Low-lying zeros of cuspidal Maass forms

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Introduction

L-functions

L-functions generalizes the Riemann zeta-function:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Explicit Formula: Relates sums over zeros to sums over primes.

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f)L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Measures of Spacings: n-Level Density

n-level density for one function

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

- Test function $\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i is even Schwartz function.
- Fourier Transforms $\widehat{\phi}$ has compact support: $(-\sigma, \sigma)$.
- Zeros scaled by L_f.
- Most of contribution is from low zeros.

Katz-Sarnak Conjecture

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Need to average *n*-level density over a family and take the limit of this parameter; as $|N| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) \quad \to \quad \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

Cuspidal Maass Forms

Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a function $f : \mathcal{H} \to \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. f vanishes at the cusps of Γ, and
3. $\Delta f = \lambda f$ for some $\lambda = s(1-s) > 0$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on $\mathcal{H}$.

- Coefficients contain information about partitions.
- For full modular group, $s = 1/2 + it_j$ with $t_j \in \mathbb{R}$.
- Test Katz-Sarnak conjecture.

L-function associated to Maass forms

Write Fourier expansion of Maass form u_j as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi|n|y) e^{2\pi i n x}.$$

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Define L-function attached to u_j as

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1}$$

where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1.$

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where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1.$
Also,

$$\lambda_j(p) \ll p^{7/64}.$$

n-level over a family

- Recall for Katz-Sarnak Conjecture,

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(L_f\gamma_E^{(j_i)}\right)$$
$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

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$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

- For Dirichlet/cuspidal newform L-functions, there are many with a given conductor.
- Problem: For Maass forms, expect at most one with a given conductor.

n-level over a family, continued

- Solution: Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$.

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$$h_{2,T}(t_j) = \exp\left(-(t_j - T)^2/L^2\right) + \exp\left(-(t_j + T)^2/L^2\right),$$

which picks out eigenvalues centered at $\pm T$.

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- Weighted 1-level density becomes

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D_{n,u_j}(\phi)$$

$$= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\gamma}{2\pi} \log R \right)$$

Results

1-Level Density

1-level density for one function

$$D(u_j;\phi) = \sum_{\gamma} \phi\left(\frac{\gamma}{2\pi}\log R\right)$$

1-Level Density

1-level density for one function

$$D(u_j;\phi)$$

$$= \text{Terms involving } \Gamma + \frac{2}{\log R}\sum_p \frac{\log p}{p}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$

$$-\sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$

$$+O\left(\frac{1}{\log R}\right)$$

1. Explicit formula.

1-Level Density

1-level density for one function

$$D(u_j;\phi)$$
$$= \hat{\phi}(0)\frac{\log(1+t_j^2)}{\log R} + \frac{2}{\log R}\sum_p \frac{\log p}{p}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$
$$- \sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$
$$+ O\left(\frac{1}{\log R}\right)$$

1. Explicit formula.
2. Gamma function identities

1-Level Density

1-level density for one function

$$D(u_j;\phi)$$
$$= \hat{\phi}(0)\frac{\log(1+t_j^2)}{\log R} + \frac{\phi(0)}{2} + O\left(\frac{\log\log R}{\log R}\right)$$
$$- \sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$

1. Explicit formula.
2. Gamma function identities
3. Prime Number Theorem

Average 1-level density

The weighted 1-level density becomes:

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} D(u_j;\phi)$$

$$= \frac{\phi(0)}{2} + O\left(\frac{\log\log R}{\log R}\right) + \frac{1}{\sum_j \frac{h_t(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} \widehat{\phi}(0) \frac{\log(1+t_j^2)}{\log R}$$

$$- \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p^{\frac{1}{2}}\log R} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p)$$

$$- \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p\log R} \widehat{\phi}\left(\frac{2\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p^2)$$

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$$-\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p^{\frac{1}{2}}\log R} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p)$$

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Kuznetsov Trace Formula

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$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)}$$
$$= \text{ some function that depends just on h, m, and n}$$

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)} + \frac{1}{4\pi}\int_{\mathbb{R}} \overline{\tau(m,r)}\tau(n,r)\frac{h(r)}{\cosh(\pi r)}dr =$$

$$\frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} r\tanh(r)h(r)dr + \frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr$$

where

$\tau(m,r) = \pi^{\frac{1}{2}+ir}\Gamma(1/2+ir)^{-1}\zeta(1+2ir)^{-1}n^{-\frac{1}{2}}\sum_{ab=|m|}\left(\frac{a}{b}\right)^{ir}$.

$$S(n,m;c) = \sum_{0\leq x\leq c-1, gcd(x,c)=1} e^{2\pi i(nx+mx^*)/c}$$

$$J_{ir}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+ir+1)}\left(\frac{1}{2}x\right)^{2m+ir}.$$

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- The only $\lambda(m)\overline{\lambda(n)}$ term that contributes is when $m = n = 1$.
- The $m = 1, n = p$ and $m = 1, n = p^2$ terms do not contribute because of the $\delta_{m,n}$ function.

Result: 1-level density

Theorem (AILMZ, 2011)

If $h_T = h_{1,T}$ or $h_{2,T}$, $T \to \infty$ and $L \ll T / \log T$, and $\sigma < 1/6$ then 1-level density is

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D(u_j;\phi) = \frac{\phi(0)}{2} + \widehat{\phi}(0) + O\left(\frac{\log\log R}{\log R}\right) + O(T^{3\sigma/2-1/4+\epsilon} + T^{\sigma/2-1/4+\epsilon}).$$

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- This matches with the orthogonal family density as predicted by Katz-Sarnak.

Support

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Can distinguish unitary and symplectic from the 3 orthogonal groups, but 1-level density cannot distinguish the orthogonal groups from each other if support in $(-1, 1)$.

2-level density can distinguish orthogonal groups with arbitrarily small support; additional term depending on distribution of signs of functional equations.

2- Level Density

To differentiate between even and odd in orthogonal family, we calculated the 2-level density:

$$D_2^*(\phi) := \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{j_1, j_2} \phi_1(\gamma^{(j_1)}) \overline{\phi_2(\gamma^{(j_2)})}$$

$$= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \prod_{i=1}^{2} \left| \frac{\phi_i(0)}{2} + \widehat{\phi}_i(0) \frac{\log(1+t_j^2)}{\log R} + O\left(\frac{\log\log R}{\log R}\right) \right.$$

$$\left. - \sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R} \widehat{\phi}_i\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R} \widehat{\phi}_i\left(\frac{2\log p}{\log R}\right) \right| .$$

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25 terms, handled by Cauchy-Schwarz or Kuznetsov.

Result: 2-level density

Theorem (AILMZ, 2011)

Same conditions as before, for $\sigma < 1/12$ have

$$\begin{aligned} D^*_{2,\mathcal{F}} &= \prod_{i=1}^{2}\left[\frac{\phi_i(0)}{2}+\widehat{\phi}_i(0)\right]+2\int_{-\infty}^{\infty}|z|\widehat{\phi}_1(z)\widehat{\phi}_2(z)dz \\ &\quad -\phi_1(0)\phi_1(0)-2\widehat{\phi_1\phi_2}(0)+(\phi_1\phi_2)(0)\mathcal{N}(-1) \\ &\quad +O\left(\frac{\log\log R}{\log R}\right). \end{aligned}$$

Note that $\mathcal{N}(-1)$ is the weighted percent that have odd sign in functional equation.

Conclusion

Recap

- We calculated 1-level for $\sigma < 1/6$.
- Calculated 2-level densities for $\sigma < 1/12$ in order to distinguish the orthogonal families.
- We showed agreement with Katz-Sarnak conjecture.

Thank you!