Extending Agreement in the Katz-Sarnak Density Conjecture

Peter Cohen (pcohen@bowdoin.edu), Steven Miller (sjm1@williams.edu), Roger Van Peski (rpeski@princeton.edu)

Joint with Oscar Gonzalez, Anand Hemmady, Geoffrey Iyer, Kevin Kwan, Carsten Sprunger, Nicholas Triantafillou, Yen Nhi Truong Vu

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Introduction

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f)L(s,f) = \epsilon_f \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings between zeros appear same as b/w eigenvalues of Complex Hermitian matrices.

Pair and n-level correlations

- Good: Remarkable agreement b/w Number Theory and GUE.
- Bad: Insensitive to finitely many zeros.

1-Level Density

- Study the behavior of zeros for L-functions near $s = 1/2$.
- We define the **1-level density** for an L-function $L(s, f)$ and ϕ an even Schwartz function, where $\widehat{\phi}$ is compactly supported, by

$$D_f(\phi) := \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log R}{2\pi}\right).$$

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log C_f}{2\pi} \gamma_E^{(j_i)} \right)$$

$$\to \int \cdots \int \phi(x) W_{n, \mathcal{G}(\mathcal{F})}(x) dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

n-Level Density: Determinant Expansions from RMT

- U(N), $U_k(N)$: $\det\left(K_0(x_j,x_k)\right)_{1\le j,k\le n}$
- USp(N): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(even): $\det\left(K_1(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(odd): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n} +$
  $\sum_{\nu=1}^{n}\delta(x_\nu)\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\neq\nu\le n}$

  where

$$K_\epsilon(x,y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \epsilon\frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

Random Matrix Theory Analogue

- For an even Schwartz function ϕ on $\mathbb{R}$ define

$$F_M(\theta) := \sum_{j=-\infty}^{\infty} \phi\left(\frac{M}{2\pi}(\theta + 2\pi j)\right).$$

- For U and $M \times M$ unitary matrix with eigenvalues $e^{i\theta_n}$ let

$$Z_\phi(U) := \sum_{n=1}^{M} F_M(\theta_n)$$

The Question

- What are the moments of $Z_\phi(U)$ for matrices from the classical compact groups for given ϕ?
- Katz and Sarnak: Compute for any test function, but often intractable.

Attacking The Question

- Rather than moments μ_n' of 1-level density, we study cumulants C_i given by

$$\log \mathbb{E}[e^{tX}] = \sum_{i=0}^{\infty} C_i \frac{t^n}{n!}$$

Related to moments by

$$\mu_n' = \sum \left(\frac{C_2}{2!}\right)^{k_2} \cdots \left(\frac{C_n}{n!}\right)^{k_n} \frac{n!}{k_2! \cdots k_n!},$$

summing over k_j such that $\sum_{j=2}^{n} j k_j = n$.

Cumulants and the Classical Compact Groups

- For $\phi \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}(\widehat{\phi}) \subseteq \left[-\frac{2}{n}, \frac{2}{n}\right]$ and $n \geq 3$, we have

$$
\begin{aligned}
C_n^{U}(\phi) &= 0 \\
C_n^{SO(even)}(\phi) &= 2^n Q_n(\phi) \\
C_n^{SO(odd)}(\phi) &= -2^n Q_n(\phi)
\end{aligned}
$$

What is $Q_n(\phi)$

$$Q_n(\phi) = -\frac{1}{2}\sum_{m=1}^{n}\sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m}\frac{n!}{\lambda_1!\cdots\lambda_m!}$$

$$\int_{\mathbb{R}^m}\left(\prod_{j=1}^{m}\phi^{\lambda_j}(x_j)\right)\times S(x_1-x_2)\cdots S(x_{m-1}-x_m)$$

$$\times S(x_m+x_1)dx_1\cdots dx_m$$

where $S(x) = \frac{\sin(\pi x)}{\pi x}$.

What is $Q_n(\phi)$

$$Q_n(\phi) = \frac{1}{4}\int_0^\infty \cdots \int_0^\infty \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n)K(y_1,\dots,y_n)dy_1\cdots dy_n,$$

What is $Q_n(\phi)$

$$K(y_1,\ldots,y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

$$\sum_{\epsilon_1,\ldots,\epsilon_n=\pm 1} \prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right|\leq 1\right\}}$$

and

$$\eta(\ell,j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^{\ell} \lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell} \lambda_k. \end{cases}$$

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- For fixed $(y_1, \ldots, y_n)$ take sum of all terms of $K(y_1, \ldots, y_n)$, subtract those which have certain vanishing χ's in them, add those which have certain pairs of vanishing χ's in them, etc.
- Example: Suppose $\mathrm{Supp}(\hat{\phi}) \subset \left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$

The Main Concept: An Example

- Suppose that $0 \leq y_j \leq \frac{1}{n-1}$ and $\sum y_j \geq 1$.

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- $\chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right| \leq 1\right\}} = 0$ iff either $\eta(l,j)\epsilon_j = 1$ for all j or $\eta(l,j)\epsilon_j = -1$ for all j.

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- We have $2m$ choices for $(\epsilon_1, \dots, \epsilon_n)$ which give

$$\prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right| \leq 1\right\}} = 0. \qquad (1)$$

- The remaining $2^n - 2m$ choices yield 1.

The Main Concept: Example

- So

$$K(y_1,\dots,y_n) = \sum_{m=1}^{n} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots!\lambda_m} (2^n - 2m)$$

if $(y_1,\dots,y_n) \in \left[0, \frac{1}{n-1}\right]$ and $\sum_{i=1}^{n} y_i \geq 1$.

The Main Concept: Combinatorial Trick

- Notice that

$$e^{-z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n.$$

- Also

$$\frac{1}{1 + e^z - 1} = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\lambda_1 + \cdots + \lambda_m = n} (-1)^m \frac{1}{\lambda_1! \cdots \lambda_m!}.$$

The Main Concept: Example Conclusion

- So

$$K(y_1, \ldots, y_n) = \sum_{m=1}^{n} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots ! \lambda_m} (2^n - 2m)$$
$$= 2(-1)^n$$

if $(y_1, \ldots, y_n) \in \left[0, \frac{1}{n-1}\right]$ and $\sum_{i=1}^{n} y_i \geq 1$.

The Main Concept: Example Conclusion

- Therefore

$$Q_n(\phi) = \frac{(-1)^n}{2} \int_0^{\frac{1}{n-1}} \cdots \int_0^{\frac{1}{n-1}} \hat{\phi}(y_1) \cdots \hat{\phi}(y_n) \chi_{\{y_1 + \ldots + y_n \geq 1\}}$$

- By Fourier computations and changes of variable

$$Q_n(\phi) \quad = \quad \frac{(-1)^{n-1}}{2} \left(\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} - \frac{1}{2}\phi(0)^n \right)$$

Inclusion-Exclusion for Support $\frac{1}{n-2}$

- Now two indicator functions in $\prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right| \le 1\right\}}$ can be zero in same y-region

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- Now two indicator functions in $\prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right| \leq 1\right\}}$ can be zero in same y-region
- Subtract off indicator of each region, add back in indicator of intersection

Inclusion-Exclusion for Support $\frac{1}{n-2}$

$$K(y_1,\dots,y_n) = \left(\sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} 2^n \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}\right) \chi_{\{y_1+\cdots+y_n>1\}}$$

$$- \left(\sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} 2m \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}\right) \chi_{\{y_1+\cdots+y_n>1\}}$$

$$- \sum_{j=1}^{n} \left(\sum_{m=1}^{n} \sum_{\substack{\lambda_1+\cdots+\lambda_m=n \\ \lambda_j\geq 1}} 2m \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}\right) \chi_{\{y_1+\cdots+y_n>1+2y_j\}}$$

$$+ \sum_{j=1}^{n} \left(\sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\substack{\lambda_1+\cdots+\lambda_{\ell-1}=j-1 \\ \lambda_\ell=1 \\ \lambda_{l+1}+\cdots+\lambda_m=n-j}} 4 \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}\right) \chi_{\{y_1+\cdots+y_n>1+2y_j\}}.$$

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- For $(\hat{\phi}) \subset [-\frac{1}{n-w}, \frac{1}{n-w}]$ with $w \geq 3$, get integrals against products of indicator functions, NOT amenable to these techniques

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- In this framework, can show elegant cancellation of most terms.
- Remaining terms have simple indicator functions and can be simplified nicely.

Final expression

Theorem

For $\phi \in \mathcal{S}(\mathbb{R})$ even such that supp$(\widehat{\phi}) \subseteq \left[-\frac{1}{n-w}, \frac{1}{n-w}\right]$ with $w \leq n/2$, we have

$$Q_n(\phi) = \sum_{\ell=0}^{w-1} \frac{(-1)^{n+\ell+1}\binom{n}{\ell}}{2} \Bigg(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_{\ell+1}) \cdots \widehat{\phi}(x_2)$$

$$\int_{-\infty}^{\infty} \phi^{n-\ell}(x_1) \frac{\sin(2\pi x_1(1+|x_2|+\cdots+|x_{\ell+1}|))}{2\pi x_1} dx_1 \cdots dx_{\ell+1} - \frac{1}{2}\phi^n(0) \Bigg)$$

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- Again, the point of all this simplification is to compare with L-functions.
- On the number theory side, we look at L-functions associated to cuspidal newforms, splitting by the sign of their functional equations.
- Can extend what is known there to test functions ϕ with $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-3}, \frac{1}{n-3}\right]$ and show agreement with random matrix theory using the previous theorem.

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How Does Soshnikov's Trick Work

We use the identities that

$$z = \log(1 + (e^z - 1)) = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{1}{\lambda_1! \cdots \lambda_m!}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n = e^{-z} = \frac{1}{1 + (e^z - 1)}$$

$$= \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j \geq 1}} (-1)^m \frac{1}{\lambda_1! \cdots \lambda_m!}$$

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Suppose that we want $Q_n(\phi)$ for $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$.

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Suppose that we want $Q_n(\phi)$ for $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$.
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- Suffices to analyze $K(y_1, \dots, y_n)$ when $0 \le y_j \le \frac{1}{n-1}$ for all j.
- If $\sum_i y_i > 1$ then $\chi_{\left\{\left|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j\right| \le 1\right\}} = 0$ if and only if all $\eta(\ell, j)\epsilon_j$ have same sign.

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- So

$$K(y_1, \dots, y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!} \times \left(2^n - 2m\chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right| \geq 1\right\}}\right).$$

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Using a combinatorial trick from Soshnikov, we use generating functions to evaluate the sum above. This gives

$$K(y_1, \dots, y_n) = 2(-1)^n \chi_{\left\{\left|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j\right| \geq 1\right\}}.$$

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Integration using standard techniques from Fourier analysis gives us,

$$Q_n(\phi) = \frac{(-1)^{n-1}}{2}\left(\int_{\mathbb{R}} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} - \frac{1}{2}\phi(0)^n\right)$$