Convergence rates in generalized Zeckendorf decomposition problems

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Summary

- Review Zeckendorf-type decompositions

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- Discuss new approaches to asymptotic behavior of variance

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- Discuss new approaches to asymptotic behavior of variance
- Discuss new results on Gaussian behavior of gaps between summands

Previous Results

Definitions: Zeckendorf Decomposition

Theorem (Zeckendorf)

Let $\{F_n\}_{n\in\mathbb{N}}$ denote the Fibonacci numbers with $F_1 = 1$ and $F_2 = 2$. Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

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Example

$$101 = 89 + 8 + 3 + 1 = F_{10} + F_5 + F_3 + F_1$$

Definitions: Positive Linear Recurrence Sequence

Definition

A *Positive Linear Recurrence Sequence (PLRS)* is a sequence $\{G_n\}$ satisfying

$$G_n = c_1 G_{n-1} + \cdots + c_L G_{n-L}$$

with nonnegative integer coefficients c_i with $c_1, c_L, L \geq 1$ and initial conditions $G_1 = 1$ and $G_n = c_1 G_{n-1} + c_2 G_{n-2} + \cdots + c_{n-1} G_1 + 1$ for $1 \leq n \leq L$.

Examples of PLRS

1. Fibonacci numbers: $L = 2, c_1 = c_2 = 1$.
   $G_1 = 1, G_2 = 2, G_3 = 3, G_4 = 5, \dots$.

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   $G_1 = 1, G_2 = b, G_3 = b^2, G_4 = b^3, \dots$.

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2. Powers of b: $L = 2, c_1 = b - 1, c_2 = b$.
   $G_1 = 1, G_2 = b, G_3 = b^2, G_4 = b^3, \ldots$.
3. d-bonacci numbers: $L = d, c_1 = c_2 = \cdots = c_d = 1$,
   $G_n = 2^{n-1}$ for $n \leq d$.

Definition: Generalized Zeckendorf Decomposition

Definition (Generalized Zeckendorf Decomposition)

Let $\{G_n\}$ be a PLRS and m be a positive integer. Then

$$m = \sum_{i=1}^{N} a_i G_{N+1-i}$$

is a **legal** decomposition if $a_1 > 0$ and the other $a_i \geq 0$, and one of the following conditions holds.

1. We have $N < L$ and $a_i = c_i$ for $1 \leq i \leq N$.
2. There exists an $s \in \{1, \dots, L\}$ such that $a_1 = c_1, a_2 = c_2, \dots, a_{s-1} = c_{s-1}$, $a_s < c_s$, and $\{b_i\}_{i=1}^{N-s}$ (with $b_i = a_{s+i}$) is either legal or empty.

Example

Consider the PLRS:

$$G_n = 3G_{n-1} + 2G_{n-2} + 2G_{n-4}.$$

Examples of legal decompositions:

- $m = 3G_9 + 2G_8 + G_6 + 3G_5 + G_4 + 2G_1$.
- $m = 3G_9 + 2G_8 + G_6 + 3G_5 + G_4 + 3G_1$.

Examples of NOT legal decompositions:

- $m = 4G_9$.
- $m = 3G_9 + 2G_8 + G_7$.
- $m = 3G_9 + 2G_8 + 2G_6$.

Theorem: Generalized Zeckendorf Decomposition

Theorem

Let $\{G_n\}$ be a PLRS. Then there is a unique legal decomposition for every positive integer m.

Definitions and Notations

Definition

- Probability Space Ω_n: The set of legal decompositions of integers in $[G_n, G_{n+1})$.
- Probability Measure: Let each of the $G_{n+1} - G_n$ legal decompositions be weighted equally.
- Random Variables K_n: Set $K_n(\omega)$ equal to the number of summands of $\omega \in \Omega_n$.

Asymptotic Behavior of Variance

Old Result

Theorem (Miller and Wang, 2012)

When $\{G_n\}$ is a PLRS, there exists constants $A, B, C, D, \gamma_1 \in (0,1), \gamma_2 \in (0,1)$ such that

$$\mathbf{E}[K_n] = An + B + o(\gamma_1^n) \tag{1}$$

$$\mathbf{Var}[K_n] = Cn + D + o(\gamma_2^n) \tag{2}$$

Remark

- *Proof of* (1) *is* ***easy***.
- *Proof of* (2) *is* ***hard***. *Key difficulty: bound of C.*

Main New Result

Theorem

When $\{G_n\}$ is a PLRS with length L, there exists $C_{min} > 0$ such that

$$\textbf{Var}[K_n] > C_{min} \cdot n$$

for all $n > L$.

Definition of Blocks

Definition

- We define a Type 1 block as an integer sequence corresponding to Condition 1.
- We define a Type 2 block as an integer sequence corresponding to Condition 2.

Example

$$G_n = 2G_{n-1} + 2G_{n-2} + 2G_{n-4}$$

Suppose $m = G_5 + 2G_3 + G_2 + 2G_1$, we write the decomposition into an integer sequence: $[1, 0, 2, 1, 2]$.

- The Type 2 Blocks: $[1], [0], [2, 1]$.
- The Type 1 Block: $[2]$.

Definition of Blocks

Definition

- We define the size of a block as the total number of summands in it.
- We define the length of a block as the total number of indices in it.

Example

$$G_n = 2G_{n-1} + 3G_{n-3} + 2G_{n-4}$$

A possible Type 2 Block is [2, 0, 3, 1]. It has size 6 and length 4.

Key Observations of Type 1 Blocks

Properties:

- It appears at most once in any legal decomposition.
- It has to be the last block if it does exist.

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Properties:

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- It has to be the last block if it does exist.

So,

- Type 1 block matters little when n is large.
- Second to last block has to be a Type 2 block.

Key Observations of Type 2 Blocks

- The length of a Type 2 block is fully determined by its size. So we can define a function $\ell(t)$ such that a Type 2 block with size t has length $\ell(t)$.

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Key Observations of Type 2 Blocks

- The length of a Type 2 block is fully determined by its size. So we can define a function $\ell(t)$ such that a Type 2 block with size t has length $\ell(t)$.
- A Type 2 block always has nonnegative size and strictly positive length.
- A legal decomposition will stay legal if we add a Type 2 block to it or remove a Type 2 block from it.

Key Idea

Lemma

For a fixed t, there is a bijection h_t between:

- *all legal decompositions with total length n and the second to last block with size t, and*
- *all legal decompositions with total length $n - \ell(t)$.*

Example

$$G_n = 2G_{n-1} + 2G_{n-2} + 0 + 2G_{n-4}.$$

$$m = G_6 + G_5 + 2G_4 + G_1.$$

Its block representation: $[1], [1], [2, 0], [0], [1]$.
After removing $[0]$: $[1], [1], [2, 0], [1]$.
The resulting legal decomposition: $G_5 + G_4 + 2G_3 + G_1$.

Key Idea

- Set $Z_n(\omega)$ equal to the size of the second to last block for the legal decomposition $\omega \in \Omega_n$. Then if $Z_n(\omega) = t$, we have
$$K_n(\omega) = K_{n-\ell(t)}(h_t(\omega)) + t.$$

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- Set $Z_n(\omega)$ equal to the size of the second to last block for the legal decomposition $\omega \in \Omega_n$. Then if $Z_n(\omega) = t$, we have
$$K_n(\omega) = K_{n-\ell(t)}(h_t(\omega)) + t.$$

- In other words, we get the conditional expectations:

$$\mathbf{E}[K_n|Z_n = t] = \mathbf{E}[K_{n-l(t)}] + t$$

$$\begin{aligned} \mathbf{E}[K_n^2|Z_n = t] &= \mathbf{E}[(K_{n-l(t)} + t)^2] \\ &= \mathbf{E}[K_{n-l(t)}^2] + 2t\,\mathbf{E}[K_{n-l(t)}] + t^2. \end{aligned}$$

Proof of Main Theorem

- We first explicitly choose C_{min}. Then, we use strong induction on n to prove $\textbf{Var}[K_n] > C_{min} \cdot n$.

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- From induction hypothesis, we can bound $\mathbf{E}[K^2_{n-\ell(t)}]$ by estimating $\mathbf{Var}[K_{n-\ell(t)}] + (\mathbf{E}[K_{n-\ell(t)}])^2$.

Proof of Main Theorem

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- From previous results, we know $\mathbf{E}[K_n] = An + B + f(n)$ for some $f(n) = o(\gamma_1^n)$.
- From induction hypothesis, we can bound $\mathbf{E}[K_{n-\ell(t)}^2]$ by estimating $\mathbf{Var}[K_{n-\ell(t)}] + (\mathbf{E}[K_{n-\ell(t)}])^2$.
- In inductive step, we will be able to bound $\mathbf{Var}[K_n]$ by estimating $\mathbf{E}[K_n^2] - (\mathbf{E}[K_n])^2$.

Further Works

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- Can we generalize to more general sequences? Even without a recurrence relation?
- Can we define legal decompositions and blocks for other sequences?

Gaussian Behavior of Gaps

Technical Note

For the rest of the talk, assume PLRS refers to recurrences

$$G_n = c_1 G_{n-1} + \cdots + c_L G_{n-L}$$

with $c_i > 0$ for $1 \leq i \leq L$.

(Previoiusly, we only needed $c_1, c_L > 0$ and $c_i \geq 0$ for $1 < i < L$.)

Gaps in Decompositions

Definition

Let m be a positive integer with decomposition

$$m = \sum_{i=1}^{N} a_i G_{N+1-i} = G_{i_1} + G_{i_2} + \cdots + G_{i_k}$$

and $i_1 \geq i_2 \geq \cdots \geq i_k$. Then the *gaps* in the decomposition of m are the numbers $i_1 - i_2, i_2 - i_3, \ldots, i_{k-1} - i_k$.

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and $i_1 \geq i_2 \geq \cdots \geq i_k$. Then the *gaps* in the decomposition of m are the numbers $i_1 - i_2, i_2 - i_3, \ldots, i_{k-1} - i_k$.

Example

The gaps for the decomposition of $m = 101$ are $5, 2, 2$.

$$101 = F_{10} + F_5 + F_3 + F_1.$$

Gaps in Decompositions

Recall:

- Let $k(m)$ be the number of summands in the decomposition of m.
- Let K_n be $k(m)$ for an m chosen uniformly in $[G_n, G_{n+1})$.

Gaps in Decompositions

- Let $k(m)$ be the number of summands in the decomposition of m.
- Let K_n be $k(m)$ for an m chosen uniformly in $[G_n, G_{n+1})$.
- Let $k_g(m)$ be the number of size-g gaps in the decomposition of m.
- Let $K_{g,n}$ be $k_g(m)$ for an m chosen uniformly in $[G_n, G_{n+1})$.

Gaps in Decompositions

- $k(m)$: number of summands in m
- $k_g(m)$: number of size-g gaps in m

Example

$$101 = F_{10} + F_5 + F_3 + F_1$$

$k(101) = 4$, $k_2(101) = 2$, $k_3(101) = k_4(101) = 0$, $k_5(101) = 1$

Gaps in Decompositions

- $k(m)$: number of summands in m
- $k_g(m)$: number of size-g gaps in m

Example

$$101 = F_{10} + F_5 + F_3 + F_1$$

$k(101) = 4$, $k_2(101) = 2$, $k_3(101) = k_4(101) = 0$,
$k_5(101) = 1$

Fact

$$k(m) = 1 + \sum_{g=0}^{\infty} k_g(m)$$

Results

Understand: Distribution of number of summands: K_n

Results

Understand: Distribution of number of summands: K_n

Don't Understand: Distribution of number of size-g gaps: $K_{g,n}$

Results: Mean and Variance

K_n: number of summands in random $m \in [G_n, G_{n+1})$

Theorem (Lekkerkerker, 1951)

When $\{G_n\}$ is Fibonacci, $\mathbf{E}[K_n] = C_{Lek} n + O(1)$ for $C_{Lek} = \frac{1}{\varphi^2+1}$. ($C_{lek} \approx 0.276$)

Theorem (Miller and Wang, 2012)

When $\{G_n\}$ is a PLRS, there exists $A > 0$, B, and $\gamma_1 \in (0,1)$ such that $\mathbf{E}[K_n] = An + B + O(\gamma_1^n)$.

Theorem (Miller and Wang, 2012)

When $\{G_n\}$ is a PLRS, there exists $C > 0$, D, and $\gamma_2 \in (0,1)$ such that $\mathbf{Var}[K_n] = Cn + D + O(\gamma_2^n)$.

Results: Mean and Variance

$K_{g,n}$: number of size-g gaps in random $m \in [G_n, G_{n+1})$

Theorem (Main Result 1: Lekkerkerker for gaps)

When $\{G_n\}$ is a PLRS, for every integer $g \geq 0$, there exists A_g, B_g and $\gamma_{g,1} \in (0, 1)$ such that $\mathbf{E}[K_{g,n}] = A_g n + B_g + O(\gamma_{g,1}^n)$. By Bower et al. 2013, A_g is known for all PLRS $\{G_n\}$ and g.

Theorem (Main Result 2: Variance is linear for gaps)

When $\{G_n\}$ is a PLRS, for every integer $g \geq 0$, there exists C_g, D_g and $\gamma_{g,2} \in (0, 1)$ such that $\mathbf{Var}[K_{g,n}] = C_g n + D_g + O(\gamma_{g,2}^n)$.

Results: Asymptotic Gaussianity

K_n: number of summands in random $m \in [G_n, G_{n+1})$
$K_{g,n}$: number of size-g gaps in random $m \in [G_n, G_{n+1})$

Theorem (Kopp, Koloğlu, Miller, and Wang, 2011)

When $\{G_n\}$ is Fibonacci, as $n \to \infty$, K_n approaches Gaussian.

Theorem (Miller and Wang, 2012)

When $\{G_n\}$ is PLRS, as $n \to \infty$, K_n approaches Gaussian.

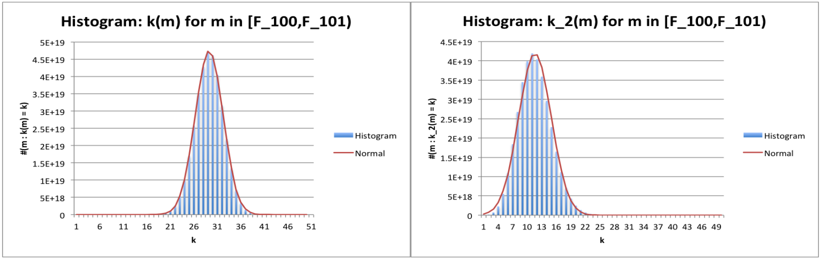
Theorem (Main Result 3: Gaussian Behavior for Gaps)

When $\{G_n\}$ is PLRS, as $n \to \infty$, $K_{g,n}$ approaches Gaussian.

Results: Summary

Theorem (Main Results, Summary)

The mean $\mu_{g,n}$ and variance $\sigma^2_{g,n}$ of $K_{g,n}$ grow linearly in n, and $(K_{g,n} - \mu_{g,n})/\sigma_{g,n}$ converges to the standard normal $N(0, 1)$ as $n \to \infty$.



Proof Sketch

Theorem (Main Result 3: Fibonacci Numbers, $g = 2$)

When $\{G_n\}$ is Fibonacci, as $n \to \infty$, $(K_{2,n} - \mu_{2,n})/\sigma_{2,n}$ converges to standard normal.

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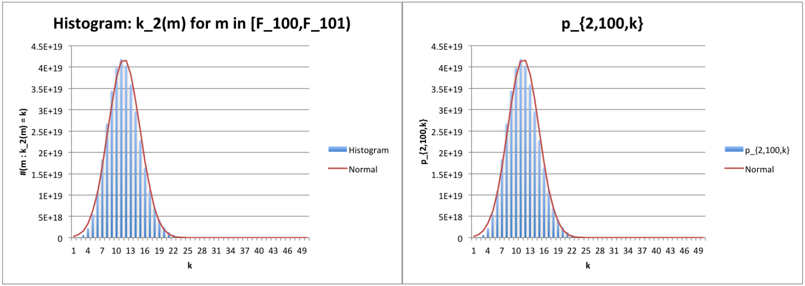
When $\{G_n\}$ is Fibonacci, as $n \to \infty$, $(K_{2,n} - \mu_{2,n})/\sigma_{2,n}$ converges to standard normal.

Let $p_{2,n,k} = \#\{m \in [F_n, F_{n+1})$ with k size-2 gaps$\}|$.
$\mathbf{Pr}[K_{2,n} = k] \propto p_{2,n,k}$.



Proof Sketch

$$p_{2,n,k} = \#\{m \in [F_n, F_{n+1}) \text{ with } k \text{ size-2 gaps}\}|.$$

n\k	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	1	0	0	0	0	0	0
3	1	1	0	0	0	0	0
4	2	1	0	0	0	0	0
5	3	1	1	0	0	0	0
6	4	3	1	0	0	0	0
7	6	5	1	1	0	0	0
8	9	7	4	1	0	0	0

Proof Sketch

$$p_{2,n,k} = \#\{m \in [F_n, F_{n+1}) \text{ with } k \text{ size-2 gaps}\}|.$$

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2	1	0	0	0	0	0	0
3	1	1	0	0	0	0	0
4	2	1	0	0	0	0	0
5	3	1	1	0	0	0	0
6	4	3	1	0	0	0	0
7	6	5	1	1	0	0	0
8	9	7	4	1	0	0	0

Lemma

$$p_{2,n,k} = p_{2,n-1,k} + p_{2,n-2,k-1} + p_{2,n-3,k-1} - p_{2,n-3,k}$$

Proof Sketch

Lemma (Key Lemma)

If $a_{n,k}$ is a "nice" two dimensional homogenous recurrence

$$a_{n,k} = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} a_{n-i,k-j}$$

for constants $t_{i,j}$, then the random variable X_n given by $\mathbf{Pr}[X_n = k] \propto a_{n,k}$ approaches Gaussian as $n \to \infty$.

Proof Sketch

Lemma (Key Lemma)

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Famous example: $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$ gives...

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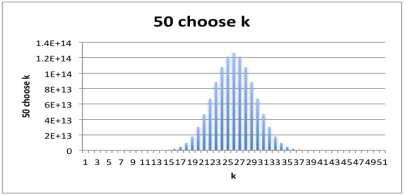
Famous example: $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$ gives...
the binomials: $a_{n,k} = \binom{n}{k}$!



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Applications

- $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$
  $X_n = $ # heads after n coin flips ($\mathbf{Pr}[X_n = k] = \frac{\binom{n}{k}}{2^n}$)
- $a_{n,k} = a_{n-1,k} + a_{n-2,k-1} + a_{n-3,k-1} - a_{n-3,k}$
  $X_n = $ # size-2 gaps of random $m \in [F_n, F_{n+1})$
  $(a_{n,k} = p_{2,n,k}, X_n = K_{2,n})$
- $a_{n,k} = a_{n-1,k} + a_{n-2,k-1}$
  $X_n = $ # summands of random $m \in [F_n, F_{n+1})$
  $(X_n = K_n)$

Proof Sketch

Lemma (Key Lemma)

If $a_{n,k}$ is a "nice" two dimensional homogenous recurrence then the random variable X_n given by $\mathbf{Pr}[X_n = k] \propto a_{n,k}$ approaches Gaussian as $n \to \infty$.

Let $\tilde{\mu}_n(m) = \mathbf{E}[(X_n - \mu_n)^m]$.

Lemma (Method of Moments)

Suffices to prove

$$\lim_{n\to\infty} \frac{\tilde{\mu}_n(2m)}{\tilde{\mu}_n(2)^m} = (2m-1)!! \qquad \lim_{n\to\infty} \frac{\tilde{\mu}_n(2m+1)}{\tilde{\mu}_n(2)^{m+\frac{1}{2}}} = 0.$$

Proof Sketch

Let $\tilde{\mu}_n(m) = \mathbf{E}[(X_n - \mu_n)^m]$. We have

$$\tilde{\mu}_n(m) = \sum_{\ell=0}^{m} \binom{m}{\ell} \sum_{t_{i,j} \neq 0} \frac{F_{n-i} t_{i,j}}{F_n} \cdot (j + \mu_{n-i} - \mu_n)^{\ell} \cdot \tilde{\mu}_{n-i}(m-\ell).$$

Proof Sketch

Let $\tilde{\mu}_n(m) = \mathbf{E}[(X_n - \mu_n)^m]$. We have

$$\tilde{\mu}_n(m) = \sum_{\ell=0}^{m} \binom{m}{\ell} \sum_{t_{i,j} \neq 0} \frac{F_{n-i} t_{i,j}}{F_n} \cdot (j + \mu_{n-i} - \mu_n)^{\ell} \cdot \tilde{\mu}_{n-i}(m-\ell).$$

Lemma

For each integer $m \geq 0$, there exist degree m polynomials Q_{2m}, Q_{2m+1} and $\gamma_{2m}, \gamma_{2m+1} \in (0,1)$ such that

$$\tilde{\mu}_n(2m) = Q_{2m}(n) + O(\gamma_{2m}^n)$$
$$\tilde{\mu}_n(2m+1) = Q_{2m+1}(n) + O(\gamma_{2m+1}^n).$$

Furthermore, if C_{2m} is the leading coefficient of Q_{2m}, then for all m, $C_{2m} = (2m-1)!! \cdot C_2^m$.

Future Work

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- Are there other meaningful two-dimensional recurrences to which we can apply our asymptotic Gaussianity result?
- Can you lift the constraint that every coefficient c_i must be positive?
- What is the rate at which our $K_{g,n}$ converges to a normal distribution?