IDENTIFYING & BREAKING THE SYMMETRY GROUP OF ZEROS OF FAMILIES OF L-FNS

ZEROS OF L-FNS WELL MODELED BY RMT

↳ Goal:
- describe some problems whose answers depend on behavior of zeros
- discuss "techniques" of RMT
- investigate zeros near central point
  - ↳ evidence that "in limit" well modeled by evalues near 1 of a classical compact group
  - ↳ show how "universal" answers are, discuss cause (2nd moment Satake params), break universality
  - ↳ if time, problems with "finite" conds

1-20-06

3 Motivating Problems

(1) $\pi(x)$ and $\pi_{a,q}(x)$

$$\zeta(s) = \sum \frac{1}{n^s} \overset{\text{UF}}{=} \prod_p (1-p^{-s})^{-1}$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum \frac{\Lambda(n)}{n^s}$$

Mellin transform, shift contour:

Explicit Formula

$$x - \sum_\rho \frac{x^\rho}{\rho} = \sum_{n \le x} \Lambda(n)$$

by Partial Summation get PNT if $\operatorname{Re}(\rho) < 1$

Similar for primes in arithm progression: $p \equiv a\ (q)$

$$L(s,\chi) = \sum \frac{\chi(n)}{n^s} \underset{\text{MULT}}{\overset{\text{UF}}{=}} \prod_p (1-\chi(p)p^{-s})^{-1}$$

and $\displaystyle \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \chi(n) = \begin{cases} 1 & n \equiv 1 \bmod q \\ 0 & \text{otherwise} \end{cases}$

Look at $\displaystyle \frac{1}{\varphi(q)} \sum_{\chi \bmod q} -\frac{L'(s,\chi)}{L(s,\chi)} \cdot \chi(\bar{a})$

$$= \sum_{p \equiv a(q)} \frac{\log p}{p^s} + \text{Good}(s)$$

Contour integral, ...

To understand $\pi_{a,q}(x)$ need all chars mod q

See the advantage of studying a family.

CHEBYSHEV'S BIAS (RUB-SAR)

$\pi_{3,4}(x) \geq \pi_{1,4}(x)$ and $\pi_{2,3}(x) \geq \pi_{1,3}(x)$ "most" of time

Use analytic density: $\mathrm{Den}_{an}(S) = \limsup \frac{1}{\log T} \int_{S \cap [2,T]} \frac{dt}{t}$

$\pi_{3,4} \geq \pi_{1,4}$ @ density .9959 (1st flip at 26861)

$\pi_{2,3} \geq \pi_{1,3}$ @ density .9990 (1st flip at $\approx 6 \cdot 10^{11}$)

Non-residues beat residues

Key ingredient: GSH (Wintner '38)

↳ <u>Structure</u> of zeros important (indep over $\mathbb{Q}$)

(have a torus, implies "full" and not degenerate: need equidistribution)

CLASS NUMBER

Imaginary Quad field $\mathbb{Q}(\sqrt{D})$, Fund Disc $D<0$

I = Group non-zero fractional ideals

P = Sub-group of principal ideals

$\mathcal{H} = I/P$ class group: measures non-principal

$h(D) = \#\mathcal{H}$ is the class number

Dirichlet: $L(1, \chi_D) = \dfrac{2\pi h(D)}{w_D \sqrt{D}}$ $\quad w_D = \begin{cases} 2 & D<-4 \\ 4 & D=-4 \\ 6 & D=-3 \end{cases}$

$(h(D)=1 \iff -D \in \{3,4,7,8,11,19,43,67,163\})$

Expect $\dfrac{\sqrt{|D|}}{\log\log|D|} \ll h(D) \ll \sqrt{|D|}\log\log|D|$

Siegel ('35): $\forall \varepsilon\ \exists C(\varepsilon)$ st ineffectively $h(D) > C(\varepsilon)|D|^{\frac{1}{2}-\varepsilon}$

Goldfeld, Gross-Zagier

Thm: f primitive cuspform weight k, level N, trivial central character

Suppose $m = \operatorname{ord}_{s=1/2} L(s,f)L(s,\chi_D) \geq 3$

Let $g = m-1$ or $m-2$ st $(-1)^g = \omega(f)\omega(f_\chi)$

Then $h(D) \underset{\text{effective}}{\gg} \Theta(D)(\log|D|)^{g-1}$, $\Theta(D) = \prod_{p|D}(1+\frac{1}{p})^{-3}(1+\frac{\lambda(p)\sqrt{p}}{p+1})^{-1}$

↳ result from "many" zeros at central point

THM: Assume a pos percent (or $\frac{cT\log T}{(\log|D|)^A}$ of zeros @ $\gamma \leq T$) of zeros of $\zeta(s)$ are at most $\frac{1}{2}-\varepsilon$ of the ave spacing from the next zero of $\zeta(s)$.

Get $h(D) \gg \sqrt{|D|}\,/\,(\log|D|)^B$, all constants computable

↳ actual spacings b/w zeros tied to number theory

Instead of $\frac{1}{2}-c$ have: RH: .68 (Mont), .5179 (Mont-Odlyzko), .5171 (Conrey-Ghosh-Gonek), .5169 (Conrey-Iwaniec)

Mont says led to Pair Corr by looking gaps zeros $\zeta(s)$ and $h(D)$

SUMMARY TO DATE

(PROBABLY SKIP)

Knowledge of Zeros Important

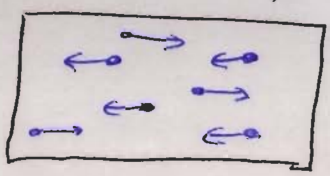
↳ more refined/sophisticated the problem, The more info need on zeros!

(1) Real Part ≤ 1 $\pi(x)$, $\pi_{a,q}(x)$

(2) GRH Chebyshev's Bias

(3) { Order vanishing at $s=\frac{1}{2}$ / Spacing zeros $\zeta(s)$ } $h(D)$

CLASSICAL RANDOM MATRIX THEORY



- 50s: WIGNER: STAT MECH

3 body intractable: Uranium?

① Quantum: $H\Psi_n = E_n \Psi_n$

↳ Approx H by $N \times N$

- entries iidrv
- symm based on physics
- calculate evalues, $\lim_{N\to\infty}$

> **Means**: $\text{Prob}(a_{ij} \in [\alpha,\beta]) = \int_\alpha^\beta p(x)\,dx$
>
> $\text{Prob}(A)\,dA = \prod_{i \le j} p(a_{ij})\,da_{ij}$

GOE, GUE

↳ later do classical compact groups @ Haar measure

more "natural" randomness than p

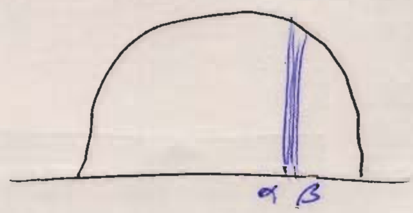
WIGNER'S SEMICIRCLE LAW

(Don't need for NT, but highlights key features)

THM: Prob distr p @ mean 0, var 1, finite higher moments

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

With Prob 1 have $\mu_{A,N} \xrightarrow[N\to\infty]{}$ Semicircle



IDEA OF PROOF

(1) EIGENVALUE TRACE LEMMA: $Tr(A^k) = \sum \lambda_i(A)^k$

↳ Converts info on matrix entries → evalues

↳ NT analogue: Explicit Formula:

(2) SCALING: CLT $\lambda_i(A)/\sqrt{N}$

↳ $k=2$: $\mathbb{E}[Tr(A^2)] = \sum_{i,j} \mathbb{E}[a_{ij}^2] \sim N^2$

$\Rightarrow N \lambda_{ave}^2(A) \sim N^2 \Rightarrow \lambda_{ave}(A) \sim \sqrt{N}$

↳ NT analogue: Analytic cond's low zeros, $T \log T$ high zeros

(3) AVERAGING FORMULAS

k^{th} moment $\mu_{A,N}$ is $\frac{1}{N} \sum_i \frac{\lambda_i(A)^k}{(2\sqrt{N})^k} = \frac{1}{2^k N^{\frac{k}{2}+1}} Tr(A^k)$

Ave. k^{th} moment $\int X^k \mu_{A,N}(x) P(A) dA = \frac{1}{2^k N^{\frac{k}{2}+1}} \int Tr(A^k) P(A) dA$

Combinatorics: $Tr(A^k)$ polynomial, Catalan #s, matching in pairs

↳ Eigenvalue Trace Lemma USELESS w/o averaging formula

↳ NT analogue: Orthog Dirichlet Chars, Petersson Formula,

$a_{t+mp}(p) = a_t(p)$ Ell Curves

Wigner's Semicircle Law: Plots

Can show some plots

(1) Entries from Gaussian

(2) Entries from Cauchy (no Semi-Circle)

Talk about spacings b/w adjacent norm evalues

↳ real symm GOE ~ xe^{-Ax^2} Comp Hermitian GUE ~ $x^2e^{-Bx^2}$

(1) Show Uniform

(2) Show Cauchy

(3) Mention d-regular graphs; show 3-regular

(4) Show Odlyzko's GUE vs $\zeta(s)$

↳ Different Density of States

n-level correlations

$\alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots$ interval I

"Pair Correlation" ~ $\lim_{N\to\infty} \frac{\#\{(i,j): \alpha_j - \alpha_i \in I,\ i,j \le N\}}{N}$

↳ n-level: (n-1) dim Box

↳ Can use smooth test fns instead

↳ know all n-level corr, know spacings

<u>RESULTS</u>: n=2 (Pair) Mont ('73) (TEA)

n=3 (Triple) Hejhal (94)

n Rud-Sar (94-96) (2nd moment universality)

↳ all auto cusp rep

Spacings Odlyzko (80s)

<u>Drawback</u>:
- $\lim_{N\to\infty}$: Can remove fin many ∅ changing answer
- misses central point, lot of NT there

<u>QUESTION</u>: IS GUE enough for NT?

Katz-Sarnak ('99)

Ensemble: Classical Comp Grps @ Haar measure

↳ N→∞ then n-level corr/ spacings same GUE

↳ new stat, n-level density, distinguish

Classical Compact Groups: Differences

- Unitary $U(N)$ $e^{i\theta}$

  Haar replaces $P(A)\,dA$

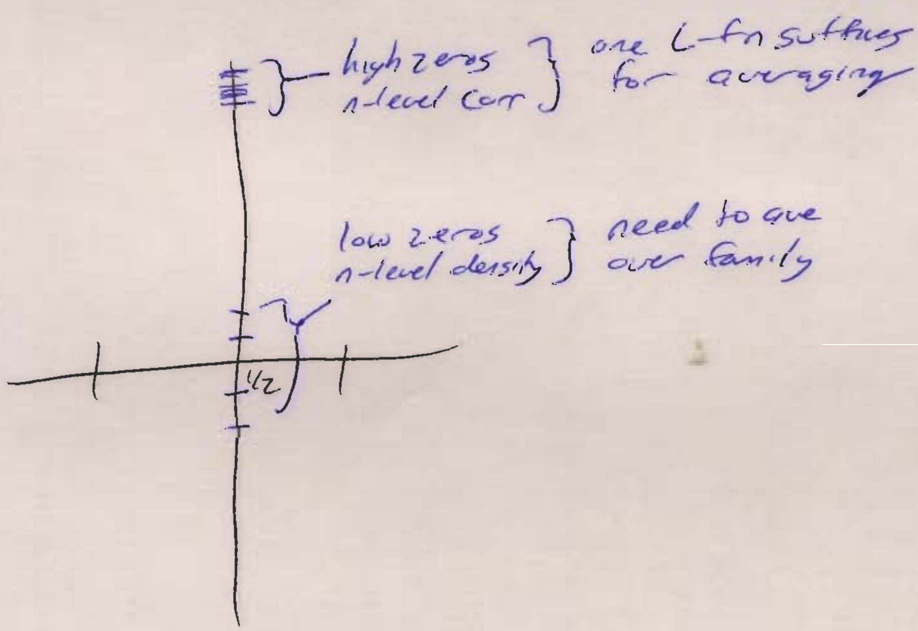
- Orthogonal

  ↳ $SO(2N)$ $\quad$ $SO(2N+1)$

  1 always evalue

- Symplectic

  $J_N = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$, $S J_N S^T = J_N$



n-LEVEL DENSITIES

EXPLICIT FORMULA

GRH: $L(s,f)$ zeros $\frac{1}{2}+i\gamma_{f,k}$, ϕ even Schwartz, $\hat{\phi}$ comp support

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} \omega_f \sum_k \phi\left(\gamma_{f,k}\frac{\log N_f}{2\pi}\right)$$

$$= A_{\mathcal{F}}(\phi) - \frac{2}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\omega_f \sum_p \sum_{m=1}^{\infty} \frac{\alpha_{f,1}(p)^m + \cdots + \alpha_{f,n}(p)^m}{p^{m/2}}$$

$$* \frac{\log p}{\log N_f}\, \hat{\phi}\left(m\frac{\log p}{\log N_f}\right)$$

where $L(s,f) = \prod (1-\alpha_{f,i}(p)p^{-s})^{-1}$

↳ Euler product essential

↳ Often replace N_f with ave cond, ω_f with $\omega_R(f)$

↳ as "family → ∞" converges to $\int \phi(x)\, W_{G(\mathcal{F})}(x)\,dx$

EX: $\mathcal{F}$ = Dirichlet chars of modulus $N\to\infty$

= Weight k cusp newforms sq free $N\to\infty$

= $y^2 = x^3 + A(T)x + B(T)$, $T\in[N,2N]\to\infty$

↳ Different families agree @ different classical comp grps

Behavior norm zeros near 1/2 ~ norm eigenvalues near 1.

(1) QUESTION: How do we determine which class comp grp?

↳ sometimes Function Field (Monodromy)

↳ in general calculate, will show ways to predict

1-Level Densities

Have Slide For This!!!

$$\widehat{W}_G(u) = \begin{cases} \delta(u) & \text{Unitary} \\ \delta(u) - \frac{1}{2}\eta(u) & \text{Symp} \\ \delta(u) + \frac{1}{2} & \text{orthog} \\ \delta(u) + \frac{1}{2}\eta(u) & SO(\text{even}) \\ \delta(u) - \frac{1}{2}\eta(u) + 1 & SO(\text{odd}) \end{cases}$$

indistinguishable if $\text{supp}(\hat{\phi}) \subset (-1,1)$

where $\eta(u) = \begin{cases} 1 & |u| < 1 \\ 1/2 & |u| = 1 \\ 0 & |u| > 1 \end{cases}$

Katz-Sarnak have det expansions

↳ somewhat intractable, but for <u>all</u> support

↳ Hughes-Miller: more tractable, but more restricted supp

<u>Key Observation</u>

2-level is diff for arbitrary small supp

easily distinguishes orthog

EXAMPLE: DIRICHLET L-FNS

Ozl-Sny, Rub, Hu-Rud, Me, . . .

Easiest case: conds easy (const or monotone), orthog relations

$$\mathcal{F}_m = \{ \chi \bmod m : m \text{ prime}, \chi \neq \chi_0 \}$$

Study $$\frac{-2}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log m/\pi} \hat{\varphi}\left(\frac{\log p}{\log m/\pi}\right) \frac{\chi(p)}{p^{1/2}} - \frac{2}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log m/\pi} \hat{\varphi}\left(\frac{\log p}{\log m/\pi}\right) \frac{\chi(p^2)}{p}$$

$$+ O\left(\frac{1}{\log m}\right)$$ ← Typically p^3 and higher give negligible contribution

Set $$\delta_m(p,1) = \begin{cases} 1 & p \equiv 1 \ (m) \\ 0 & \text{otherwise} \end{cases}$$, Note $$\sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} (m-1)\delta_m(p,1) - 1 & \text{if } p \neq m \\ 0 & p = m \end{cases}$$

1st sum basically $$-\frac{2}{m-2} \sum_p \frac{\log p}{\log m/\pi} \hat{\varphi}\left(\frac{\log p}{\log m/\pi}\right) \left[\frac{(m-1)\delta_m(p,1)}{p^{1/2}} - \frac{1}{p^{1/2}} \right]$$

$$\ll \frac{1}{m} \sum_p^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p \equiv 1 (m)}^{m^\sigma} p^{-\frac{1}{2}}$$

$$\ll \frac{1}{m} \sum_k^{m^\sigma} k^{-\frac{1}{2}} + \sum_{\substack{k \equiv 1 (m) \\ k \geq m+1}}^{m^\sigma} k^{-\frac{1}{2}}$$

$$\ll \frac{1}{m} \sum_k^{m^\sigma} k^{-\frac{1}{2}} + \frac{1}{m} \sum_k^{m^\sigma} k^{-\frac{1}{2}} \ll \frac{m^{\sigma/2}}{m}$$

- Is negligible for $\sigma < 2$
- This is not exploiting any cancellation!
- Similar argument handles p^2 terms, get negligible contribution for all support

$\Longrightarrow$ Agrees with unitary

- χ quad char, $m \in [N, 2N]$ get Symplectic: note $\chi(p^2) = 1$

EXAMPLE: DIRICHLET L-FNS

Generalizations: (M_e):

Fix r, take sq-free m @ exactly r factors, $m \to \infty$

↳ can take m sq-free by alae + book-keeping (divisor-fn)

Extending Support

$$\psi(x;q,a) = \sum_{\substack{n \le x \\ n \equiv a(q)}} \Lambda(n) = \frac{\psi(x)}{\varphi(q)} + \underbrace{E(x;q,a)}$$

roughly $x^{\frac{1}{2}}(xq)^{\varepsilon}$ under GRH

(Mont?) Expect $E(x;q,a) \ll \sqrt{\frac{x}{\varphi(q)}} \cdot (xq)^{\varepsilon}$

↳ have $\varphi(q)$ residue classes

$\sum_a E(x;q,a) \ll x^{\frac{1}{2}+\varepsilon}$

Sq root cancellation expect each of size $\frac{x^{\frac{1}{2}}}{\varphi(q)^{\frac{1}{2}}}(xq)^{\varepsilon}$

CONJ: $\exists\, \theta < \frac{1}{2}$ st $E(x;q,1) \ll q^{\theta}\sqrt{\frac{x}{\varphi(q)}}\,(xq)^{\varepsilon}$

↳ Implies agrees @ unitary for any finite support

CONJ: $\exists\, \theta \le 1$ st for $N^2 \ll u \ll N^{4-2\theta}$ have

$$\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} E(u;m,1)^2 \ll \frac{N^{\theta}}{N} \sum_{\substack{m=N \\ m\text{ prime}}}^{2N} \sum_{a=1}^{m-1} E(u;m,a)^2$$

↳ implies agree @ unitary for $\sigma < 4-2\theta$

↳ trivially true for $\theta = 1$

↳ need to use Goldston-Vaughan bound to get supp

$V(x,q) = \sum_{a(q)} \left|\psi(x,q,a) - \frac{x}{\varphi(q)}\right|^2$, $\sum_{\substack{q \le Q \\ \text{prime}}} V(x,q) \ll_{GRH} Qx\log Q + Q^{\frac{7}{4}}x^{\frac{1}{4}+\varepsilon} + x^{\frac{3}{2}+\varepsilon}$

↳ replace $x^{\frac{3}{2}+\varepsilon}$ with $x^{1+\varepsilon+h}$ then $\sigma < \frac{2-\theta}{h}$ so $h \to 0$ get $\sigma \to \infty$ (even for $\theta = 1$!)

CONVOLVING FAMILIES OF L-FNS

RMT-GOOD FAMILY OF L-FNS $\mathcal{F}$

- primitive auto L-fns for $GL_n(\mathbb{A}_\mathbb{Q})$, $\mathcal{F}_N \subset \mathcal{F}$ is finite
- Cardinality: $|\mathcal{F}_N| \to \infty$
- Conds: analy conds $f \in \mathcal{F}_N$ essentially const

  $\hookrightarrow \log Q_f = \log R_N + o(\log R_N)$, $R_N \to \infty$

  $\hookrightarrow |\mathcal{F}_N| \geq R_N^{\delta_0}$

- prime sums: $L(s,f) = \prod_{j=1}^{n} \left(1 - \alpha_{f,j}(p)\, p^{-s}\right)^{-1}$

$$\Lambda(s,f) = L_\infty(s,f)\, L(s,f) = \varepsilon(f)\, \Lambda(1-s, \tilde{f})$$

$$b_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu$$

$$\hookrightarrow -2 \sum_p \frac{1}{\sqrt{p}} \frac{\log p}{\log R_N}\, \phi\!\left(\frac{\log p}{\log R_N}\right) \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} b_f(p) = r_{\mathcal{F}}\, \phi(0) + o(1)$$

$$-2 \sum_p \frac{1}{p} \frac{\log p}{\log R_N}\, \phi\!\left(2 \frac{\log p}{\log R_N}\right) \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} b_f(p^2) = -C_{\mathcal{F}} \frac{\phi(0)}{2} + o(1)$$

sums over higher prime powers negligible

THM (Dueñez–Miller)

$r_{\mathcal{F}} = 0$, then if $C_{\mathcal{F}} = \begin{cases} 0 & \text{unitary} \\ 1 & \text{symp} \\ -1 & \text{orthog} \end{cases}$

If $r_{\mathcal{F}} \neq 0$ trivially modify with $r_{\mathcal{F}} \times r_{\mathcal{F}}$ identity block

CONVOLVING FAMILIES OF L-FNS

$f \leftrightarrow \{\alpha_{f,i}\}_{i=1}^{n}$ $g \leftrightarrow \{\alpha_{g,j}\}_{j=1}^{m}$ $f \times g = \{\alpha_{f,i} \cdot \alpha_{g,j}\}_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$

Key observation: $b_{f \times g}(p^r) = b_f(p^r)\, b_g(p^r)$

Allows us to average over each separately

π_f, π_g acts cusp rep GL_n and GL_m, often need to assume functorial lift $\pi_1 \times \pi_2$ to GL_{nm} exists

↳ can prove in some cases: $GL(1)$, $GL(2)$

THM (Dueñez–Miller)

$$\mathcal{C}_{\mathcal{F} \times \mathcal{G}} = \mathcal{C}_{\mathcal{F}} \cdot \mathcal{C}_{\mathcal{G}} \quad \text{and} \quad \text{rank} = 0$$

Aside: ell curves with rank: generalizes Goldfeld quad twists

Arose from studying $\{\phi \times f\}$ and $\{\phi \times \text{Sym}^2 f\}$ with ϕ fixed even Maass form and $f \in H_k^*(1)$

↳ first has symp symmetry, second SO(even)

↳ both all even signs, no corr odd family

↳ Disproves folklore conj: Low-Lying Zeros is Signs Final Eqs

UNIVERSALITY IS SIMILAR TO RUD–SAR
COMES FROM SECOND MOMENT SATAKE PARAMS

LOWER ORDER CORRECTION TERMS

To first order agree w/ RMT

Find lower order terms which depend on arithm

↳ Me (thesis, now), Fouvry–Iwaniec, Young

Ell Curves: 1st order, only rank matters

Modified/Variant Exp Formula: Tractable for GL_2

↳ terms like $$-\frac{2\hat{\phi}(0)}{\log R}\sum_{r=3}^{\infty}\sum_{p}\frac{p^{r/2}(p-1)\log p}{(p+1)^{r+1}}A_{r,\mathcal{F}}(p)$$

$$A_{r,\mathcal{F}}(p) \text{ ``$=$'' } \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} w_f\, a_f(p)^r$$

Idea: expand Satake params into poly Fourier Coeff

- interchange $\sum_r \sum_r$: Abel Summation
- lin comb polylogarithm fns arise simplify to nice rational function $x^{\ell}(1+x)/(1-x)^{2\ell+1}$

↳ can use geom series to simplify r-sum

Advantages:
- Everything in terms of moments Fourier Coeff
- If know/conj distr, can evaluate r-sum with generating function
- see different behavior for CM/no-CM

ONE PARAMETER FAMILIES OF ELL CURVES

$\mathcal{E}: Y^2 = X^3 + A(T)X + B(T) \qquad A(T), B(T) \in \mathbb{Z}[T]$

- Rosen-Silverman: $\lim_{X \to \infty} \frac{1}{X} \sum_{p \le X} \left(\frac{1}{p} \sum_{t(p)} a_t(p) \right) \log p = -r_{\mathcal{E}}$

  ↳ Conj by Nagao, proved uncond for rational ell surface, cond on Tate's Conj otherwise

  ↳ relates the first moment to rank over $\mathbb{Q}(T)$

- MICHEL: no CM: $\frac{1}{p} \sum_{t(p)} a_t(p)^2 = p + O(\sqrt{p})$

  ↳ Me: sharp: $Y^2 = X^3 + TX^2 + 1$

These key ingredients to calculate 1-level density: O-Prog

↳ easier if use ave-log conductor (global rescaling)

Me: Thesis: local rescaling: conds

↳ need to do some sieving to get "almost" arithm prog

↳ conds monotone there: essential for error terms: "Bounded Variation": monotone means traverse once, bound indep N

$\sum_{n=0}^{N/d} g_\varrho(c_n)$: bound indep N

AVE FORMULA: $a_t(p) = a_{t+mp}(p)$: Enough to do Complete sums

↳ Much weaker than Petersson, O-Prog (DI-), RMT

ONE PARAM FAMILIES ELL CURVES

What is the right "Finite Conductor" Model?

Keating - Snaith: Zeros of $\zeta(s)$ at height T modeled by evalues of matrices of size $N \sim \log T$

(1) Show excess rank results

↳ mention Watkins $x^3 + y^3 = d$

(2) Me: study first zero above central point

↳ "faster convergence" but expect like log(Cond)

↳ amalgamate different families (like nuclear phys: nuclei ⊙ similar quantum numbers)

L-Functions and Random Matrix Theory (Random Matrix Theory Slides)

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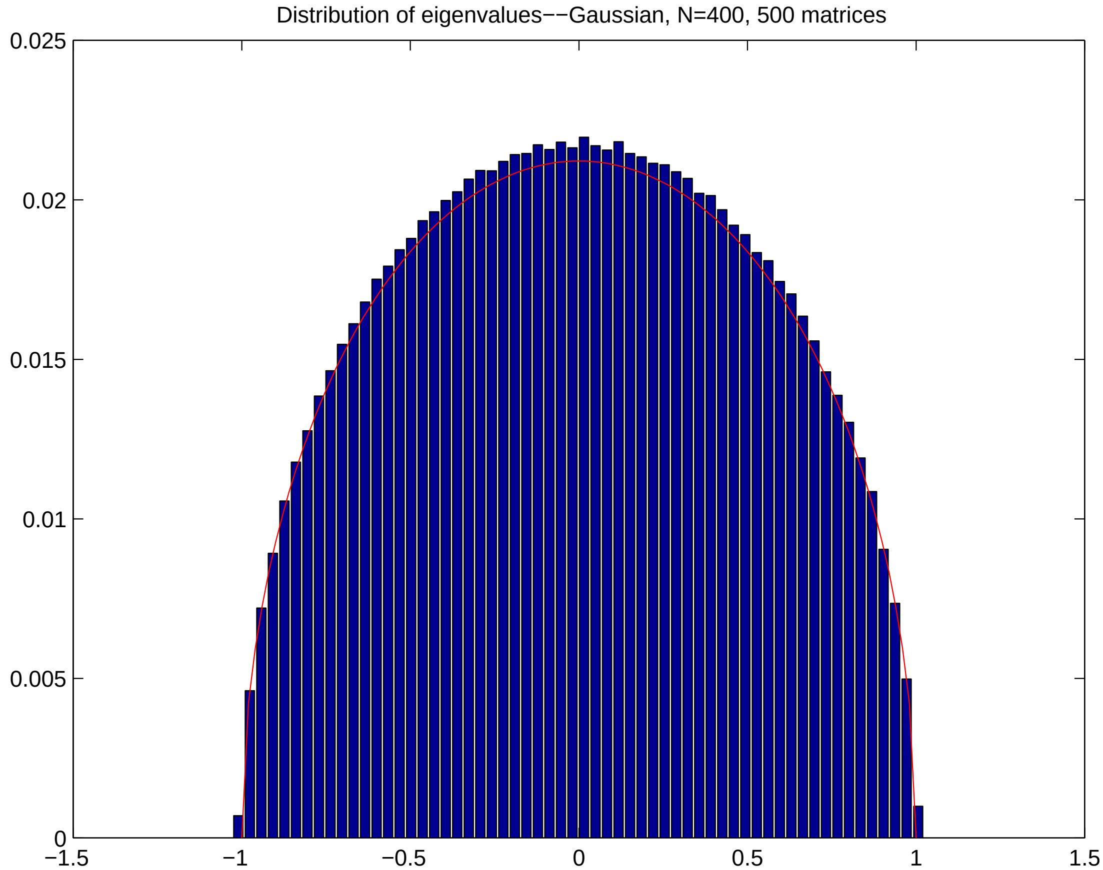
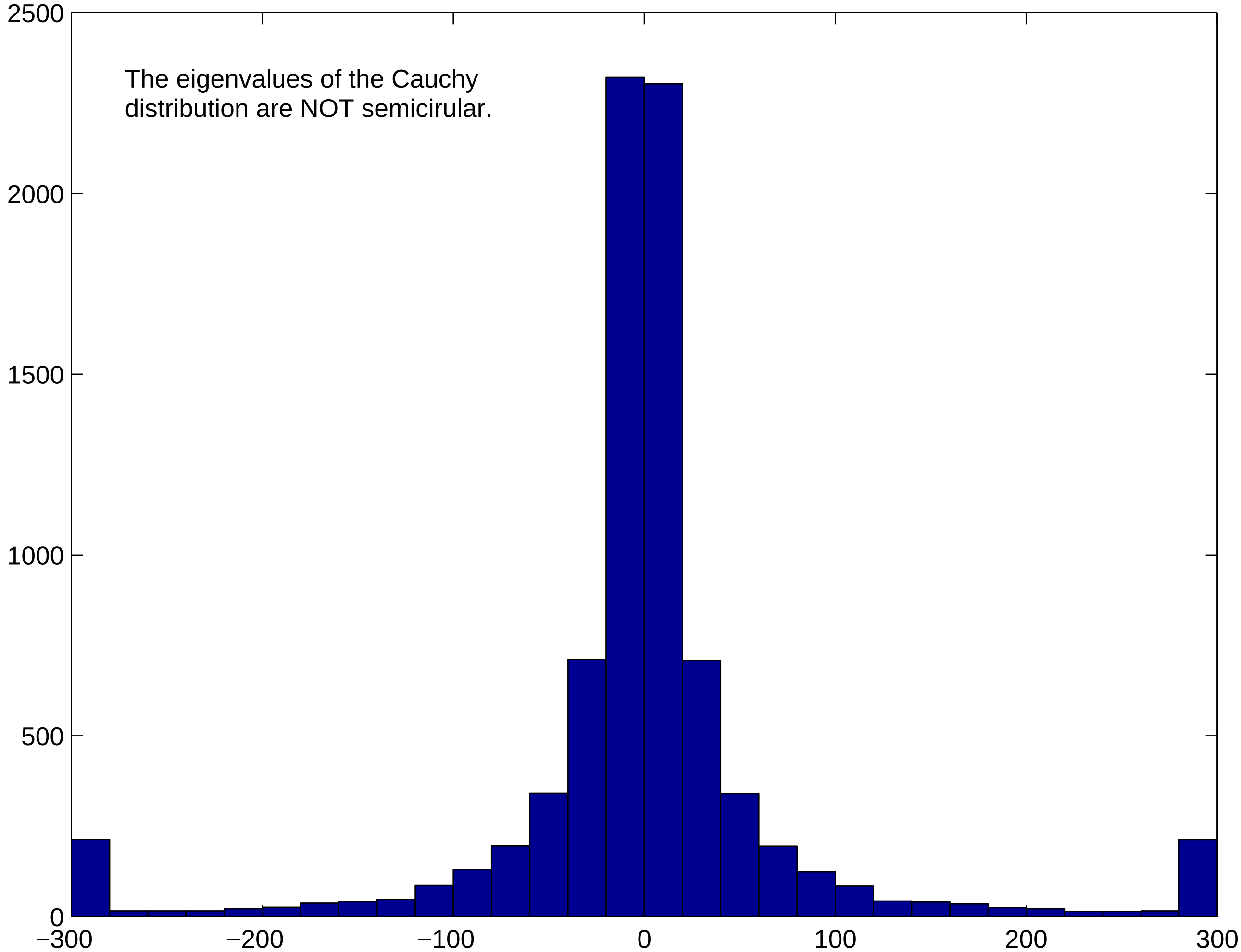
Random Matrix Theory: Semi-Circle Law



500 Matrices: Gaussian 400×400

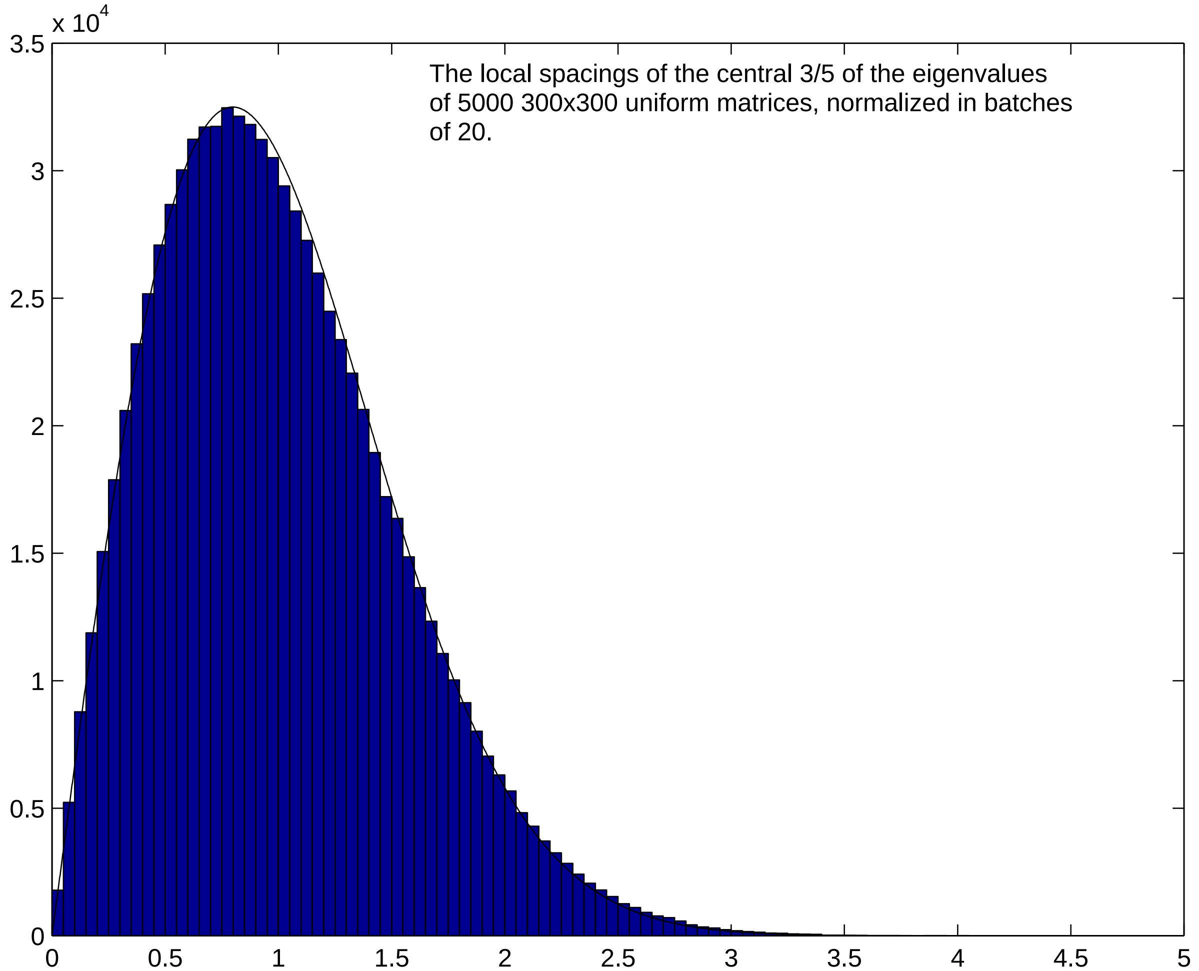
$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Random Matrix Theory: Semi-Circle Law



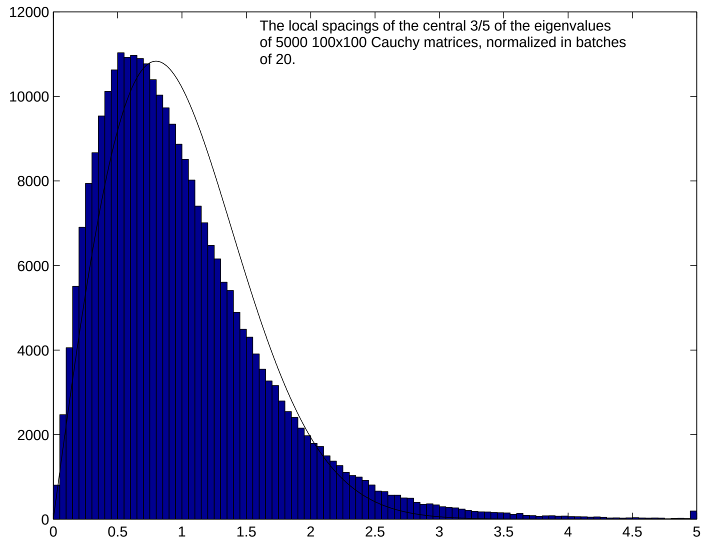
Cauchy Distr: $p(x) = \frac{1}{\pi(1+x^2)}$

Uniform Distribution: $p(x) = \frac{1}{2}$ for $|x| \leq 1$

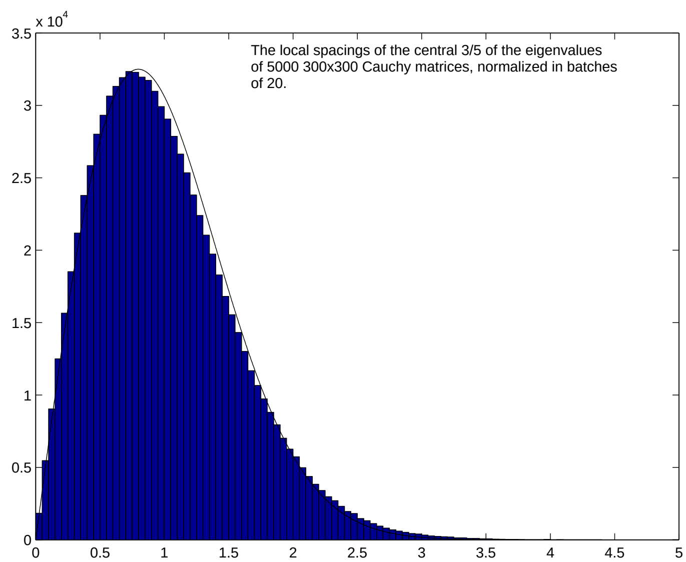


5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

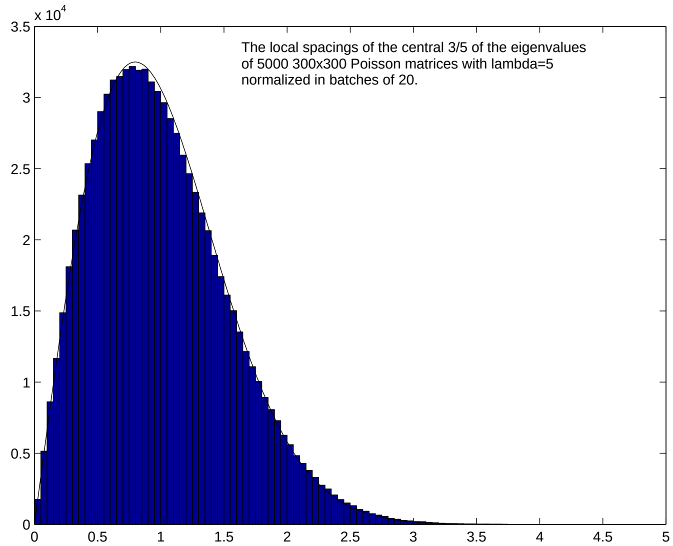


5000: 100×100 Cauchy

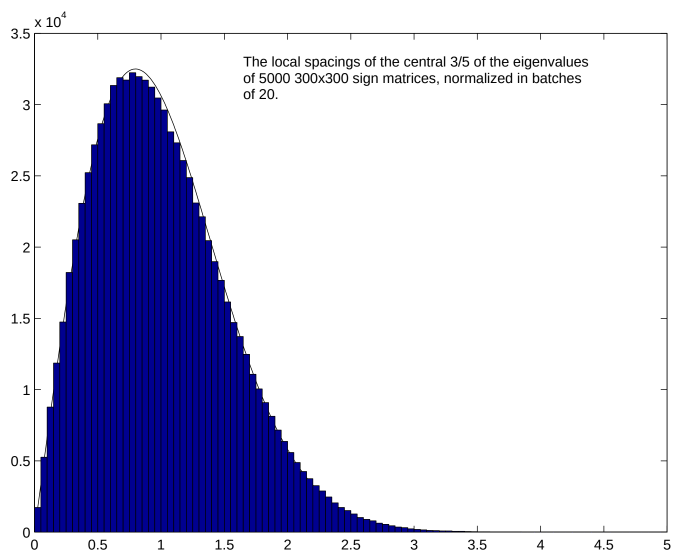


5000: 300×300 Cauchy

Poisson Distribution: $p(n) = \frac{\lambda^n}{n!} e^{-\lambda}$



5000: 300×300 Poisson, $\lambda = 5$

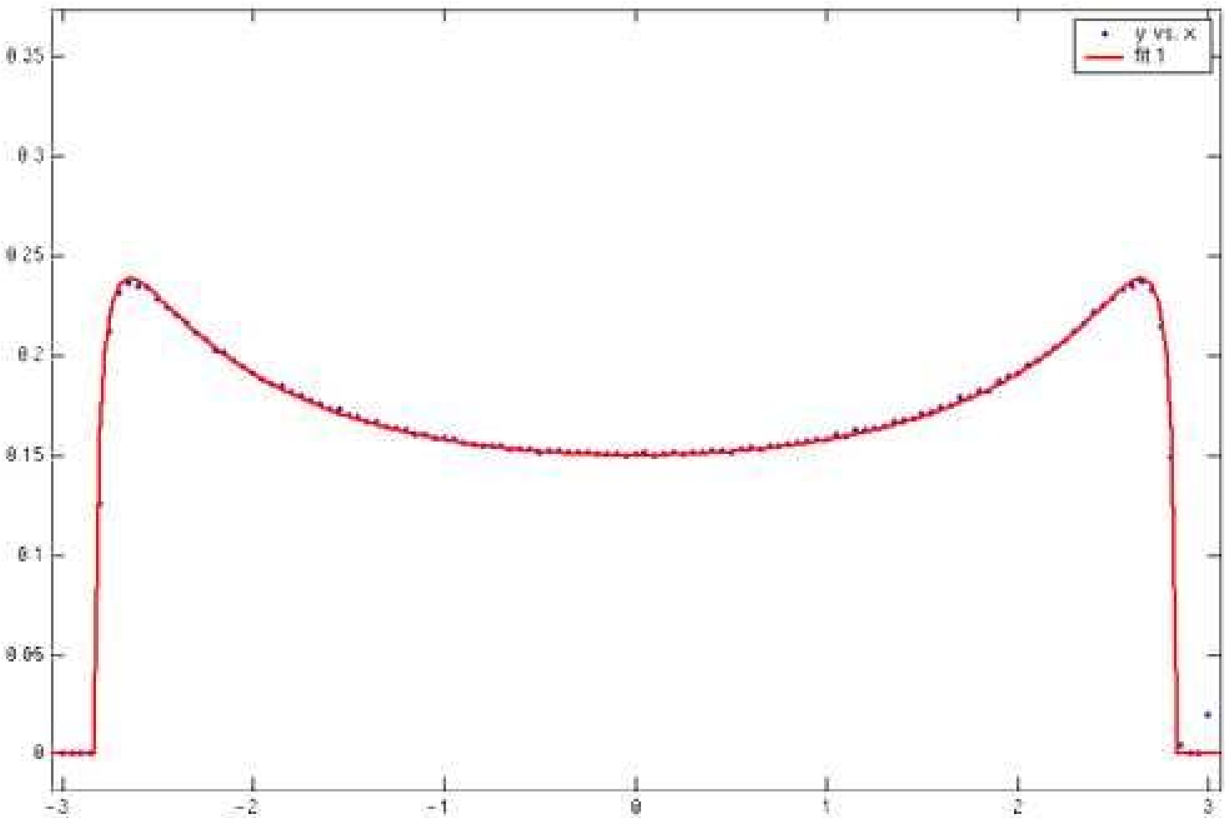


5000: 300×300 Poisson, $\lambda = 20$

McKay's Law (Kesten Measure)

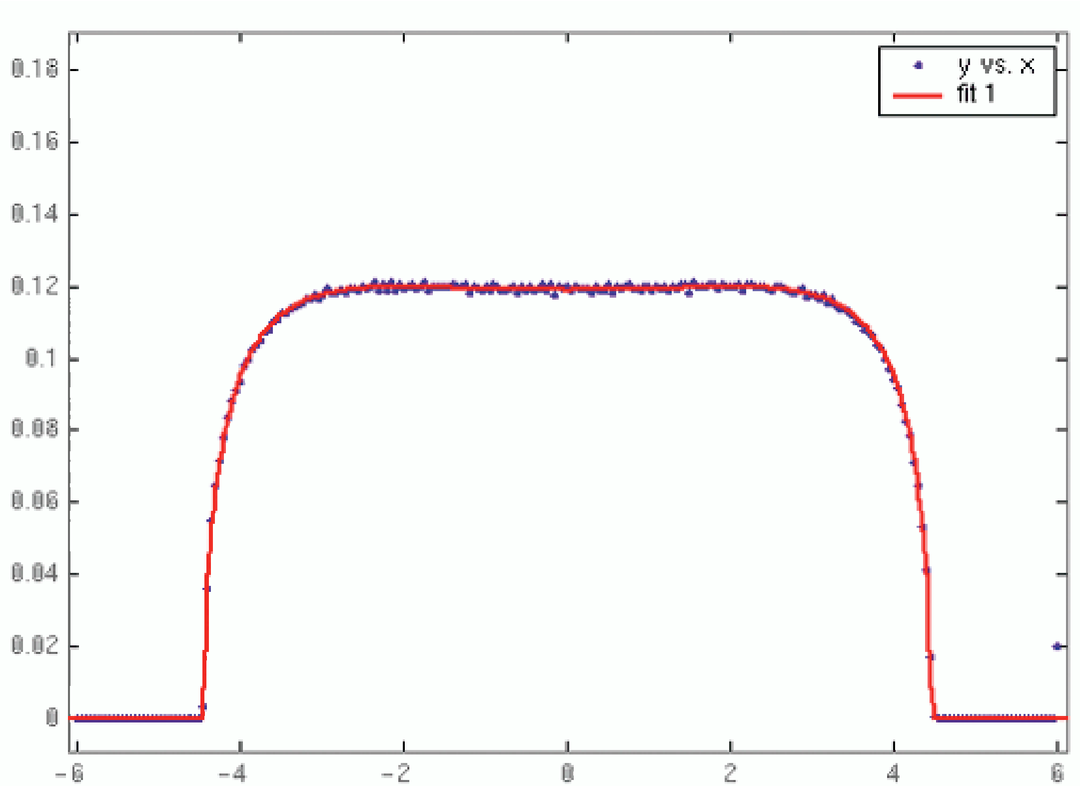
Density of Eigenvalues for d-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



$$d = 3.$$

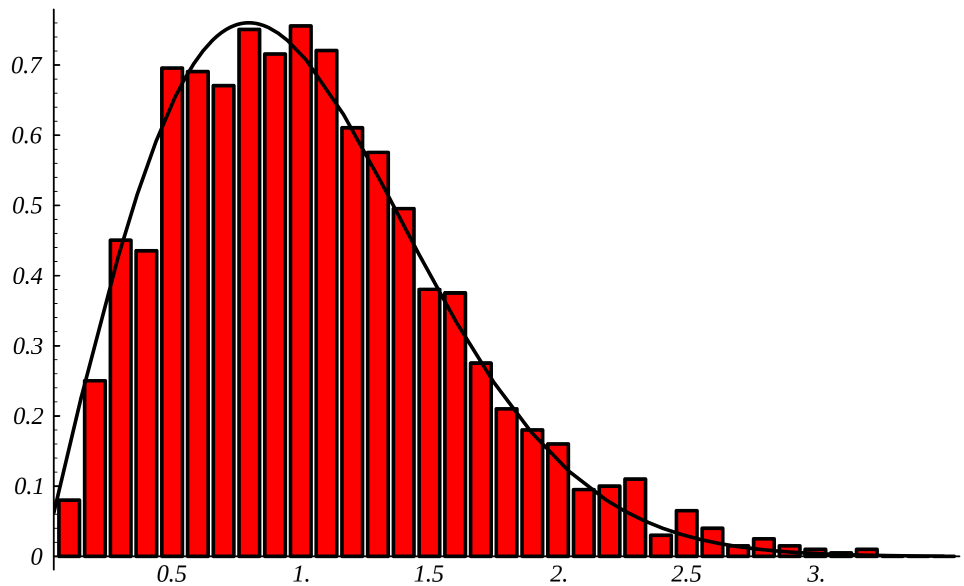
McKay's Law (Kesten Measure)



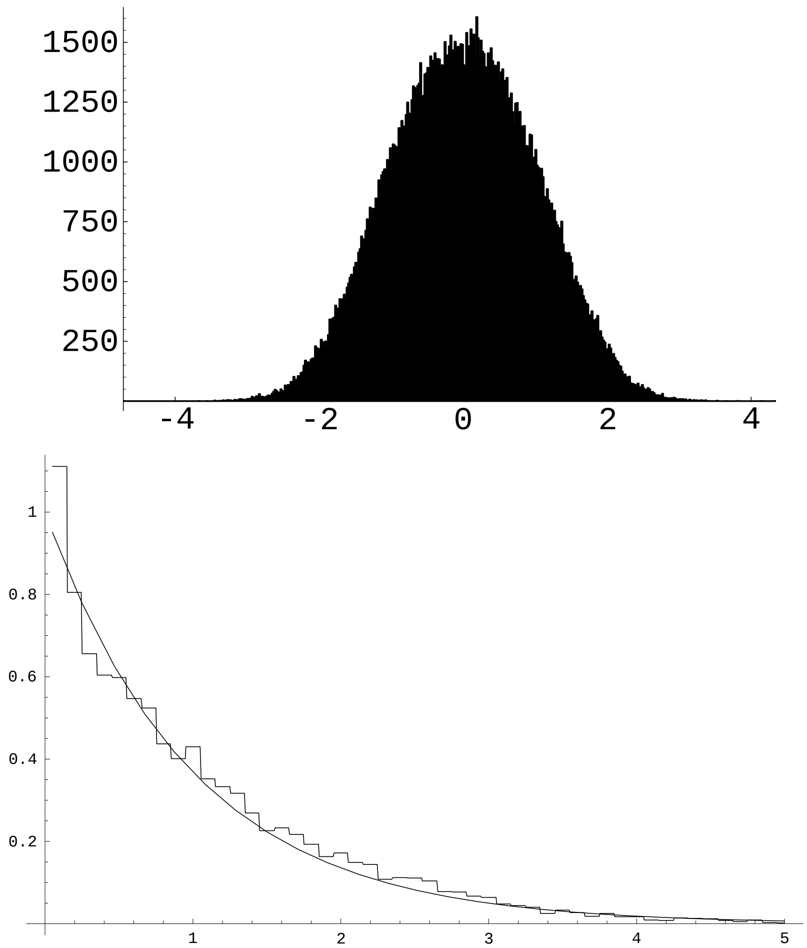
$$d = 6.$$

Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.

3-Regular, 2000 Vertices and GOE: Jakobson-SDMiller-Rivin-Rudnick



Real Symmetric Toeplitz Matrices (Hammond-M–)



(i) Density of normalized eigenvalues.

(ii) Spacings b/w the middle 11 normalized eigenvalues of 1000 Toeplitz matrices (1000×1000), with entries i.i.d.r.v. from the standard normal vs Poissonian behavior.

L-Functions and Random Matrix Theory (Elliptic Curve Slides)

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Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Testing Random Matrix Theory Predictions

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: what percent have rank $\geq r+2$?

2. **First (Normalized) Zero above Central Point:** Do extra zeros at the central point affect the distribution of zeros near the central point?

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(T)$.

RMT $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r	$\approx$	32%
Percent with rank r+1	$\approx$	48%
Percent with rank r+2	$\approx$	18%
Percent with rank r+3	$\approx$	2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$.

Data on Excess Rank

$$y^2 + a_1xy + a_3y \;=\; x^3 + a_2x^2 + a_4x + a_6$$

Family: $a_1 : 0$ to 10, rest -10 to 10.

Percent with rank 0	=	28.60%
Percent with rank 1	=	47.56%
Percent with rank 2	=	20.97%
Percent with rank 3	=	2.79%
Percent with rank 4	=	.08%

14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).

Data on Excess Rank

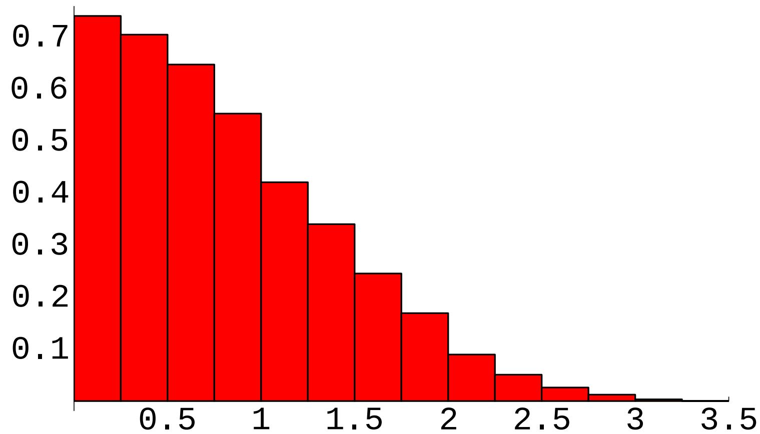
$$y^2 + y = x^3 + Tx$$

Each data set 2000 curves from start.

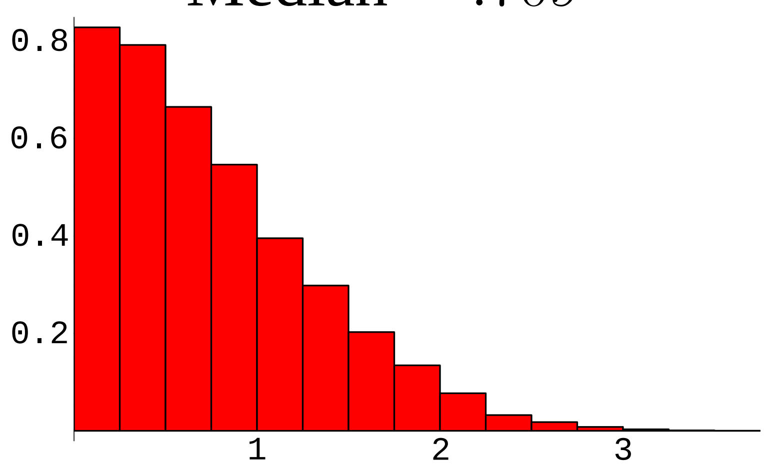
t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{17}, but on logarithmic scale still small.

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

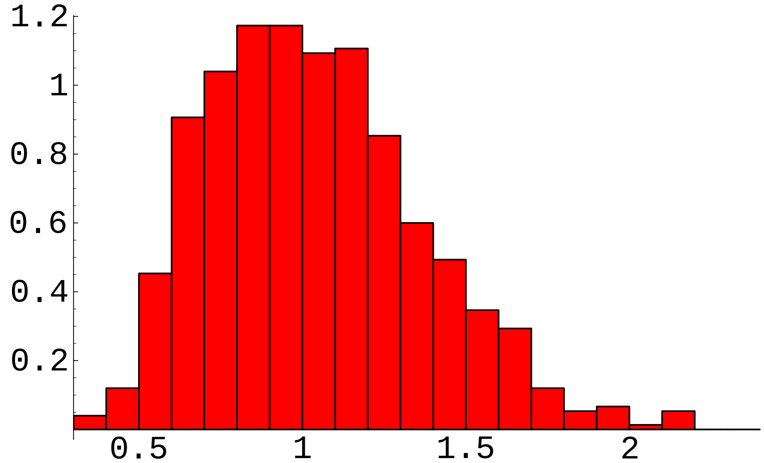


1st norm. evalue above 1: 23,040 SO(4) matrices
Mean $= .709$, Std Dev of the Mean $= .601$,
Median $= .709$



1st norm. evalue above 1: 23,040 SO(6) matrices
Mean $= .635$, Std Dev of the Mean $= .574$,
Median $= .635$

Rank 0 Curves: 1st Normalized Zero above Central Point

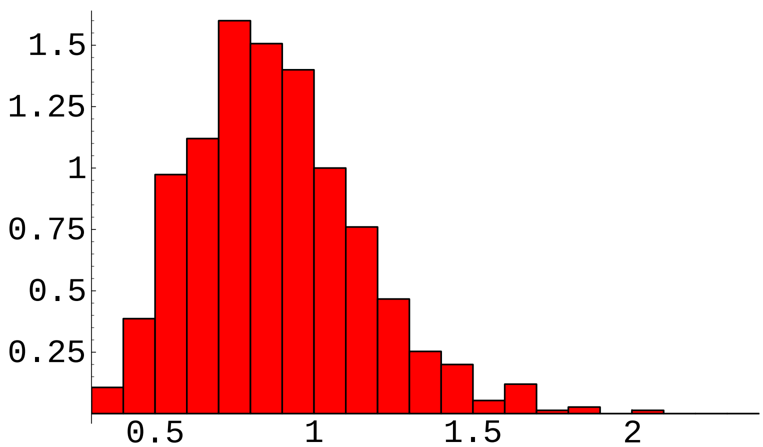


750 rank 0 curves from

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

$\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$,

$\sigma_\mu = .32$

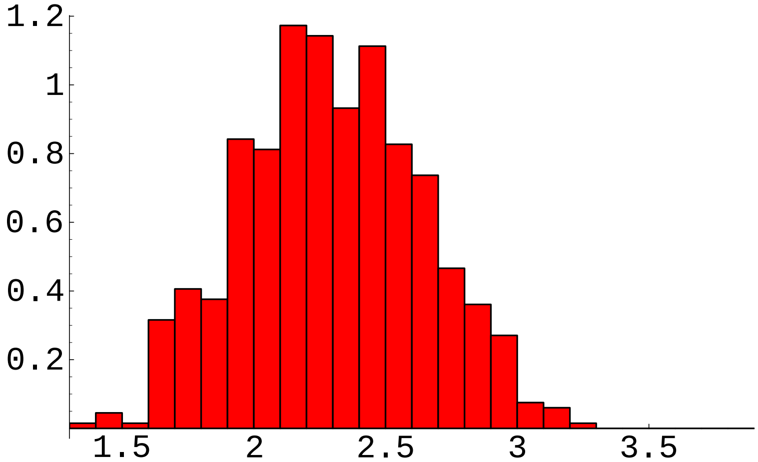


750 rank 0 curves from

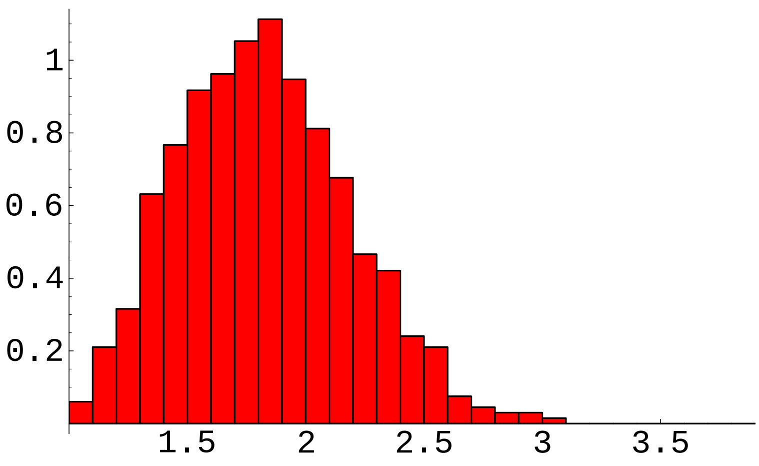
$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

$\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

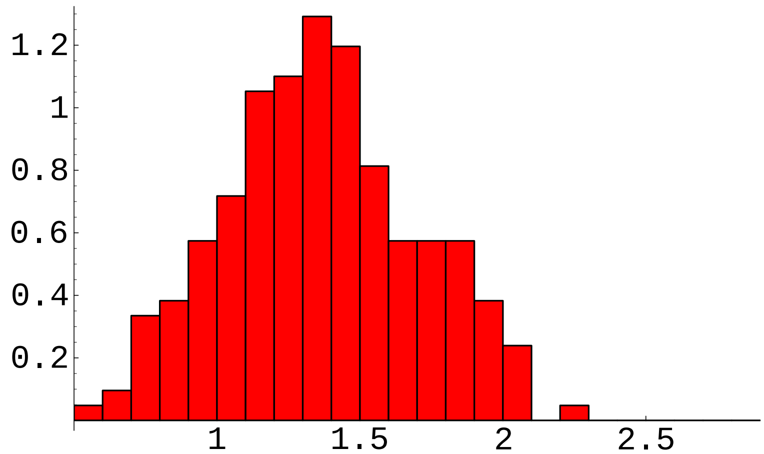


665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

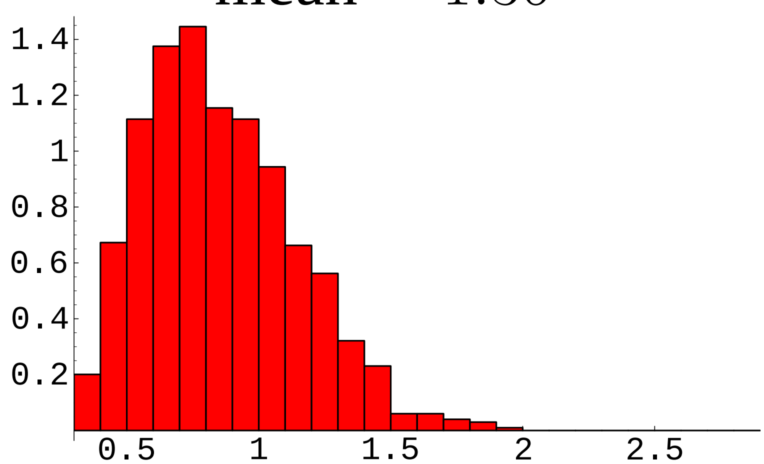


665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0



209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$



996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	$\widetilde{\mu}$	μ	**StDev** σ_μ	**log(cond)**	**Number**
1: [0,1,1,1,T]	1.28	1.33	0.26	[3.93, 9.66]	7
2: [1,0,0,1,T]	1.39	1.40	0.29	[4.66, 9.94]	11
3: [1,0,0,2,T]	1.40	1.41	0.33	[5.37, 9.97]	11
4: [1,0,0,-1,T]	1.50	1.42	0.37	[4.70, 9.98]	20
5: [1,0,0,-2,T]	1.40	1.48	0.32	[4.95, 9.85]	11
6: [1,0,0,T,0]	1.35	1.37	0.30	[4.74, 9.97]	44
7: [1,0,1,-2,T]	1.25	1.34	0.42	[4.04, 9.46]	10
8: [1,0,2,1,T]	1.40	1.41	0.33	[5.37, 9.97]	11
9: [1,0,-1,1,T]	1.39	1.32	0.25	[7.45, 9.96]	9
10: [1,0,-2,1,T]	1.34	1.34	0.42	[3.26, 9.56]	9
11: [1,1,-2,1,T]	1.21	1.19	0.41	[5.73, 9.92]	6
12: [1,1,-3,1,T]	1.32	1.32	0.32	[5.04, 9.98]	11
13: [1,-2,0,T,0]	1.31	1.29	0.37	[4.73, 9.91]	39
14: [-1,1,-3,1,T]	1.45	1.45	0.31	[5.76, 9.92]	10
All Curves	1.35	1.36	0.33	[3.26, 9.98]	209
Distinct Curves	1.35	1.36	0.33	[3.26, 9.98]	196

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	$\widetilde{\mu}$	μ	**StDev** σ_μ	**log(cond)**	**#**
1: [0,1,1,1,T]	0.80	0.86	0.23	[15.02, 15.97]	49
2: [1,0,0,1,T]	0.91	0.93	0.29	[15.00, 15.99]	58
3: [1,0,0,2,T]	0.90	0.94	0.30	[15.00, 16.00]	55
4: [1,0,0,-1,T]	0.80	0.90	0.29	[15.02, 16.00]	59
5: [1,0,0,-2,T]	0.75	0.77	0.25	[15.04, 15.98]	53
6: [1,0,0,T,0]	0.75	0.82	0.27	[15.00, 16.00]	130
7: [1,0,1,-2,T]	0.84	0.84	0.25	[15.04, 15.99]	63
8: [1,0,2,1,T]	0.90	0.94	0.30	[15.00, 16.00]	55
9: [1,0,-1,1,T]	0.86	0.89	0.27	[15.02, 15.98]	57
10: [1,0,-2,1,T]	0.86	0.91	0.30	[15.03, 15.97]	59
11: [1,1,-2,1,T]	0.73	0.79	0.27	[15.00, 16.00]	124
12: [1,1,-3,1,T]	0.98	0.99	0.36	[15.01, 16.00]	66
13: [1,-2,0,T,0]	0.72	0.76	0.27	[15.00, 16.00]	120
14: [-1,1,-3,1,T]	0.90	0.91	0.24	[15.00, 15.99]	48
All Curves	0.81	0.86	0.29	[15.00,16.00]	996
Distinct Curves	0.81	0.86	0.28	[15.00,16.00]	863

Rank 2 Curves: 1st Norm Zero: 21 1-Param of Rank 0

first set $\log(\text{cond}) \in [15, 15.5)$; second set $\log(\text{cond}) \in [15.5, 16]$.

Family	$\widetilde{\mu}$	μ	σ_μ	**Number**	$\widetilde{\mu}$	μ	σ_μ	**Number**
1: [0,1,3,1,T]	1.59	1.83	0.49	8	1.71	1.81	0.40	19
2: [1,0,0,1,T]	1.84	1.99	0.44	11	1.81	1.83	0.43	14
3: [1,0,0,2,T]	2.05	2.03	0.26	16	2.08	1.94	0.48	19
4: [1,0,0,-1,T]	2.02	1.98	0.47	13	1.87	1.94	0.32	10
5: [1,0,0,T,0]	2.05	2.02	0.31	23	1.85	1.99	0.46	23
6: [1,0,1,1,T]	1.74	1.85	0.37	15	1.69	1.77	0.38	23
7: [1,0,1,2,T]	1.92	1.95	0.37	16	1.82	1.81	0.33	14
8: [1,0,1,-1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
9: [1,0,1,-2,T]	1.74	1.74	0.43	14	1.82	1.90	0.40	14
10: [1,0,-1,1,T]	2.00	2.00	0.32	22	1.81	1.94	0.42	18
11: [1,0,-2,1,T]	1.97	1.99	0.39	14	2.17	2.14	0.40	18
12: [1,0,-3,1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
13: [1,1,0,T,0]	1.89	1.88	0.31	20	1.82	1.88	0.39	26
14: [1,1,1,1,T]	2.31	2.21	0.41	16	1.75	1.86	0.44	15
15: [1,1,-1,1,T]	2.02	2.01	0.30	11	1.87	1.91	0.32	19
16: [1,1,-2,1,T]	1.95	1.91	0.33	26	1.98	1.97	0.26	18
17: [1,1,-3,1,T]	1.79	1.78	0.25	13	2.00	2.06	0.44	16
18: [1,-2,0,T,0]	1.97	2.05	0.33	24	1.91	1.92	0.44	24
19: [-1,1,0,1,T]	2.11	2.12	0.40	21	1.71	1.88	0.43	17
20: [-1,1,-2,1,T]	1.86	1.92	0.28	23	1.95	1.90	0.36	18
21: [-1,1,-3,1,T]	2.07	2.12	0.57	14	1.81	1.81	0.41	19
All Curves	1.95	1.97	0.37	350	1.85	1.90	0.40	388
Distinct Curves	1.95	1.97	0.37	335	1.85	1.91	0.40	366

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

- Observe the medians and means of the small conductor set to be larger than those from the large conductor set.

- For all curves the Pooled and Unpooled Two-Sample t-Procedure give t-statistics of 2.5 with over 600 degrees of freedom.

- For distinct curves the t-statistics is 2.16 (respectively 2.17) with over 600 degrees of freedom (about a 3% chance).

- Provides evidence against the null hypothesis (that the means are equal) at the $.05$ confidence level (though not at the $.01$ confidence level).

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

Apply non-parametric tests to further support our claim that the repulsion decreases as the conductors increase.

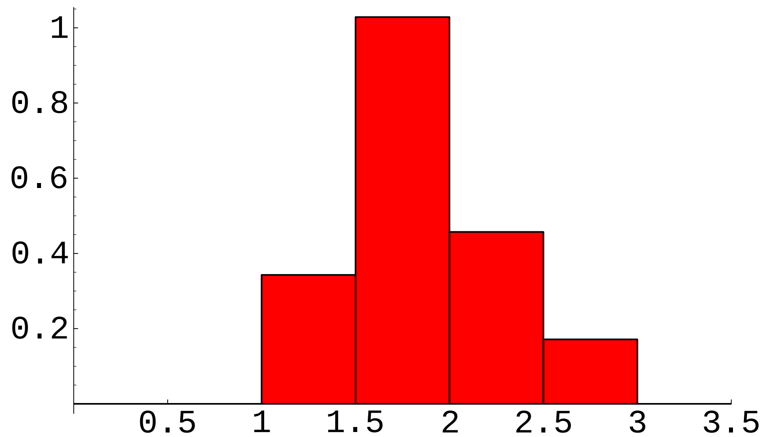
- Write a plus sign if the small conductor set has a larger mean and a minus sign if not.

- Observe four minus signs and seventeen plus signs.

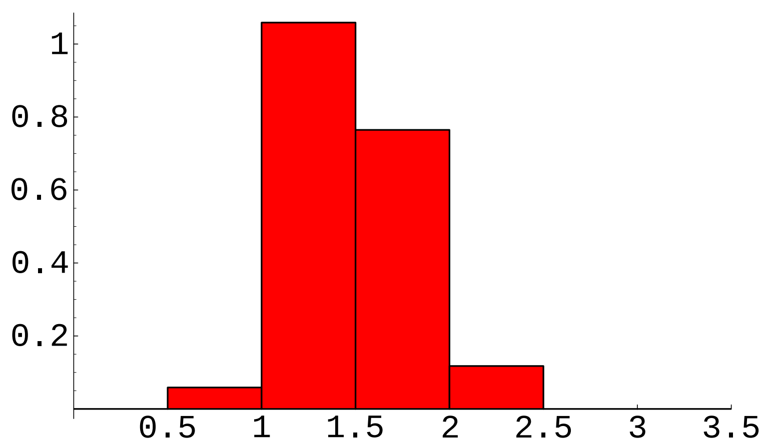
- The null hypothesis is that each is equally likely to be larger; thus the number of minus signs is a random variable from a binomial distribution with $N = 21$ and $\theta = \frac{1}{2}$.

- The probability of observing four or fewer minus signs is about 3.6%, supporting the claim of decreasing repulsion with increasing conductor.

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point



35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$



34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 2 over $\mathbb{Q}(T)$

$\log(\text{cond}) \in [15, 16]$, $t \in [0, 120]$, median is 1.64.

Family	Mean	Standard Deviation	log(cond)	Number
1: [1,T,0,-3-2T,1]	1.91	0.25	[15.74,16.00]	2
2: [1,T,-19,-T-1,0]	1.57	0.36	[15.17,15.63]	4
3: [1,T,2,-T-1,0]	1.29		[15.47, 15.47]	1
4: [1,T,-16,-T-1,0]	1.75	0.19	[15.07,15.86]	4
5: [1,T,13,-T-1,0]	1.53	0.25	[15.08,15.91]	3
6: [1,T,-14,-T-1,0]	1.69	0.32	[15.06,15.22]	3
7: [1,T,10,-T-1,0]	1.62	0.28	[15.70,15.89]	3
8: [0,T,11,-T-1,0]	1.98		[15.87,15.87]	1
9: [1,T,-11,-T-1,0]				
10: [0,T,7,-T-1,0]	1.54	0.17	[15.08,15.90]	7
11: [1,T,-8,-T-1,0]	1.58	0.18	[15.23,25.95]	6
12: [1,T,19,-T-1,0]				
13: [0,T,3,-T-1,0]	1.96	0.25	[15.23, 15.66]	3
14: [0,T,19,-T-1,0]				
15: [1,T,17,-T-1,0]	1.64	0.23	[15.09, 15.98]	4
16: [0,T,9,-T-1,0]	1.59	0.29	[15.01, 15.85]	5
17: [0,T,1,-T-1,0]	1.51		[15.99, 15.99]	1
18: [1,T,-7,-T-1,0]	1.45	0.23	[15.14, 15.43]	4
19: [1,T,8,-T-1,0]	1.53	0.24	[15.02, 15.89]	10
20: [1,T,-2,-T-1,0]	1.60		[15.98, 15.98]	1
21: [0,T,13,-T-1,0]	1.67	0.01	[15.01, 15.92]	2
All Curves	1.61	0.25	[15.01, 16.00]	64

Repulsion or Attraction?

Conductors in $[15, 16]$; first set is rank 0 curves from 14 one-parameter families of rank 0 over $\mathbb{Q}$; second set rank 2 curves from 21 one-parameter families of rank 0 over $\mathbb{Q}$. The t-statistics exceed 6.

Family	**2nd vs 1st**	**3rd vs 2nd**	**Number**
Rank 0 Curves	2.16	3.41	863
Rank 2 Curves	1.93	3.27	701

The additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Comparison b/w One-Param Families of Different Rank

First normalized zero above the central point.

- The first family is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;
- the second family is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

Family	Median	Mean	Std. Dev.	#
Rank 0 Families	1.926	1.936	0.388	701
Rank 2 Families	1.642	1.610	0.247	64

- t-statistic is 6.60, indicating the means differ.
- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on *how* we choose the curves.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0	**701 Rank 2**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- While the normalized zeros are repelled from the central point (and by different amounts for the two sets), the *differences* between the normalized zeros are statistically independent of this repulsion (t-statistics < 2).

- While for a given range of log-conductors the average second normalized zero of a rank 0 curve is close to the average first normalized zero of a rank 2 curve, they are not the same and the additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Spacings b/w Norm Zeros: Rank 2 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2	**23 Rank 4**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2	64 Rank 2	t-Statistic
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Conclusions and Future Work

- Theoretical supports the Independent Model and Birch and Swinnerton-Dyer Conjecture for one-parameter families over $\mathbb{Q}(T)$ as the conductors tend to infinity.

- Experimental suggests a different answer for finite conductors:

  ◇ First normalized zero is repelled by zeros at the central point.

  ◇ The more central point zeros the greater the repulsion.

  ◇ Repulsion decreases as the conductor increases.

  ◇ Difference b/w adjacent normalized zeros stat. indep. of the repulsion.

- What is the right model for rank $r + 2$ curves from rank r one-parameter families over $\mathbb{Q}(T)$: Independent, Interaction or other?

- Unlike the excess rank investigations, noticeable convergence to the limiting theoretical results as we increase the conductors.