$ecrets of the Tax $ystem: Keep More of Your MOOLA

Steven J. Miller
sjm1@williams.edu,
Steven.Miller.MC.96@aya.yale.edu

http://web.williams.edu/Mathematics/sjmiller/public_html/

Williams College, $eptember 19, 2016

https://docs.google.com/document/d/1BOA2lvjMRBq_Zjm5tA-IRfFOcbEeGuh2ihO9d85z278/

Outline

- Extending Your MOOla
- Protecting Your MOOla
- Highering Your MOOla

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- **E**xtending Your MOOla
- **P**rotecting Your MOOla
- **H**ighering Your MOOla

Extending Your MOOla

Calculus Friends....

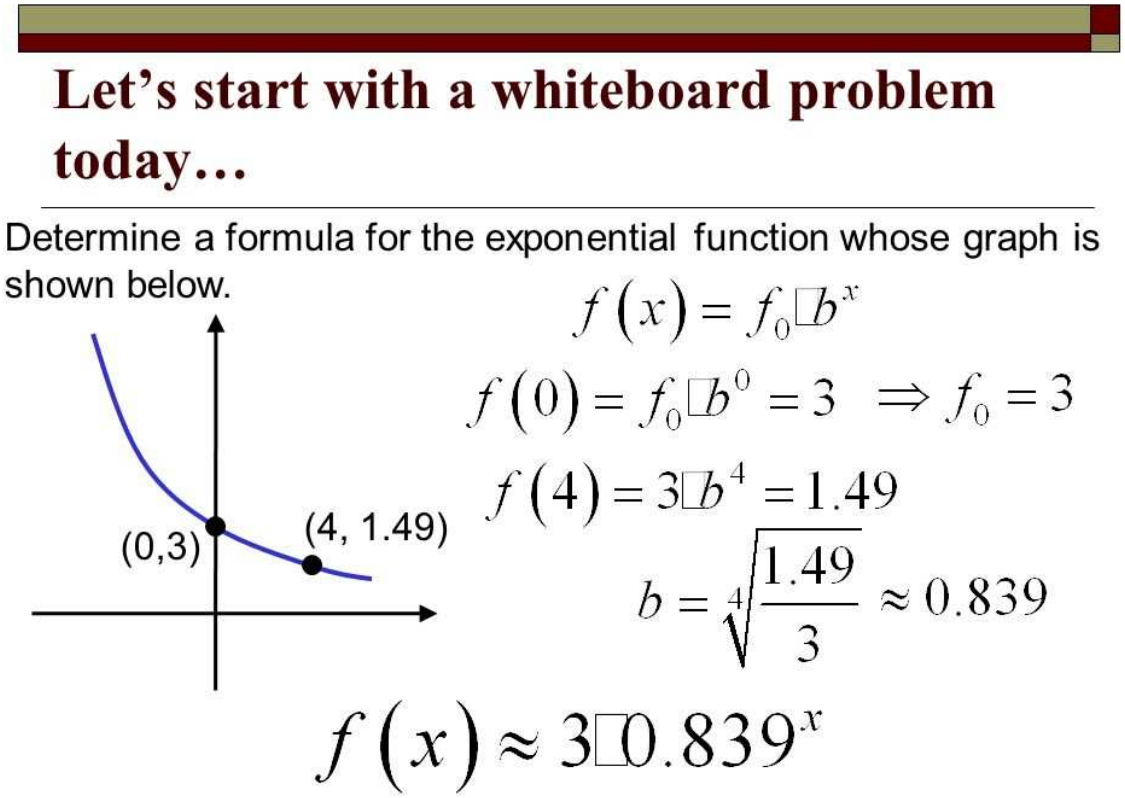

Figure: A typical calculus problem (image from `http://images.slideplayer.com/26/8809809/slides/slide_2.jpg`).

Compound Interest: Theory

Many (most?) of you have heard about compound interest, amazing how it works in practice.

Given by exponential function: start with P principal in year 0, interest rate of r, then amount in year n given by

$$A(n) = Pe^{rt}.$$

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This is very Pe$^{\text{rt}}$inent for your future....

Compound Interest: Example: Add $1000 a year at r%, continuously compounded

Year	Amount (2%)
0	1000
1	2020
2	3061
3	4123
4	5206
5	6311
6	7439
7	8589
8	9763
9	10960
10	12181

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Year	Amount (2%)	Amount (4%)
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1	2020	2041
2	3061	3124
3	4123	4252
4	5206	5425
5	6311	6647
6	7439	7918
7	8589	9241
8	9763	10618
9	10960	12051
10	12181	13543

Compound Interest: Example: Add $1000 a year at *r*%, continuously compounded

Year	Amount (2%)	Amount (4%)	Amount (8%)
0	1000	1000	1000
1	2020	2041	2083
2	3061	3124	3257
3	4123	4252	4528
4	5206	5425	5905
5	6311	6647	7397
6	7439	7918	9013
7	8589	9241	10764
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6	7439	7918	9013	13799
7	8589	9241	10764	17854
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12	??	??	??	??
14	??	??	??	??
16	??	??	??	??
18	??	??	??	??
20	??	??	??	??

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In 20yrs: one 20% beats ten 2% (getting $1000 vs $10,000 annually!).

Compound Lesson

Takeaways.

- Small changes can escalate quickly.
- Optimize time: spend a bit more thinking about stocks!
- Beware debt! Think about this when spending, when financing a house or car ....

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Winter Study Talks (all in Mabie Room):

- 1st week: Budgeting 101.
- 2nd week: Life After Williams.
- 3rd week: Understanding First Paycheck & Benefits.
- 4th week: Credit, Debt, & Loans – OH MY!

Protecting Your MOOla

Intellectual Exercises

No one here is unscrupulous, of course, but elsewhere in the world....

Goal is to see how math can flag behavior.

Important: protect your identity, detect when things are going wrong (being given faulty advice), allocate resources well.

Will concentrate on Benford's law: one of MANY ways (and one most are not aware of).

Interesting Question

Motivating Question: For a nice data set, such as the Fibonacci numbers, stock prices, street addresses of college employees and students, ..., what percent of the leading digits are 1?

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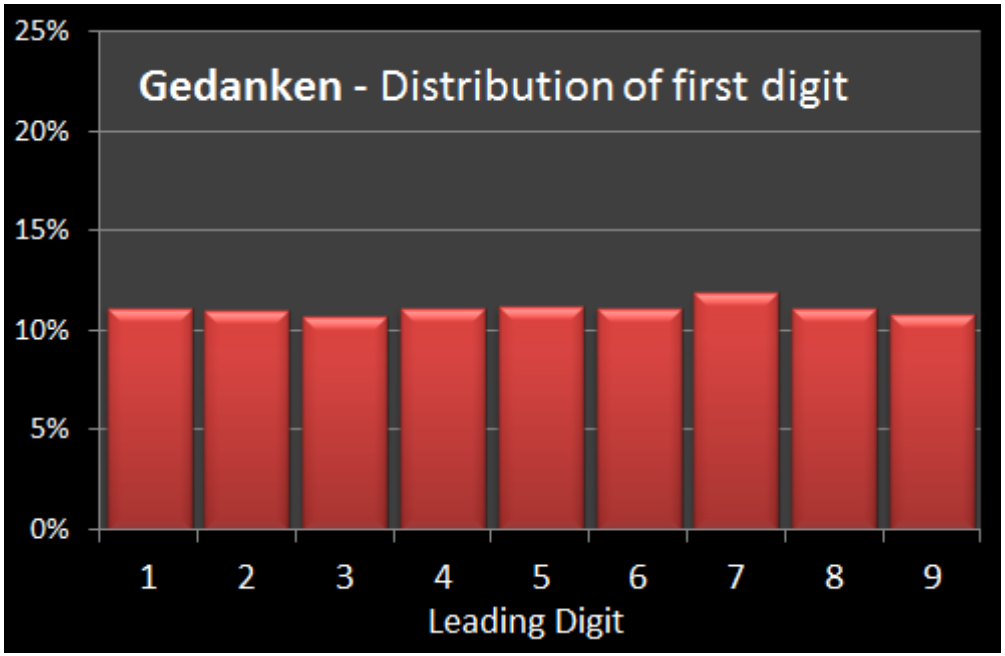

Natural guess: 10% (but immediately correct to 11%!).

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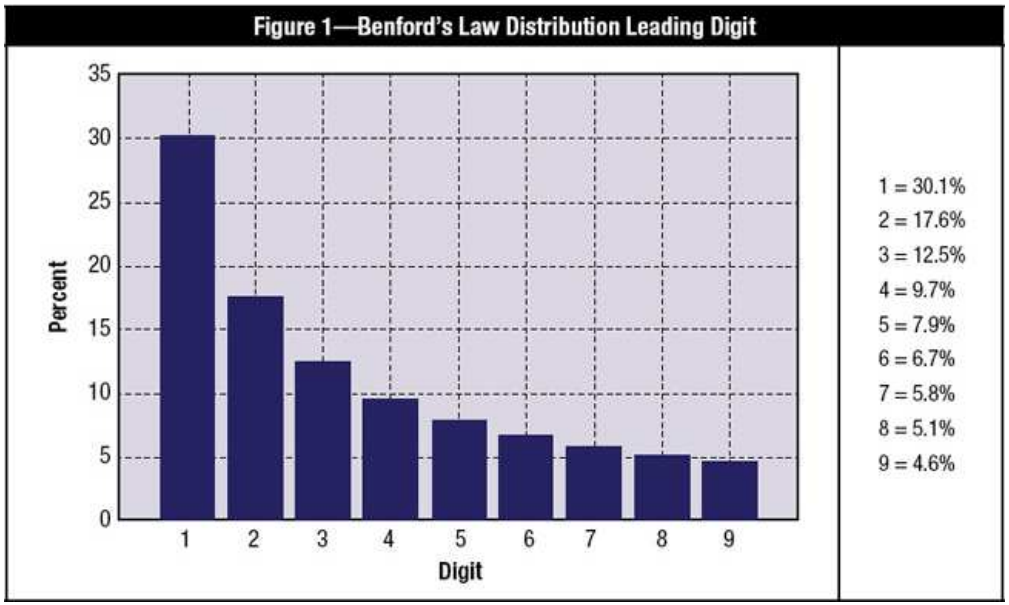
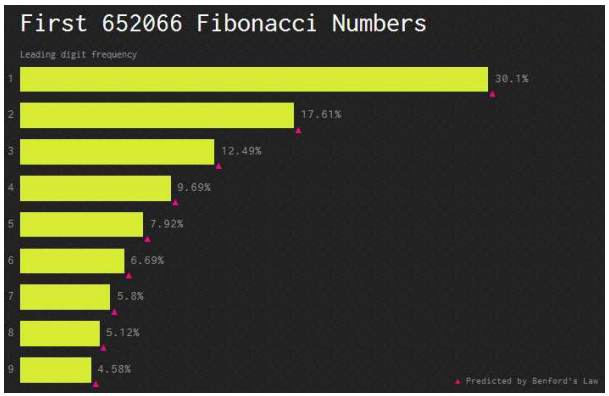


Answer: Benford's law!

Examples with First Digit Bias

Fibonacci numbers



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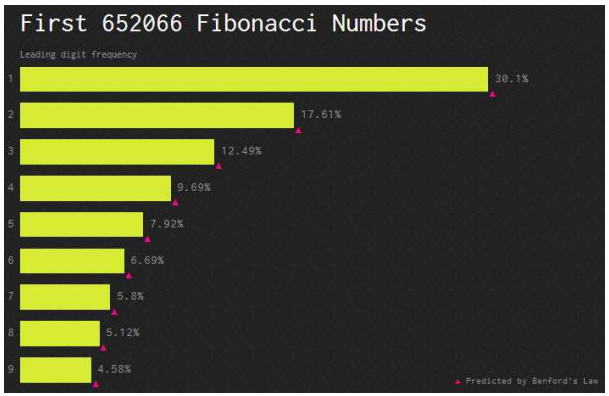
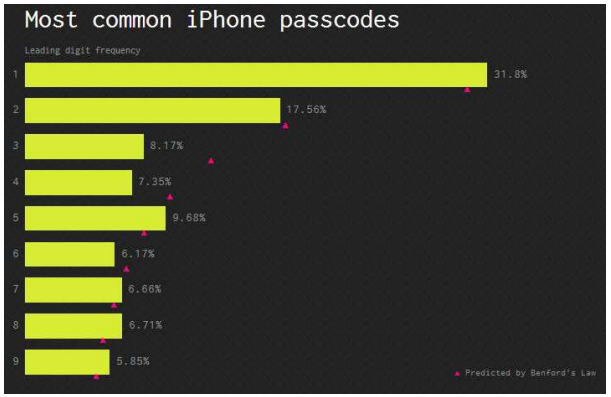


Most common iPhone passcodes



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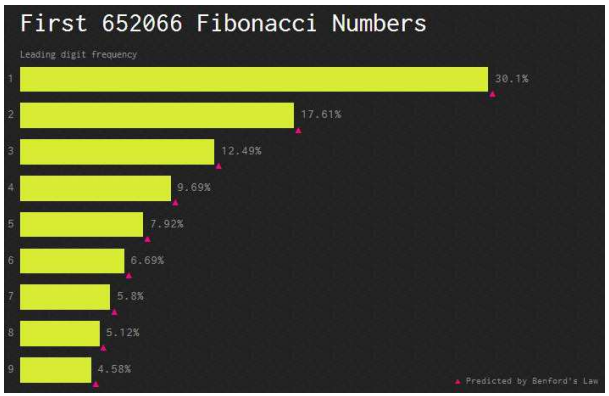
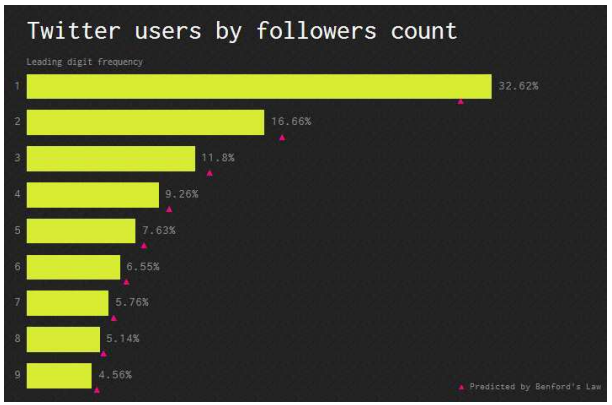


Twitter users by # followers



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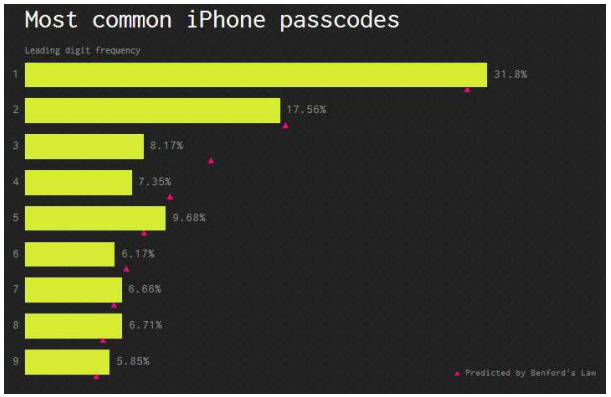

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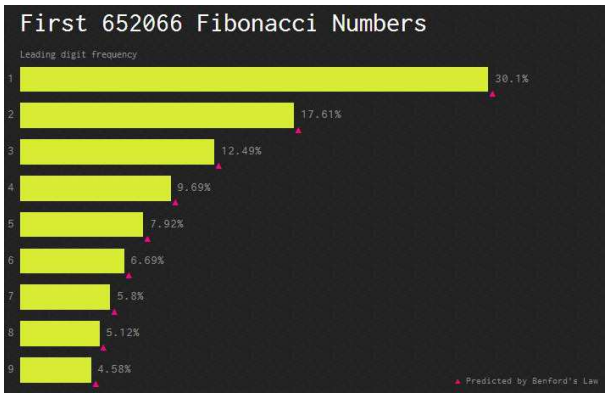

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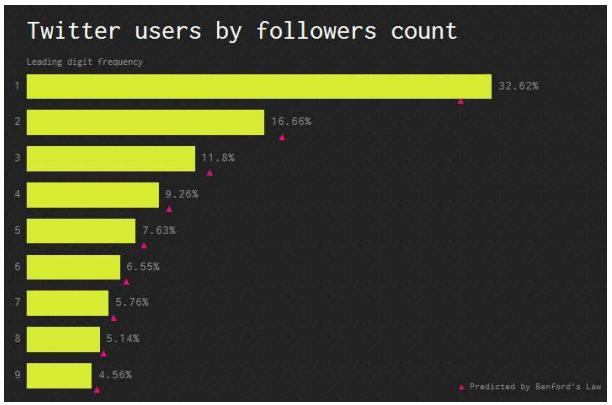

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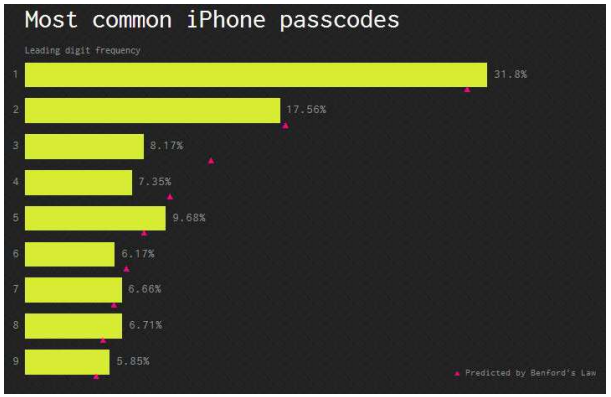
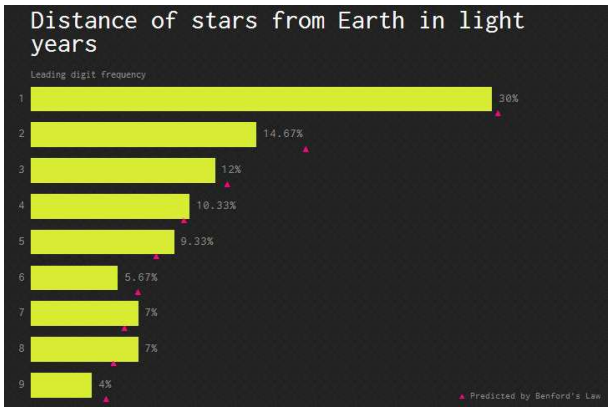


Distance of stars from Earth



Summary

- Explain Benford's Law.
- Discuss examples and applications.

Caveats!

- A math test indicating fraud is *not* proof of fraud: unlikely events, alternate reasons.

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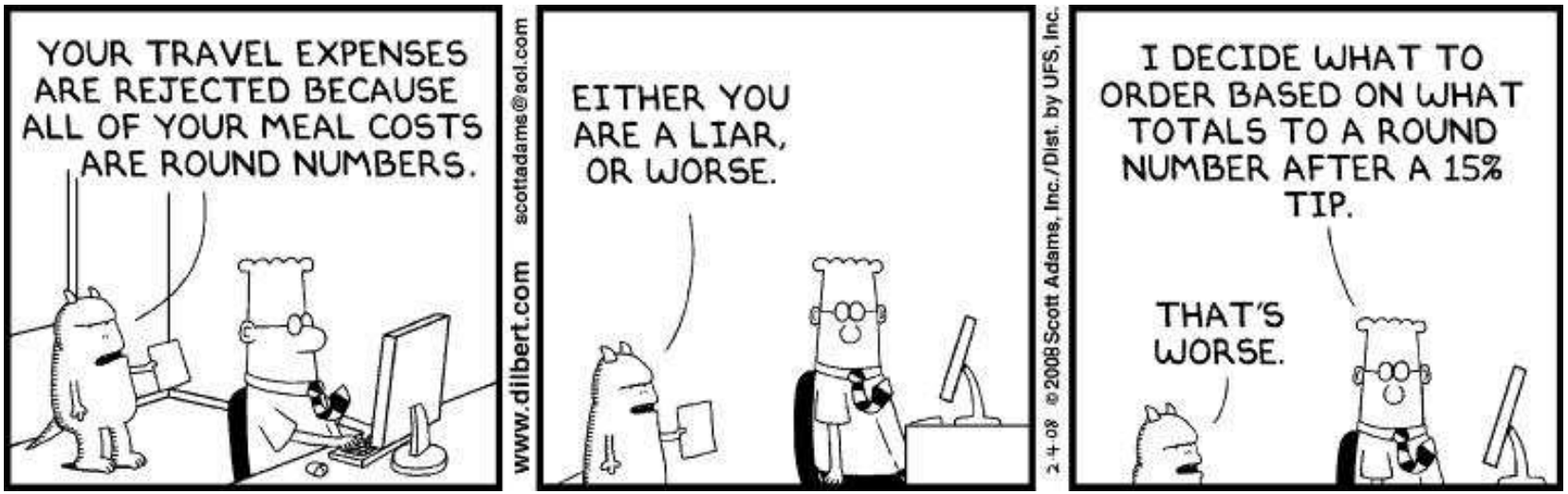


Some Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- financial data
- many, many more....

Applications

- Analyzing round-off errors.
- Determining the optimal way to store numbers.
- Detecting tax and image fraud, and data integrity.

Applications: Images (Steganography)

Cover image.

Applications: Images (Steganography)

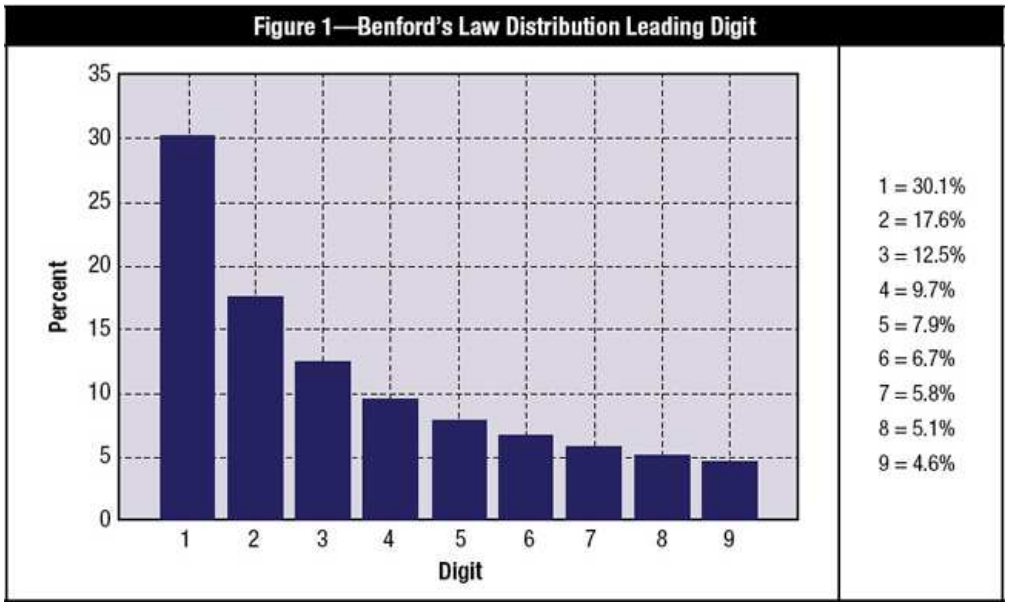
Cover image.

Extracted image.

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d is $\log_{10}\left(\frac{d+1}{d}\right)$; thus about 30% are 1's.



Benford's Law (probabilities)

Background Material

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- $S_{10}(x) = S_{10}(\widetilde{x})$ if and only if x and $\widetilde{x}$ have the same leading digits. Note $\log_{10} x = \log_{10} S_{10}(x) + k$.
- Key observation: $\log_{10}(x) = \log_{10}(\widetilde{x}) \mod 1$ if and only if x and $\widetilde{x}$ have the same leading digits.

Thus often study $y = \log_{10} x \mod 1$.
Advanced: $e^{2\pi i u} = e^{2\pi i (u \mod 1)}$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

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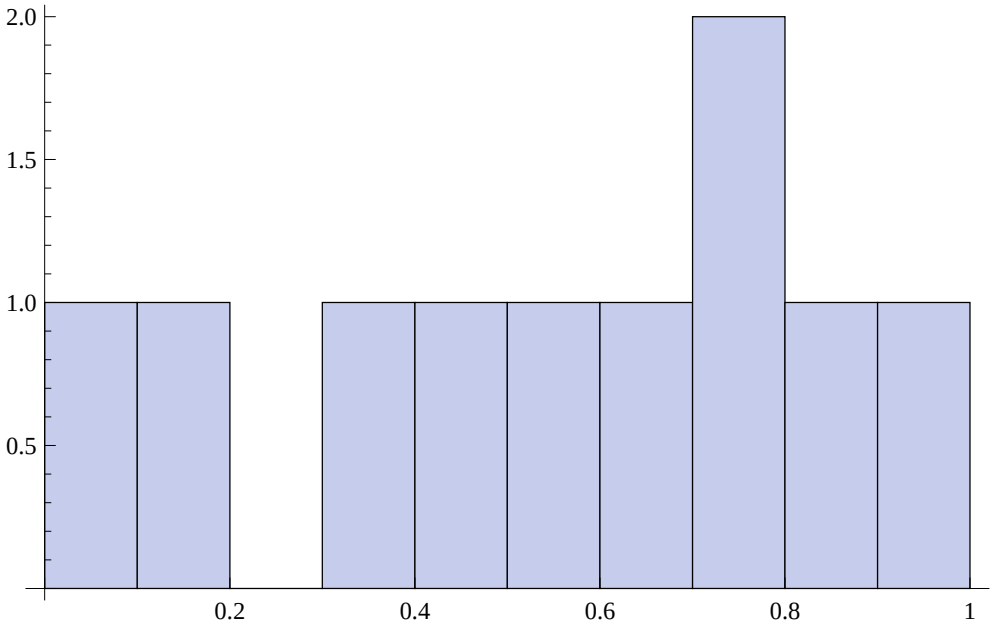
Theorem: If α is an irrational number, then $n\alpha \bmod 1$ is equidistributed.

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



$n\sqrt{\pi}$ mod 1 for $n \leq 10$

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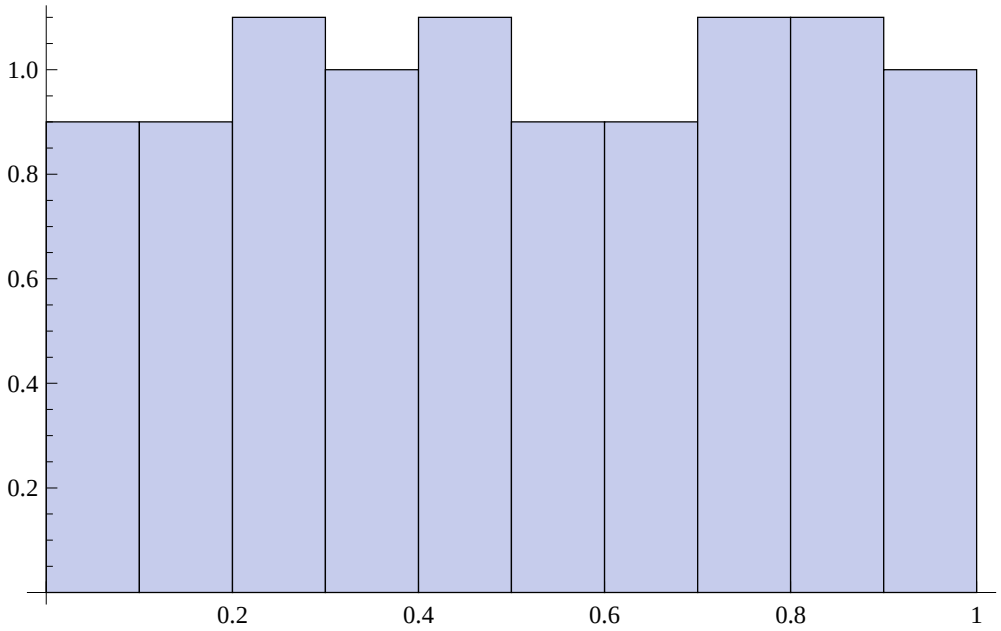
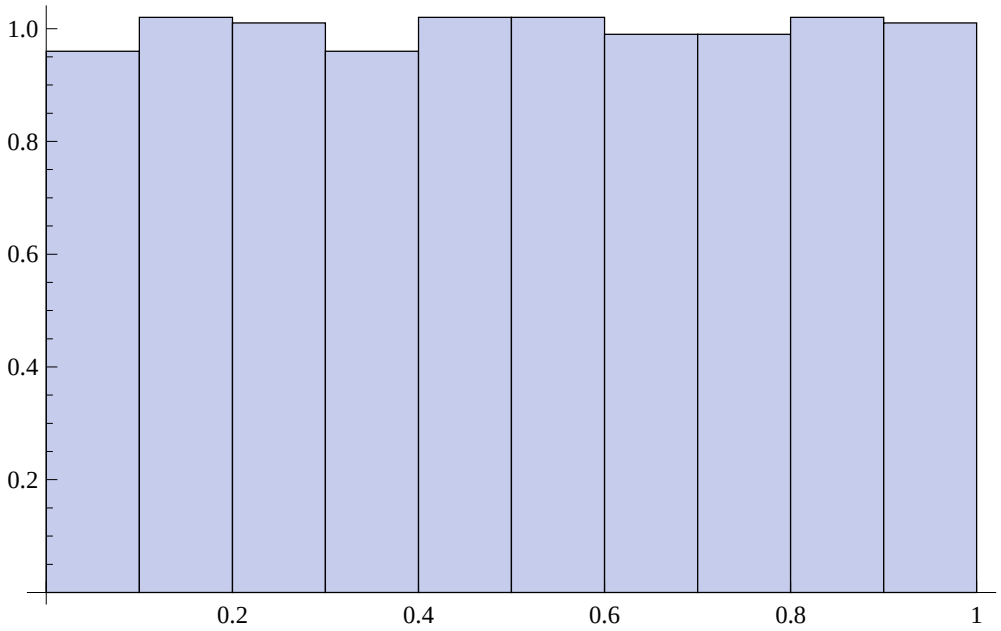


$n\sqrt{\pi}$ mod 1 for $n \leq 100$

Example of Equidistribution: $n\sqrt{\pi}$ mod 1



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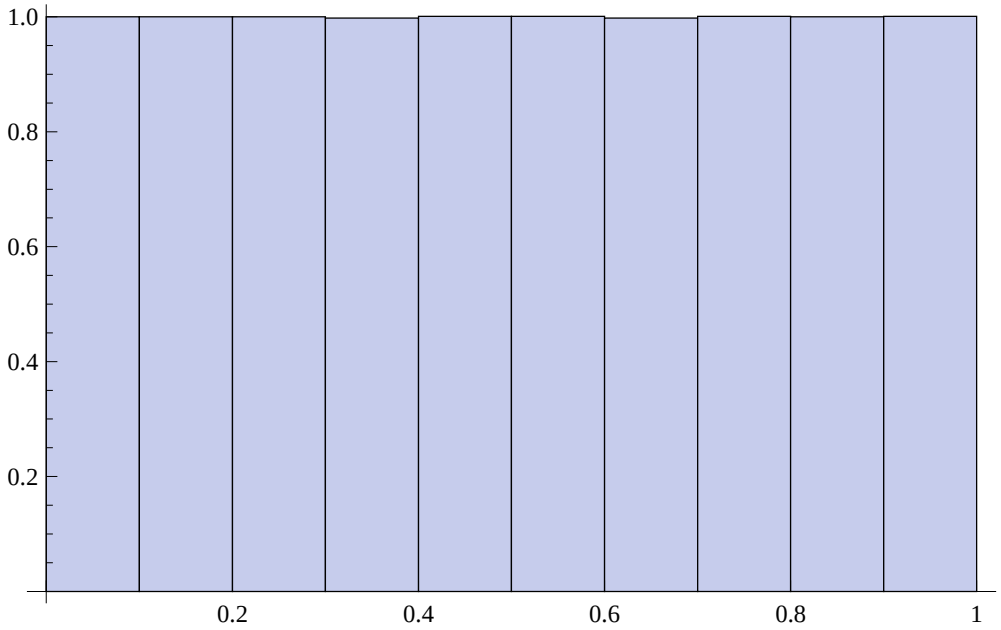

$n\sqrt{\pi} \bmod 1$ for $n \leq 10,000$

Logarithms and Benford's Law

Fundamental Equivalence

$\{x_i\}$ is Benford if and only if $\{\log_{10} x_i \bmod 1\}$ is equidistributed.

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Compound connection: *Exponential growth is Benford!*
Note $\log_{10} e^{rn}$ is $n \cdot r \log_{10} e$.

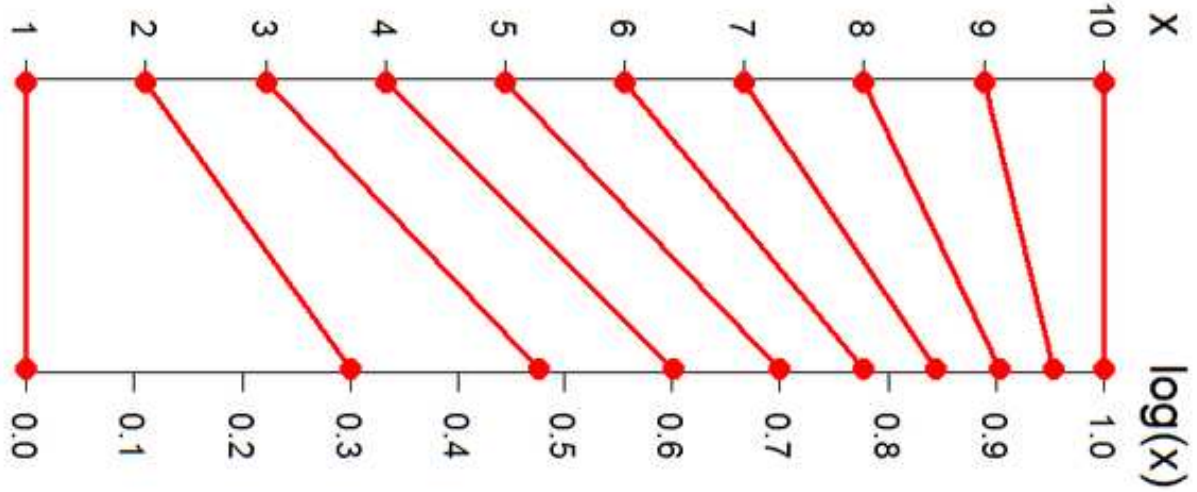
Logarithms and Benford's Law

Fundamental Equivalence

$\{x_i\}$ is Benford if and only if $\{\log_{10} x_i \bmod 1\}$ is equidistributed.

$$x = S_{10}(x) \cdot 10^k \text{ then}$$

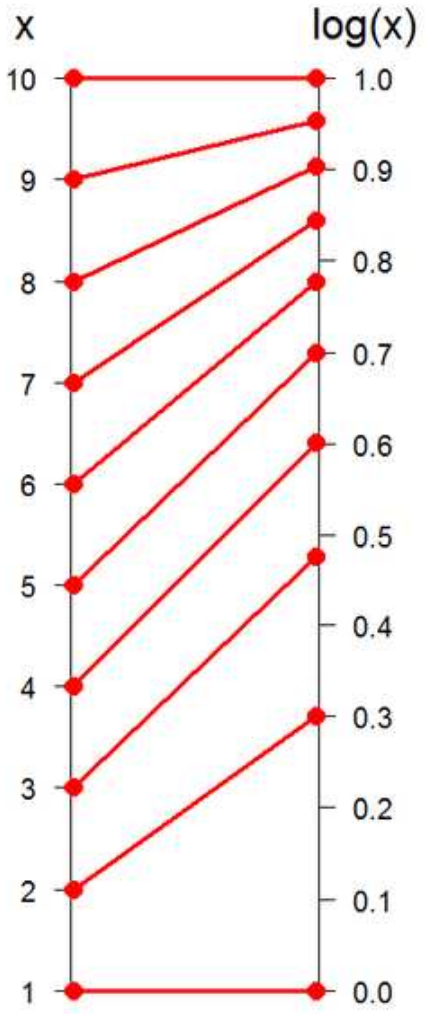
$$\log_{10} x = \log_{10} S_{10}(x) + k = \log_{10} S_{10} x \bmod 1.$$



Logarithms and Benford's Law

$$\begin{aligned}&\text{Prob}(\text{leading digit } d)\\ &= \log_{10}(d+1) - \log_{10}(d)\\ &= \log_{10}\left(\tfrac{d+1}{d}\right)\\ &= \log_{10}\left(1+\tfrac{1}{d}\right).\end{aligned}$$

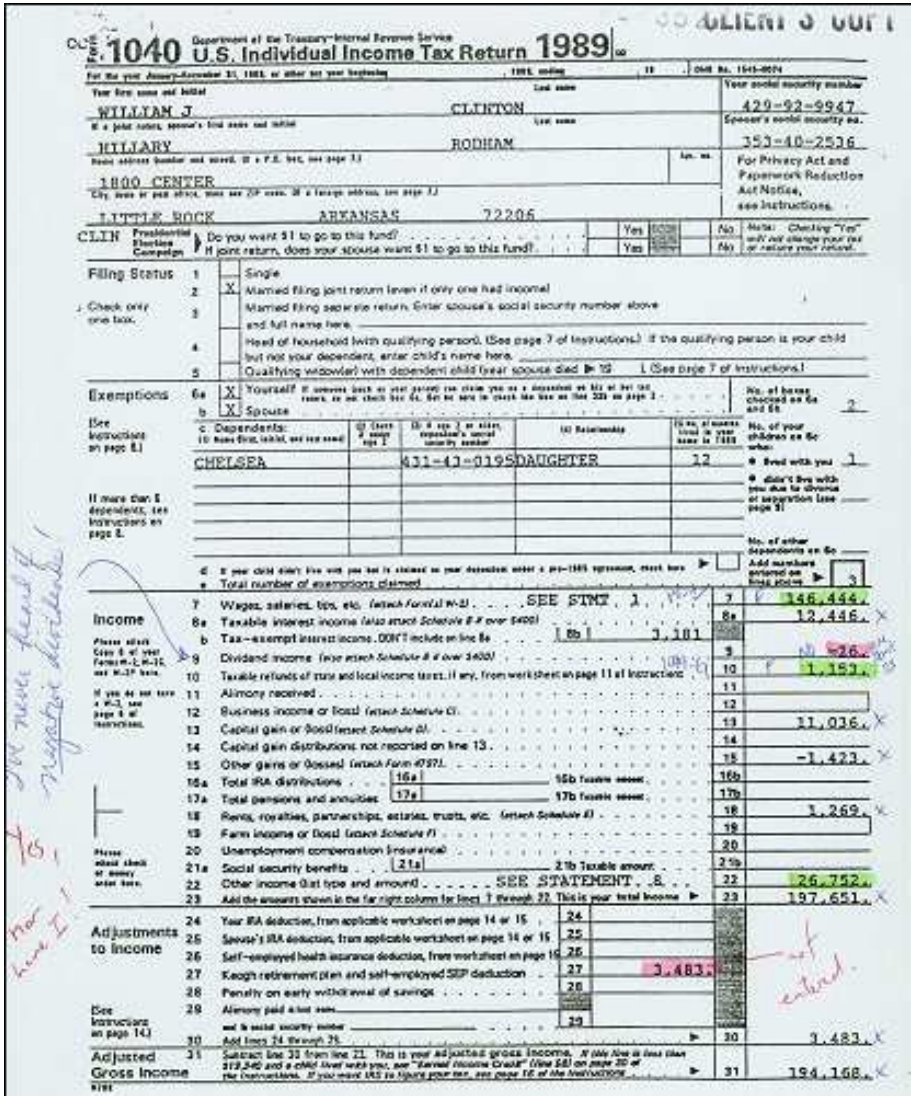
Have Benford's law $\leftrightarrow$ mantissa of logarithms of data are uniformly distributed



Applications for the IRS: Detecting Fraud

A Tale of Two Steve Millers....

Applications for the IRS: Detecting Fraud



Form 1040 U.S. Individual Income Tax Return 1989

WILLIAM J CLINTON — 429-92-9947
HILARY RODHAM — 353-40-2536
1800 CENTER
LITTLE ROCK ARKANSAS 72206

Filing Status: X Married filing joint return (even if only one had income)

Exemptions: X Yourself; X Spouse; 2

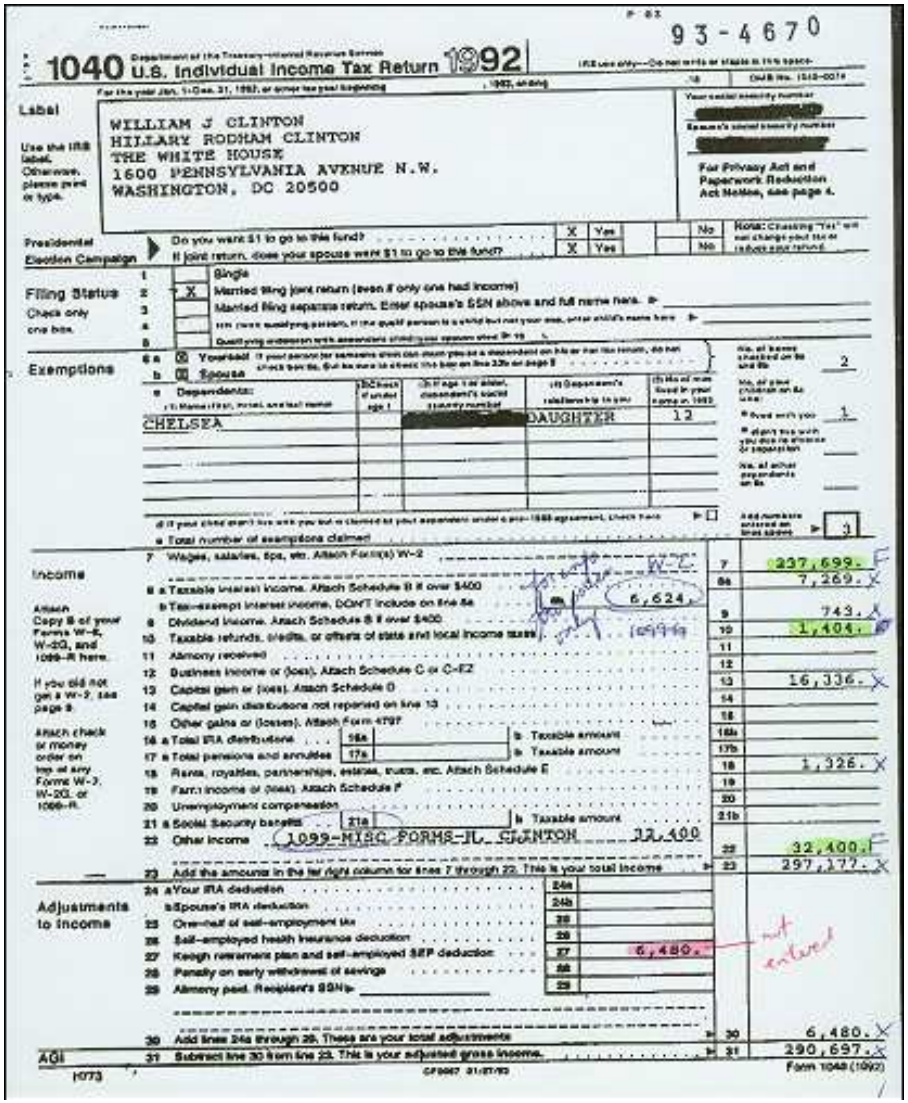
Dependents: CHELSEA 431-47-0195 DAUGHTER 12 — lived with you: 1

Income:
7 Wages, salaries, tips, etc. SEE STMT 1 — 146,444.
8a Taxable interest income — 12,446.
b Tax-exempt interest income 3,101
9 Dividend income — 26.
10 Taxable refunds of state and local income taxes, if any — 1,153.
11 Alimony received
12 Business income or (loss)
13 Capital gain or (loss) — 11,036.
14 Capital gain distributions not reported on line 13
15 Other gains or (losses) — -1,423.
16a Total IRA distributions
17a Total pensions and annuities — 1,269.
18 Rents, royalties, partnerships, estates, trusts, etc.
19 Farm income or (loss)
20 Unemployment compensation (insurance)
21a Social security benefits
22 Other income (list type and amount) SEE STATEMENT 8 — 26,752.
23 Add the amounts shown in the far right column for lines 7 through 22. This is your total income — 197,651.

Adjustments to Income:
24 Your IRA deduction
25 Spouse's IRA deduction
26 Self-employed health insurance deduction
27 Keogh retirement plan and self-employed SEP deduction — 3,483.
28 Penalty on early withdrawal of savings
29 Alimony paid
30 Add lines 24 through 29 — 3,483.

Adjusted Gross Income:
31 Subtract line 30 from line 23. This is your adjusted gross income — 194,168.

Applications for the IRS: Detecting Fraud



93-4670

1040 U.S. Individual Income Tax Return 1992

Label
WILLIAM J CLINTON
HILLARY RODHAM CLINTON
THE WHITE HOUSE
1600 PENNSYLVANIA AVENUE N.W.
WASHINGTON, DC 20500

Presidential Election Campaign
Do you want $1 to go to this fund? X Yes No
If joint return, does your spouse want $1 to go to this fund? X Yes No

Filing Status
2 X Married filing joint return (even if only one had income)

Exemptions
6a X Yourself
b X Spouse
c Dependents: CHELSEA — DAUGHTER — 12
Number of boxes checked on 6a and 6b: 2
Number of children listed in your name in 1992: 1
d Total number of exemptions claimed: 3

Income
7 Wages, salaries, tips, etc. Attach Form(s) W-2: 237,699.
8a Taxable interest income. Attach Schedule B if over $400: 7,269.
b Tax-exempt interest income. DON'T include on line 8a: 6,624.
9 Dividend income. Attach Schedule B if over $400: 743.
10 Taxable refunds, credits, or offsets of state and local income taxes: 1,404.
13 Capital gain or (loss). Attach Schedule D: 16,336.
18 Rents, royalties, partnerships, estates, trusts, etc. Attach Schedule E: 1,326.
22 Other income: 1099-MISC FORMS-H. CLINTON 32,400: 32,400.
23 Add the amounts in the far right column for lines 7 through 22. This is your total income: 297,177.

Adjustments to Income
27 Keogh retirement plan and self-employed SEP deduction: 6,480.
30 Add lines 24a through 29. These are your total adjustments: 6,480.
31 Subtract line 30 from line 23. This is your adjusted gross income: 290,697.

Detecting Fraud

Bank Fraud

- Audit of a bank revealed huge spike of numbers starting with

Detecting Fraud

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Detecting Fraud

Bank Fraud

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- Write-off limit of $5,000. Officer had friends applying for credit cards, ran up balances just under $5,000 then he would write the debts off.

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Tuesday 11/1: 6:30pm OIT Computer Lab: Mary Kate Shea (Financial Aid): Know What (And Why) You Owe: Become a tuition bill interpreter! Learn the basics of how to read and pay a bill.

Targeting Consumers

TARGET COUPONS PUT DAD ONTO DAUGHTER'S PREGNANCY

February 17, 2012 - By Brande Victorian

15 COMMENTS

Protecting Lesson

Takeaways.

- Can flag questionable events for further study.
- Allocate resources optimally.
- Can take corrective action early (and saw advantages of early).

Highering Your MOOla

Gameplan

Talked about growing and protecting money.

Now discuss how to *get* money.

Get hired where happy, productive, and benefits....

General Advice: How can you use your tools?

Law of the Hammer:

- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.

General Advice: How can you use your tools?

The Pythagorean Theorem

American League

Select favorite team | Standings as of | Jun | 5 | 2008 | Go

East	W	L	PCT	GB	L10	STRK	INT	HOME	ROAD	X W-L	LAST GAME	NEXT GAME
Boston	37	25	.597	-	6-4	W2	3-0	23-5	14-20	36-26	6/4 v TB, W 5-1	6/5 v TB, 6:05P
Tampa Bay	35	24	.593	0.5	6-4	L2	1-2	24-10	11-14	32-27	6/4 @ BOS, L 1-5	6/5 @ BOS, 6:05P
Toronto	32	29	.525	4.5	6-4	L1	2-1	15-11	17-18	34-27	6/4 @ NYY, L 1-5	6/5 @ NYY, 1:05P
New York	29	30	.492	6.5	5-5	W1	0-2	15-13	14-17	28-31	6/4 v TOR, W 5-1	6/5 v TOR, 1:05P
Baltimore	28	30	.483	7.0	4-6	L1	2-1	17-11	11-19	27-31	6/4 @ MIN, L 5-7	6/5 @ MIN, 1:10P
Central	**W**	**L**	**PCT**	**GB**	**L10**	**STRK**	**INT**	**HOME**	**ROAD**	**X W-L**	**LAST GAME**	**NEXT GAME**
Chicago	32	26	.552	-	6-4	W2	3-0	15-9	17-17	34-24	6/4 v KC, W 6-4	6/5 v KC, 8:11P
Minnesota	31	28	.525	1.5	7-3	W1	1-2	19-15	12-13	29-30	6/4 v BAL, W 7-5	6/5 v BAL, 1:10P
Cleveland	27	32	.458	5.5	4-6	W1	0-3	16-16	11-16	31-28	6/4 @ TEX, W 15-9	6/5 @ TEX, 8:05P
Detroit	24	35	.407	8.5	3-7	L3	1-2	12-14	12-21	27-32	6/4 @ OAK, L 2-10	6/6 v CLE, 7:05P
Kansas City	23	36	.390	9.5	2-8	L2	2-1	12-16	11-20	23-36	6/4 @ CWS, L 4-6	6/5 @ CWS, 8:11P
West	**W**	**L**	**PCT**	**GB**	**L10**	**STRK**	**INT**	**HOME**	**ROAD**	**X W-L**	**LAST GAME**	**NEXT GAME**
Los Angeles	37	24	.607	-	7-3	W5	2-1	18-13	19-11	31-30	6/4 @ SEA, W 5-4	6/6 @ OAK, 10:05P
Oakland	33	27	.550	3.5	6-4	W4	1-2	20-13	13-14	35-25	6/4 v DET, W 10-2	6/6 v LAA, 10:05P
Texas	30	31	.492	7.0	5-5	L1	2-1	15-14	15-17	29-32	6/4 v CLE, L 9-15	6/5 v CLE, 8:05P
Seattle	21	39	.350	15.5	3-7	L4	2-1	14-19	7-20	24-36	6/4 v LAA, L 4-5	6/6 @ BOS, 7:05P

National League

East	W	L	PCT	GB	L10	STRK	INT	HOME	ROAD	X W-L	LAST GAME	NEXT GAME
Philadelphia	35	26	.574	-	8-2	L1	1-2	20-13	15-13	36-25	6/4 v CIN, L 0-2	6/5 v CIN, 1:05P
Florida	32	26	.552	1.5	4-6	W1	1-2	18-12	14-14	29-29	6/4 @ ATL, W 6-4	6/5 @ ATL, 7:00P
New York	30	28	.517	3.5	7-3	W2	2-0	17-11	13-17	30-28	6/4 @ SF, W 5-3	6/5 @ SD, 10:05P

General Advice: How can you use your tools?

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Consulting can be lucrative: sometimes for $, sometimes for experience, sometimes for academic job!

Two relevant talks:

- Tuesday 10/25: 7pm: Weston: Let'$ Talk About It: Grads Get Real About Their MOOLA. How do you learn to "do" money? Why doesn't anyone talk about it? Grads tell their money stories.

- Tuesday 11/8: 7pm: Griffin 2: Janine Burt: How to Negotiate Anything: MOOLA and More Higher salary? More vacation or flex time? Learn vital negotiation skills from an experienced HR and Career Services professional.

General Advice: Perks (Travel)

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http://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/MMProblem10.pdf

General Advice: Perks (Travel)

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Okay, it's important not to get *too* greedy...

General Advice: Perks (Travel)

Okay, it's important not to get *too* greedy...
...but Canada, India, Japan....

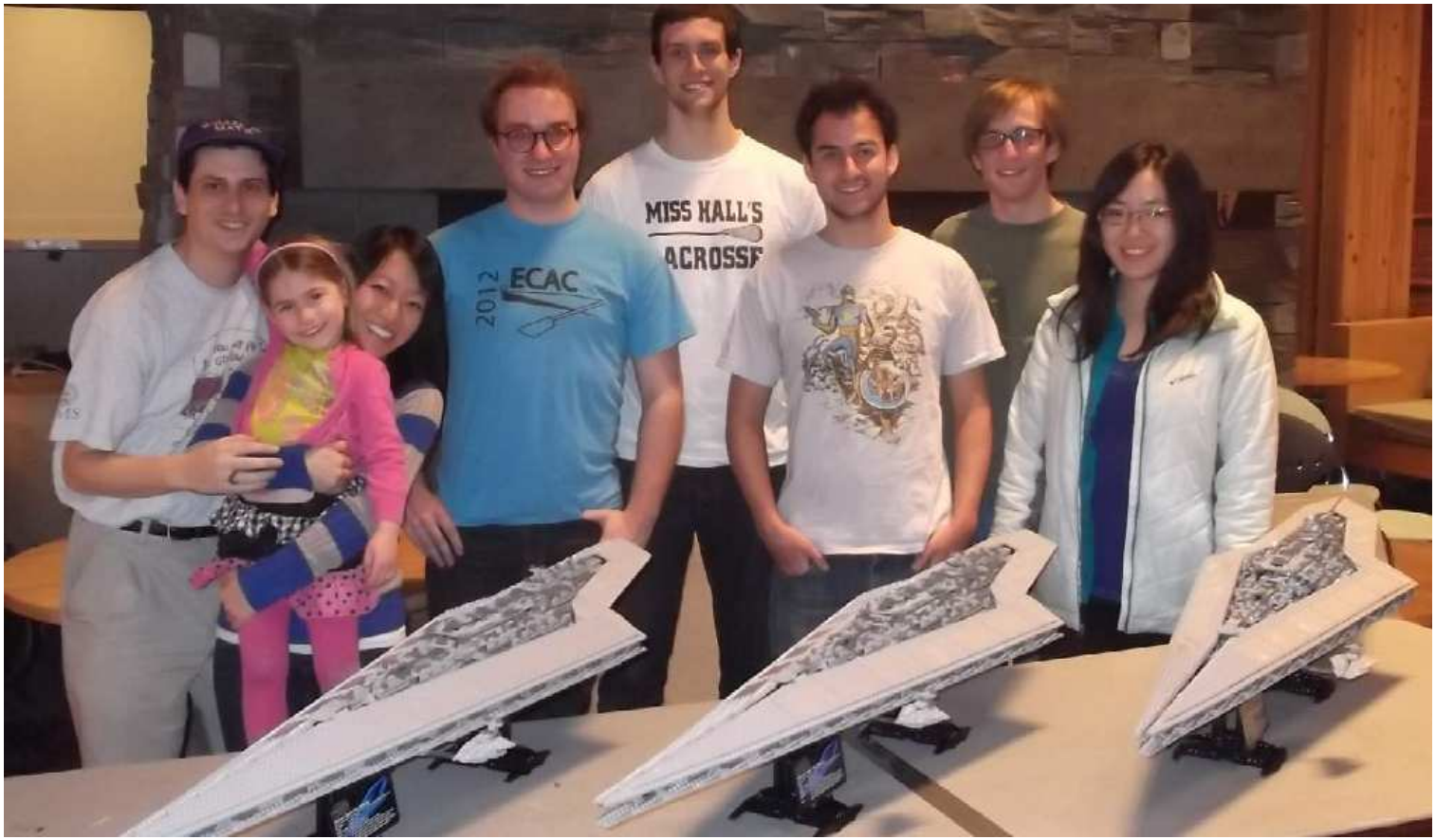
General Advice: Perks (Travel)

Brings up grad school: often *paid* to go (and travel!).

Monday 10/3: 7pm in Mabie: Molly Magavern: MOOLA for Grad School: How To Manage It?

General Advice: Combine Work and Play

General Advice: Combine Work and Play



For $10,000 we can go for an official Guinness Record....

http://web.williams.edu/Mathematics/sjmiller/public_html/legos/index.htm

General Advice: Combine Work and Play

Many thanks to the Elsevier Educational Outreach Fund!