The Explicit Sato-Tate Conjecture in Arithmetic Progressions

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October 7, 2018,
Québec-Maine Number Theory Conference

Motivation

Theorem (Prime Number Theorem)

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Theorem

Refinement to arithmetic progressions: Let a, q be such that $\gcd(a, q) = 1$. Then

$$\pi(x; q, a) := \#\{p \leq x : p \text{ prime and } p \equiv a \bmod q\} \sim \frac{1}{\varphi(q)}\mathrm{Li}(x).$$

Modular Forms

- Recall that a modular form of weight k on $SL_2(\mathbb{Z})$ is a function $f : \mathbb{H} \to \mathbb{C}$ with

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \;\; q = e^{2\pi i z}$$

and

$$f(\gamma z) = (cz + d)^k f(z) \text{ for all } \gamma \in SL_2(\mathbb{Z}).$$

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- By restricting to the action of a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of level N, we can associate that level to our modular form $f(z)$.
- We say a modular form is a cusp form if it vanishes at the cusps of Γ; hence $a_f(0) = 0$ for a cusp form $f(z)$.

Newforms

- We say f is a Hecke eigenform if it is a cusp form and

$$T_n f = \lambda(n) f \text{ for } n = 1, 2, 3, \ldots,$$

where T_n is the Hecke operator.

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- For a newform, the coefficients $a_f(n)$ are multiplicative.
- We consider holomorphic cuspidal newforms of even weight $k \geq 2$ and squarefree level N.

The Ramanujan Tau Function

- Ramanujan tau function:

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Conjecture (Lehmer)

For all $n \geq 1, \tau(n) \neq 0$.

The Sato-Tate Law

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  for some angle $\theta_p \in [0, \pi]$.
- Natural question: What is the distribution of the sequence $\{\theta_p\}$?

The Sato-Tate Law (Continued)

Theorem(Barnet-Lamb, Geraghty, Harris, Taylor)

Let $f(z) \in S_k^{new}(\Gamma_0(N))$ be a non-CM newform. If $F : [0, \pi] \to \mathbb{C}$ is a continuous function, then

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} F(\theta_p) = \int_0^\pi F(\theta) d\mu_{ST}$$

where $d\mu_{ST} = \frac{2}{\pi}\sin^2(\theta)d\theta$ is the Sato-Tate measure. Further

$$\pi_{f,I}(x) := \#\{p \leq x : \theta_p \in I\} \sim \mu_{ST}(I)\mathrm{Li}(x).$$

Symmetric Power L-functions

- We begin by writing

$$f(z) = \sum_{m=1}^{\infty} a_f(m) q^m = \sum_{m=1}^{\infty} m^{\frac{k-1}{2}} \lambda_f(m) q^m.$$

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- From this normalization, we have

$$L(s, f) = \prod_p \left(1 - e^{i\theta_p} p^{-s}\right)^{-1} \left(1 - e^{-i\theta_p} p^{-s}\right)^{-1},$$

and the n-th symmetric power L-function

$$L\left(s, \mathrm{Sym}^n f\right) = \left(\prod_{p \nmid N} \prod_{j=0}^{n} \left(1 - e^{ij\theta_p} e^{(j-n)i\theta_p} p^{-s}\right)^{-1} \right) \left(\prod_{p \mid N} L_p(s)^{-1} \right)$$

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- To pass to arithmetic progressions, we consider $L(s, \mathrm{Sym}^n f \otimes \chi)$.

Previous Work

- Define $\pi_{f,I}(x) = \#\{p \leq x : \theta_p \in I\}$ and let $\mu_{ST}(I)$ denote the Sato-Tate measure of a subinterval $I \subset [0, \pi]$.

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- Rouse and Thorner (2017): under certain analytic hypotheses on the symmetric power L-functions,

$$|\pi_{f,I}(x) - \mu_{ST}(I) Li(x)| \leq 3.33x^{3/4} - \frac{3x^{3/4}\log\log x}{\log x} + \frac{202x^{3/4}\log \mathfrak{q}(f)}{\log x}$$

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- Rouse-Thorner also leads to an explicit upper bound for the Lang-Trotter conjecture, which predicts the asymptotic of the number of primes for which $a_f(p) = c$ for a fixed constant c.

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  - The Generalized Riemann Hypothesis for the twisted symmetric power L-functions $L(s, \mathrm{Sym}^n f \otimes \chi)$.
  - The existence of an analytic continuation of $L(s, \mathrm{Sym}^n f \otimes \chi)$ to an entire function on $\mathbb{C}$ (and a corresponding functional equation).
  - Assumptions about the form of the above completed L-function, including its gamma factor and conductor.

Our Results

Assuming the aforementioned hypotheses, we prove:

Sato-Tate Conjecture for Primes in Arithmetic Progressions

Fix a modulus q. Let $\phi(t)$ be a compactly supported C^∞ test function, and set $\phi_x(t) = \phi(t/x)$. For $x \geq \max\{3.5 \times 10^7, 7400(q \log q)^2\}$:

$$\left| \sum_{\substack{p \nmid N,\, \theta_p \in I \\ p \equiv a(q)}} \log(p)\phi_x(p) - \frac{x\mu_{ST}(I)}{\varphi(q)} \left(\int_{-\infty}^{\infty} \phi(t)dt \right) \right| \leq \frac{Cx^{3/4}\sqrt{\log x}}{\sqrt{\varphi(q)}}$$

for some computable constant C depending on ϕ.

Our Results (continued)

Theorem

Let $\tau(n)$ be the Ramanujan tau function. Then for $x \geq 10^{50}$,

$$\#\{x < p \leq 2x \mid \tau(p) = 0\} \leq 5.973 \times 10^{-7} \frac{x^{3/4}}{\sqrt{\log x}}.$$

Our Results (continued)

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As a consequence, we obtain the following strong evidence in favor of Lehmer's conjecture:

Theorem

Let $\tau(n)$ be the Ramanujan tau function. Then

$$\lim_{X \to \infty} \frac{\#\{n \leq X \mid \tau(n) \neq 0\}}{X} > 1 - 5.2 \times 10^{-14}.$$

Proof Outline: Bounding $\#\{x < p \leq 2x \mid \tau(p) = 0\}$

- If $\tau(p) = 0$, then $\theta_p = \pi/2$ and, by the work of Serre (1981), p is in one of 33 possible residue classes modulo

$$q = 24 \times 49 \times 3094972416000.$$

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- If we let $\phi_x(t) = \phi(t/x)$, where $\phi(t) \in C_c^\infty$ is a test function such that $\phi(t) \geq \chi_{[1,2]}$, then we have

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$$\frac{33}{\log x} \sum_{\substack{p \\ \theta_p = \pi/2 \\ p \equiv a(q)}} \log(p)\phi_x(p) \geq \#\{x < p \leq 2x \mid \tau(p) = 0\}.$$

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Bounding the $\theta_p \in [\pi/2, \pi/2]$ condition

Rouse-Thorner (2017) construct trigonometric polynomials

$$F^{\pm}_{I,M}(\theta) = \sum_{n=0}^{M} \hat{F}^{\pm}_{I,M}(n) U_n(\cos\theta)$$

which satisfy $\forall x \in [0, \pi]$,

$$F^{-}_{I,M}(x) \leq \chi_I(x) \leq F^{+}_{I,M}(x)$$

and best approximate the indicator function for any interval $I \in [0, \pi]$. Using these we can expand out the sum from the previous slide.

Proof Outline: Bounding $\#\{p < x \leq 2x \mid \tau(p) = 0\}$

Fourier Expansion

Therefore, setting $I = [\pi/2, \pi/2]$:

$$\sum_{\substack{p \\ \theta_p = \pi/2 \\ p \equiv a(q)}} \frac{\log p}{\log x} \phi_x(p)$$

$$\leq \frac{1}{\log x} \sum_{n=0}^{M} |\hat{F}_{I,M}^{+}(n)| \frac{1}{\varphi(q)} \sum_{\chi(q)} \overline{\chi}(a) \left| \sum_{p} U_n(\cos\theta_p) \log(p) \chi(p) \phi_x(p) \right|.$$

Through contour integration we can bound this innermost sum, and consequently, obtain a bound for the entire expression.

Proof Outline: The Contour Integral

The innermost sum is related to the contour integral of the n-th symmetric L-function twisted by χ:

$$\sum_{p^j} U_n(\cos(j\theta_p))\chi(p^j)\log(p) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \mathrm{Sym}^n f\otimes\chi)\Phi_x(s)\,ds.$$

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By pushing this contour to $-\infty$ and summing the residues from the zeros of $L(s, \mathrm{Sym}^n f \otimes \chi)$, we have

$$\sum_{p} U_n(\cos\theta_p)\log(p)\chi(p)\phi_x(p) = \delta_{\substack{n=0\\ \chi=\chi_0}}\Phi(1)x - \sum_{\rho}\Phi(\rho)x^{\rho} + O(n\sqrt{x}).$$

Proof Outline: From the Contour Integral to the Final Bound

Evaluates to

$$\left| \sum_{p} U_n(\cos\theta_p) \log(p)\chi(p)\phi_x(p) \right| \leq \delta_{\substack{n=0 \\ \chi=\chi_0}} \Phi(1)x + O(n \log n\sqrt{x})$$

where we can compute explicit bounds for the error term. Then,

$$\sum_{\substack{p \\ \theta_p=\pi/2 \\ p\equiv a(q)}} \frac{\log p}{\log x}\phi_x(p) \leq \frac{1}{\log x}\left(\frac{1.33x}{\varphi(q)M} + 7.63M\log M\sqrt{x} + O(M\sqrt{x})\right).$$

Selecting $M = 6.894 \times 10^{-9}\frac{x^{1/4}}{\sqrt{\log x}}$, gives us our final bound. □

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Acknowledgements

This work was supported by NSF grants DMS1659037 and DMS1561945, Princeton University, and Williams College, specifically the John and Louise Finnerty Fund, and we are deeply grateful to our excellent advisors, Steven J. Miller and Jesse Thorner, for valuable guidance.