Distinct Angles and Angle Chains in $\mathbb{R}^3$

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History

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- Instead, let's restrict the points to general position:
  - No 3 colinear points
  - No 4 cocircular points

Distinct Angles in Two Dimensions

Theorem (FKMPPW 2022): Let A_{gen} be the minimum number of distinct angles formed by n points on a plane, with no three points on a line and no four points on a circle. Then, $A_{\text{gen}} = O(n^2)$.

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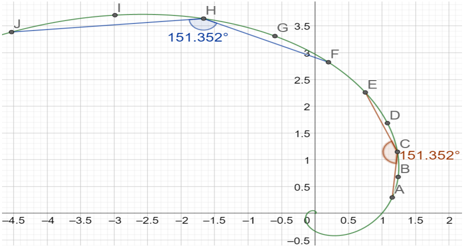

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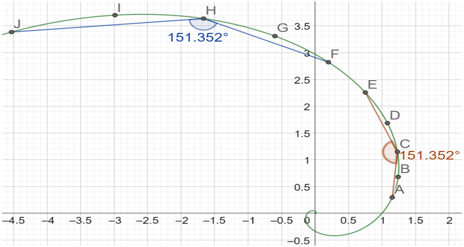

Rotate three points along the spiral to repeat the same angle!

- **Self-similarity**: Any angle formed by three of the points can also be formed using a special point A as one of the points.
- $\binom{n}{2}$ ways to choose the remaining two points, so $O(n^2)$ angles.

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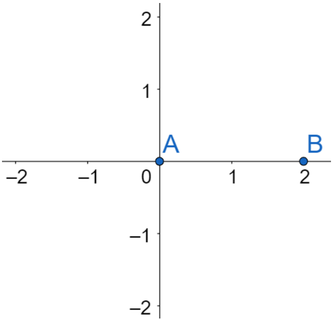
Theorem: Let A_{gen} be the minimum number of distinct angles formed by n points on a plane, with no three points on a line and no four points on a circle. Then, $A_{\text{gen}} = \Omega(n)$.

- Fix two points A and B and consider only angles with A as an endpoint and B as the center point.



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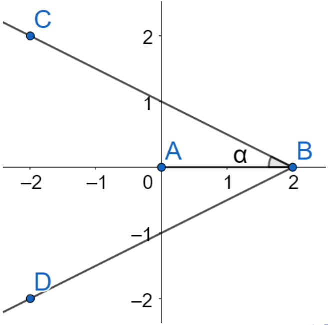
- We can only form a given angle twice without putting three points on a line.



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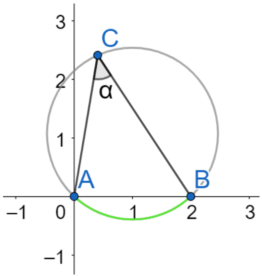

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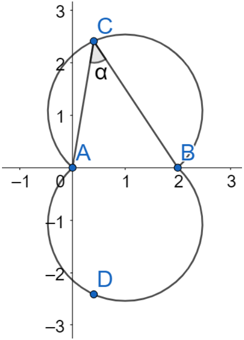

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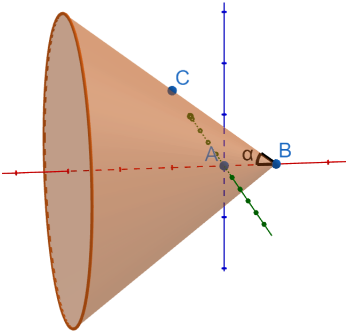

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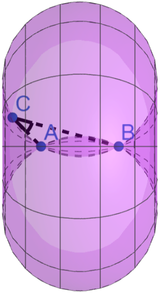
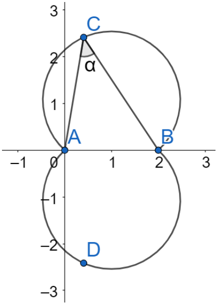
- Now if we fix A as an endpoint and B as the center point, we can put all remaining points on a cone to form only one distinct angle.
- If we fix A and B as endpoints, we can put all remaining points on a spindle torus to form only one distinct angle.





Distinct Angles in Three Dimensions

1. What lower bound can we get on the number of distinct angles in three dimensions with no three points on a line and no four points on a circle?
2. Using the extra space that we have in 3D, can we find a construction with $o(n^2)$ distinct angles?

Pinned Center Point

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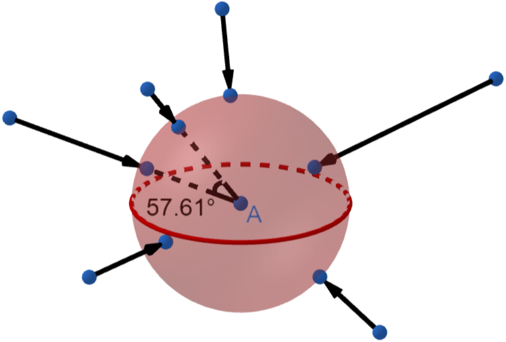

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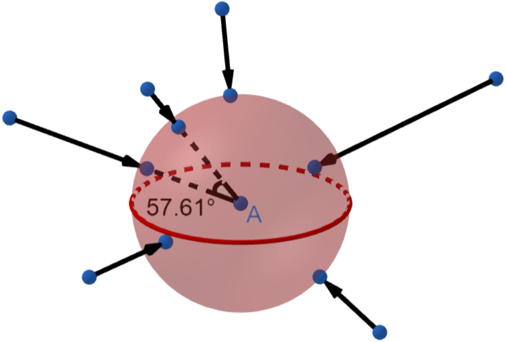
- The measure of $\angle BAC$ is a constant multiple of the spherical distance between B and C.



Pinned Center Point

Theorem (Guth and Katz, 2015)

A set of n points in the plane determines $\Omega\left(\frac{n}{\log n}\right)$ distinct distances.

Generalizing to sphere (Tao)

A set of n points on a sphere determines $\Omega\left(\frac{n}{\log n}\right)$ distinct distances.

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Determining the number of distinct angles with fixed center point A is equivalent to determining the number of distinct distances for these points lying on a sphere of radius 1 centered at A.

Corollary

The number of distinct angles for n points in general position in $\mathbb{R}^3$ *with a fixed center point is* $\Omega\left(\frac{n}{\log n}\right)$.

This is also the best known lower bound for distinct angles in three dimensions in general, counting all angles.

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Note: By distributing points along a circle on the sphere, we get an $O(n)$ upper bound on the minimum number of distinct angles with a fixed center point. The lower and upper bounds are very close together!

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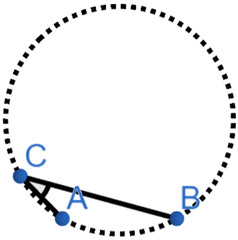

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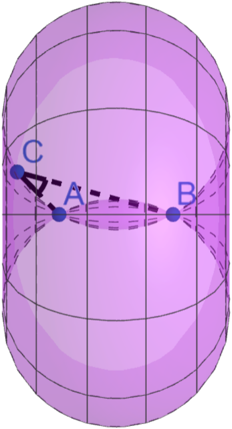

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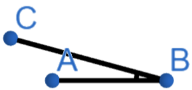

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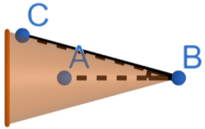

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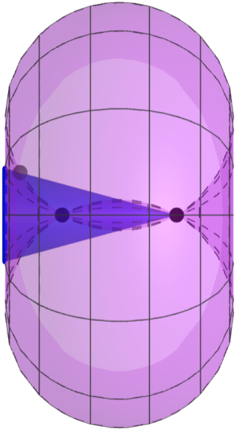

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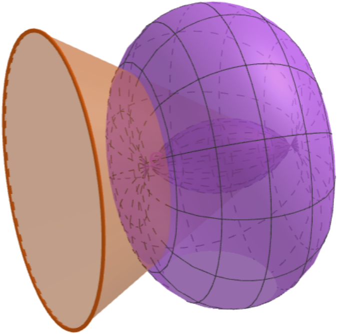

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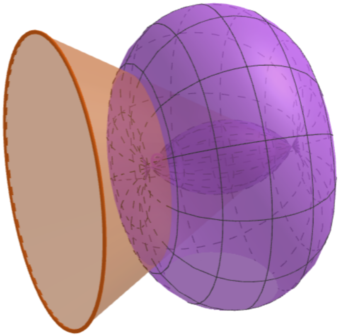

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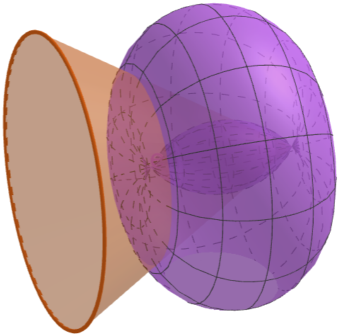

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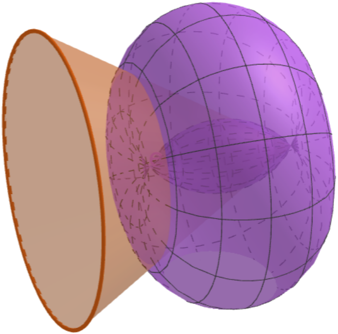
- There are at least $\max(\#\{\text{cones}\}, \#\{\text{s. tori}\})$ distinct angles.
- To minimize this, $\#\{\text{cones}\} = \#\{\text{s. tori}\} = \sqrt{(n-2)/3}$.



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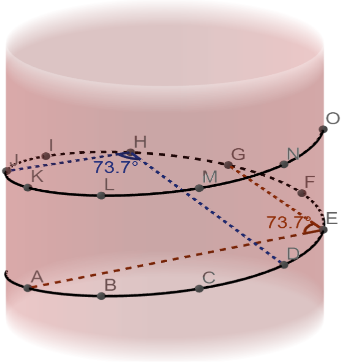
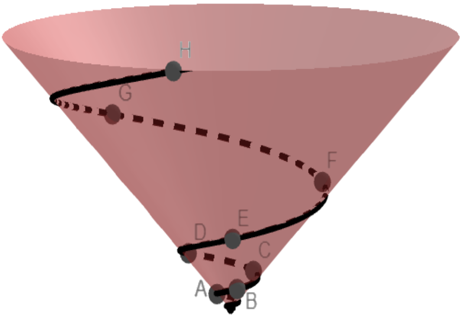
Conjecture

For any construction in general position in $\mathbb{R}^3$, there are $\Omega(n^2)$ distinct angles formed when an endpoint is pinned - the same (up to a constant) as with no pinned points.

- Even without a proof of this conjecture, it's clear that pinning an endpoint and pinning a center point lead to radically different results.

3D Constructions





To the left, points are distributed along a **cylindrical helix**, parametrized by $(\cos(t), \sin(t), t)$. To the right, points are distributed on a **conchospiral**, parametrized by $(e^t \cos(t), e^t \sin(t), e^t)$. Due to their symmetry, both of these point configurations exhibit self-similarity and thus have $O(n^2)$ distinct angles.

"General Position" in 3D

- In two dimensions, general position means no three colinear points and no four cocircular points.
- In three dimensions, we keep this definition rather than disallowing 4 coplanar points or 5 cospherical points.
  - Our constructions (upper bound) are still valid under this restriction.
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- In three dimensions, we keep this definition rather than disallowing 4 coplanar points or 5 cospherical points.
  - Our constructions (upper bound) are still valid under this restriction.
  - No obvious improved lower bound.
- We could disallow having too many points on any surface of degree 2.
  - Doing so would improve our lower bound from $\Omega(n/\log n)$ to $\Omega(n)$.
  - But, all our constructions would be invalid and our upper bound would worsen to $O(n^2 2^{C\sqrt{\log n}})$.

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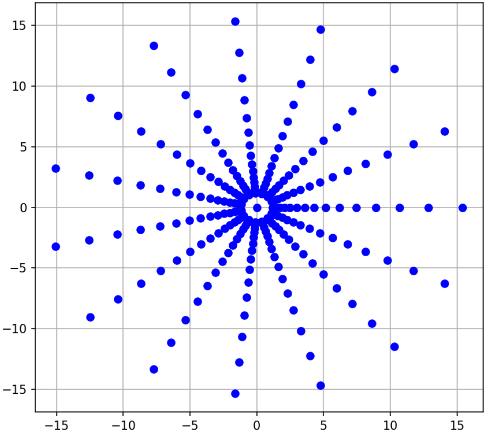
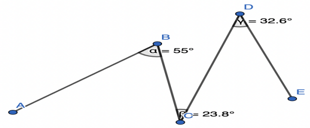


Permitting $O(\sqrt{n})$ points on lines and circles allows for a configuration with $O(n)$ distinct angles with a pinned endpoint.

Distinct Angle Chains

- A *k-chain* is a $(k+2)$-tuple of points $(x_1, \ldots, x_{k+2})$ along with the associated k-tuple of angles

$$(\alpha_1, \ldots, \alpha_k) = (\angle x_1 x_2 x_3, \ldots, \angle x_k x_{k+1} x_{k+2}).$$



A sample three-chain in $\mathbb{R}^2$

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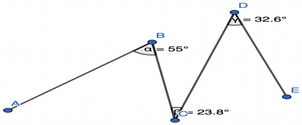

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- If $k = 1$, this is just the question we already asked.

Distinct Angle Chains in 2D

- Recall: In 2D, if one endpoint and the center point of the angle are fixed, we get $\Omega(n)$ angles since no three points are on a line.
- So, adding one leg to the chain must multiply the number of distinct angle chains by $\Omega(n)$.

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- So, adding one leg to the chain must multiply the number of distinct angle chains by $\Omega(n)$.
- By induction:

Theorem (RBLLMMPRV 2022)

For n points in general position in two dimensions, there are $\Omega(n^k)$ distinct k-tuples of angles with associated k-chains.

Distinct Angle Chains in 2D

- The logarithmic spiral provides the best upper bound we could hope for in two dimensions.

Theorem (RBLLMMPRV 2022)

With points distributed on the logarithmic spiral, there are $O(n^{k+1})$ distinct k-tuples of angles with associated k-chains.

Distinct Angle Chains in 3D

- In 3D, it is no longer true that adding a leg to the chain creates n choices for the new angle.
- We have the following weaker lower bound on the number of distinct angle k-chains.

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Theorem (RBLLMMPRV 2022)

In three dimensions, the number of distinct k-tuples of angles with associated k-chains is bounded below by:

$$\begin{cases} \Omega\left(\frac{n^{(k+2)/3}}{(\log n)^{(k+2)/3}}\right) & \text{if } k = 1 \mod 3; \\ \Omega\left(\frac{n^{(k+1)/3}}{(\log n)^{(k-2)/3}}\right) & \text{if } k = 2 \mod 3; \\ \Omega\left(\frac{n^{k/3+1/2}}{(\log n)^{k/3}}\right) & \text{if } k = 0 \mod 3. \end{cases}$$

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- Takeaway: The lower bound gets multiplied by $n/\log(n)$ every time the chain gets 3 longer (compared to n^3 for 2D).

Future Work

- Going forward, we hope to make progress in raising the lower bound for distinct angles in $\mathbb{R}^3$ with a pinned endpoint.
- This would improve bounds for the number of distinct angle chains in 3D, as would any further improvements on A_{gen}.
- The ultimate goal would be to come up with explicit constructions that minimize the number of distinct k-chains for a given k and n.

Another Approach

- Another possible approach at improving the lower bound:
- For a point configuration $\mathcal{P}$, the *energy* is given by

$$E(\mathcal{P}) = |\{(A, B, C, D, E, F) \in \mathcal{P}^6 : \angle ABC = \angle DEF\}|.$$

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- Then, if $\mathcal{A}$ is the set of all angles formed, we have

$$E(\mathcal{P}) = \sum_{\alpha \in \mathcal{A}} N_\alpha^2 \quad \text{and} \quad \sum_{\alpha \in \mathcal{A}} N_\alpha = |\mathcal{P}|^3 = n^3.$$

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which means

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An Incidence Problem

- An upper bound on $E(\mathcal{P})$ would entail a lower bound on the number of distinct angles.

$$E(\mathcal{P}) = |\{(A, B, C, D, E, F) \in \mathcal{P}^6 : \angle ABC = \angle DEF\}|.$$

- Note that $m\angle ABC = m\angle DEF$ if and only if

$$\frac{(A-B)\cdot(C-B)}{|A-B||C-B|} = \frac{(D-E)\cdot(F-E)}{|D-E||F-E|}.$$

Squaring both sides and rearranging:

$$\begin{aligned}((A-B)\cdot(C-B))^2|D-E|^2|F-E|^2 \\ -((D-E)\cdot(F-E))^2|A-B|^2|C-B|^2 = 0. \qquad (1)\end{aligned}$$

- This defines a "nice" surface. How many times can Equation (1) be satisfied by a point configuration?

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