Dynamics of the Fibonacci Order of Appearance Map

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The **Fibonacci sequence** with initial conditions $F_0 = 0$ and $F_1 = 1$ is defined recursively for $n > 1$ as $F_n = F_{n-1} + F_{n-2}$.

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55, \ldots$$

The **order of appearance** $z(n)$ for a natural n is the smallest positive integer ℓ such that $n \mid F_\ell$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$z(n)$	1	3	4	6	5	12	8	6	12	15	10	12

Table 1: The order of appearance $z(n)$

Example

Let $n = 4$. Then $z(4) = 6$ and the 6^{th} Fibonacci number is 8, which is the first Fibonacci that divides 4.

Fixed Points

A **fixed point** in the Fibonacci sequence occurs when $z(n) = n$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$z(n)$	1	3	4	6	5	12	8	6	12	15	10	12

Table 2: Highlighted values are fixed points

Theorem (Marques)
$z(n) = n$ if and only if $n = 5^k$ or $n = 12 \cdot 5^k$ for some $k \geq 0$.

The **fixed point order** $z^k(n)$ for a natural n is the smallest positive integer k such that $z^k(n)$ is a fixed point. If n is a fixed point, the fixed point order is 0.

Fixed Point Order

$n \setminus k$	1	2	3	4
1	1			
2	3	4	6	12
3	4	6	12	
4	6	12		
5	5			
6	12			
7	8	6	12	
8	6	12		
9	12			
10	15	20	30	60

Table 3: Iterations of z on n, highlighted values are fixed points

Theorem (FMV)
For all positive integers k, there exist infinitely many n with fixed point order k.

Lemma (FMV)
Suppose $z^k(5^a \cdot n) = c_k 5^{a_i}$, where $\gcd(c_k, 5) = 1$. For all non-negative integers a, the coefficient c_k remains constant.

Lemma (FMV)
For all integers k and m with $k \geq 0, m \geq 4$ and $2k + 2 \leq m$, we have

$$z^k(10^m) \;=\; 3 \cdot 5^m \cdot 2^{m-2k}.$$

Lemma 1

Lemma (FMV)
Suppose $z^k\left(5^a \cdot n\right) = c_k 5^{a_i}$, *where* $\gcd\left(c_k, 5\right) = 1$. *For all non-negative integers* a, *the coefficient* c_k *remains constant.*

Example
Suppose $n = 11$. Observe that

$$z(11) = 10 = 10 \cdot 5^0$$

$$z(11 \cdot 5) = z(55) = 10 = 10 \cdot 5^0$$

$$z(11 \cdot 5^2) = z(275) = 50 = 10 \cdot 5^1$$

$$z(11 \cdot 5^3) = z(1375) = 250 = 10 \cdot 5^2$$

Lemma 2

Lemma (FMV)

For all integers k and m with $k \geq 0, m \geq 4$ and $2k+2 \leq m$, we have

$$z^k\left(10^m\right) \;=\; 3 \cdot 5^m \cdot 2^{m-2k}.$$

Proof.

When $k = 1$

$$z\left(10^m\right) = \operatorname{lcm}\left(z\left(2^m\right), z\left(5^m\right)\right) = \operatorname{lcm}\left(3 \cdot 2^{m-2}, 5^m\right) = 3 \cdot 5^m \cdot 2^{m-2}.$$

For $k > 1$

$$\begin{aligned}
z^{k+1}\left(10^m\right) &= z\left(z^k\left(10^m\right)\right) \\
&= \operatorname{lcm}\left(z(3), z\left(5^m\right), z(2^{m-2k})\right) \\
&= \operatorname{lcm}\left(4, 5^m, 2^{m-2k-2} \cdot 3\right) \\
&= 3 \cdot 5^m \cdot 2^{m-2(k+1)} \qquad \text{since } m \geq 2(k+1)+2 = 2k+4
\end{aligned}$$

□

Proof Sketch of Theorem

Theorem (FMV)
For all positive integers k, there exist infinitely many n with fixed point order k.

Proof.
Let $r \in \mathbb{Z}_{>0}$ be arbitrary.

Case 1: n goes to a fixed point of the form 5^a in k-steps. Then $z^{k-1}(n) = c \cdot 5^b$ for $c, b \in \mathbb{Z}_{>0}$. Applying Lemma 1, we know $5^r \cdot n$ exactly k iterations to reach a fixed point.

Case 2: n goes to a fixed point of the form $12 \cdot 5^a$ in k-steps. Then $z^{k-1}(5^r \cdot n) = c \cdot 5^b$ for $c, b \in \mathbb{Z}_{>0}$. Thus, $5^r \cdot n$ requires exactly k iterations to reach a fixed point. $\square$

Future Directions

1. Where are fixed points located when initial conditions are varied?
2. How does $z^k(n)$ behave for related sequences?
   - e.g., Lucas numbers and Tribonacci sequence
3. For a given integer, can the fixed point order be bounded as a function of n?

k	1	2	3	4	5	6	7	8	9	10
n	1	4	3	2	11	89	1069	2137	4273	59833
FP	1	12	12	12	60	60	60	60	60	60

Table 4: First n that takes k iterations to reach a fixed point

References

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