent iff $|i-j| \in \{1,3,4\}$.

Triangular quilt-like sequence:

a_j and a_i are adjacent iff $|i-j| \in \{1,5\}$.

S-LID decompositions

Fix a set S of positive integers. (For example, $\{1,3,4\}$ or $\{1,5\}$.)

Definition (S-LID decomposition)

An S**-Legal Index Difference (**S**-LID) decomposition** using $\{a_i\}_{i\in I\subseteq\mathbb{Z}_{>0}}$ is a sum of the form

$$N = \sum_{\ell\in L} a_\ell$$

for finite $L \subseteq I$ such that $|\ell_1 - \ell_2| \notin S$ for all $\ell_1, \ell_2 \in L$.

Example

Let $S = \{2\}$ and $\{a_i\}_{i\in I} = \{a_1, a_2, a_3, a_4\}$. Then

- $a_1 + a_2$ is an S-LID decomposition using $\{a_i\}_{i\in I}$,
- $a_1 + a_2 + a_3$ is not an S-LID decomposition using $\{a_i\}_{i\in I}$ because $|3-1| = 2 \in S$.

S-LID sequences

We use S-LID decompositions to construct a sequence.

Definition (S-LID sequence)

The S-**LID sequence** $\{a_i\}_{i=1}^{\infty}$ is defined by:

- a_n is the smallest positive integer that does not have an S-LID decomposition using $\{a_i\}_{i=1}^{n-1}$.

Example

- The $\{\}$-LID sequence is $\{2^{i-1}\}_{i=1}^{\infty}$,
- The $\{1\}$-LID sequence is the Fibonacci sequence,
- Understanding the $\{1, 3, 4\}$ and $\{1, 5\}$-LID sequences will help us understand the Fibonacci and triangular quilt sequences.

Lower bound

Lemma

Let $\{a_i\}_{i=1}^{\infty}$ be the S-LID sequence, and let $k = \max S$. Then, for all $n \geq k+1$,

$$a_{n+1} \geq a_n + a_{n-k}.$$

To prove it, we show that all numbers smaller than $a_n + a_{n-k}$ have an S-LID decomposition using $a_1, \ldots, a_n$.

Proof of lower bound

- 0 has S-LID decomposition using $a_1, \ldots, a_{n-k-1}$.
- 1 has S-LID decomposition using $a_1, \ldots, a_{n-k-1}$.

$\vdots$

- $a_{n-k} - 1$ has S-LID decomposition using $a_1, \ldots, a_{n-k-1}$.

We add a_n to each decomposition.
Since indexes n and $n-k-1$ are more than $k = \max S$ apart, we obtain S-LID decompositions.

Proof of lower bound

- numbers $< a_n$ have S-LID decompositions using $a_1, \ldots, a_{n-1}$.
- a_n has S-LID decomposition using $a_1, \ldots, a_n$.
- $a_n + 1$ has S-LID decomposition using $a_1, \ldots, a_n$.

$\vdots$

- $a_n + a_{n-k} - 1$ has S-LID decomposition using $a_1, \ldots, a_n$.

Since a_{n+1} is the smaller number without a S-LID dec. using $a_1, \ldots, a_n$, thus

$$a_{n+1} \geq a_n + a_{n-k}.$$

Equality of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The $\{1\}$-LID sequence ($k = 1$) is the Fibonacci sequence, which satisfies

$$a_{n+1} = a_n + a_{n-1}.$$

Equality of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The first terms of the $\{1, 2, 4\}$-LID sequence ($k = 4$) are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	$\cdots$
1	2	3	4	6	7	9	12	16	22	29	38	50	$\cdots$

- $a_1 + a_5 = 1 + 6 = 7 = a_6$
- $a_2 + a_6 = 2 + 7 = 9 = a_7$
- $a_3 + a_7 = 3 + 9 = 12 = a_8$
- and so on... (up to our computing power, $n \approx 50$)

Weirdness of lower bound ($a_{n+1} \geq a_n + a_{n-k}$)

Example

The first terms of the $\{1,5\}$-LID sequence ($k = 5$) are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	$\cdots$
1	2	3	5	8	13	14	21	24	43	51	67	105	$\cdots$

$a_3 + a_8 = 3 + 21 = 24 = a_9$

$a_4 + a_9 = 5 + 24 < 43 = a_{10}$

$a_5 + a_{10} = 8 + 43 = 51 = a_{11}$

$a_6 + a_{11} = 13 + 51 < 67 = a_{12}$

$a_7 + a_{12} = 14 + 67 < 105 = a_{13}$

For $6 \leq n \leq 47$,

$$a_{n+1} = a_n + a_{n-5}$$

iff $n \in \{6, 8, 10, \quad 21, 23, 25, 27, 29, \quad 34, 36, 38, 40, 42, 44, 46\}$.

Dream False Theorem

Dream Goal (this may be false)

The S-LID sequence ($k = \max S$) satisfies

$$a_{n+1} = a_n + a_{n-k}$$

for all sufficiently large n.

Still, the $\{1, 2, \ldots, k\}$-LID sequence and a couple others satisfy

$$a_{n+1} = a_n + a_{n-k}$$

for all sufficiently large n.

Recurrence Theorem

Theorem (SMALL '22)

Fix any finite set T of positive integers with $c = \max T$. Let $k \gg 0$, and $S = \{1, \ldots, k\} \setminus (k - T)$. Then, the S-LID sequence satisfies

$$a_{n+1} = a_n + a_{n-k}$$

for all $n > k + c$.

Proof is by simultaneous induction on three statements:

$$\begin{aligned} A(n): \quad & a_{n+1} = a_n + a_{n-k}, \\ B(n): \quad & a_n + a_{n-k} > a_{n-1} + a_{n-1-(k-c)} + a_{n-1-2(k-c)} + \cdots, \\ C_d(n): \quad & a_{n+1} + a_n > a_{n+c} + a_{n-d}. \end{aligned}$$

Bounding possible sums

To prove that $a_{n+1} = a_n + a_{n-k}$ in certain cases, we bound the largest possible S-LID decomposition using $a_1, \ldots, a_n$.

Lemma

For any finite set S and $c > 0$, if $k = \max S$, then for all $n \geq 1$ we have

$$a_n > a_{n-(k-c)} + a_{n-2(k-c)} + \cdots$$

whenever $k \geq 2(c+1)$.

We prove this by iteratively applying the lower bound lemma.

Bounding possible sums

Using the previous lemma and the inequality

$$C_d(n-k-1): \; a_{n-k} + a_{n-k-1} \; > \; a_{n-k-1+c} + a_{n-k-1-d},$$

where $d > 0$ is a constant not depending on k, we obtain

$$B(n): \; a_n + a_{n-k} \; > \; a_{n-1} + a_{n-1-(k-c)} + a_{n-1-2(k-c)} + \cdots .$$

The RHS is an upper bound on any S-LID decomposition using $a_1, \dots, a_{n-1}$, hence

$$A(n): \quad a_{n+1} \; = \; a_n + a_{n-k}.$$

Finishing the proof

Terms appearing in $C_d(n-k-1)$ are much farther back in the sequence than n, so we can prove it using the recurrence relation!

Theorem (SMALL '22)

Suppose that there exists $d > 0$ such that

- $k \geq 2d - 4c + 2$,
- $C_d(n)$ *holds for* $c + d + 1 \leq n \leq k + c + 1 + d$, *and*
- $B(n)$ *holds for* $k + c + 1 \leq n \leq 2k + c + d + 2$.

Then

$$a_{n+1} = a_n + a_{n-k}$$

for all integers $n > k + c$.

We proved that these base cases hold for all $k \gg 0$ for fixed T.

Future work

- Understand behavior for S that do not give a recurrence relation, as well as infinite sets S.
- Study statistics for the number of S-LID decompositions of integers for fixed S.
- Find more efficient algorithms to generate S-LID sequences.

Acknowledgments

Thank you!

This research was conducted as part of the **2022 SMALL REU program** at Williams College, and was funded by **NSF Grant DMS1947438** and **Williams College** funds. We thank our colleagues from the 2022 SMALL REU program for many helpful conversations.

arXiv: 2210.10577.