# Chains of distributions and Benford's Law

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## Problem Outline

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- If $f_{n,k}(x_n)$ is the probability density for $X_n$, then

$$f_{n,k}(x_n) = \begin{cases} \frac{\log^{n-1}(k/x_n)}{k\Gamma(n)} & \text{if } x_n \in [0, k] \\ 0 & \text{otherwise.} \end{cases}$$

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**Theorem (JKKKM)**

*As $n \to \infty$ the distribution of digits of $X_n$ rapidly tends to Benford's Law.*

## Uniform Density Example: $n = 10$ with 10,000 trials

$\chi^2 = 7.35$ (critical threshold at 5% with 8 d.f. is 15.5)

| Digit | Observed Probability | Expected Probability |
|---|---|---|
| 1 | 0.298 | 0.301 |
| 2 | 0.180 | 0.176 |
| 3 | 0.127 | 0.125 |
| 4 | 0.097 | 0.097 |
| 5 | 0.080 | 0.079 |
| 6 | 0.071 | 0.067 |
| 7 | 0.056 | 0.058 |
| 8 | 0.048 | 0.051 |
| 9 | 0.044 | 0.046 |

## Sketch of the proof

- First prove the claim for density $f_{n,k}$ by induction.
- Use Mellin Transforms and Poisson Summation to analyze probability.

## Proof by Induction: Base Case: Calculating CDF

$$F_{2,k}(x_2) \;=\; \int_0^k \mathrm{Prob}\left(X_{2,k} \in [0, x_2] | X_{1,k} = x_1\right) \mathrm{Prob}(X_{1,k} = x_1) dx_1$$

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Differentiating yields $f_{2,k}(x_2) = \frac{\log(k/x_2)}{k}$.

## Further Comments

- Other distributions: exponential, one-sided normal.
- Densities of the form $f(x;\theta) = \theta^{-1} g(x/\theta)$.
- Weibull distribution: $f(x;\gamma) = \gamma x^{\gamma-1} \exp(-x^{\gamma})$.
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- Further areas of research - Two parameter distribution, closed form for other single variable distributions

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Conditions

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$$\lim_{n\to\infty} \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} \prod_{m=1}^{n} (\mathcal{M} f_{\mathcal{D}_{p(m)}(1)}) \left(1 - \frac{2\pi i \ell}{\log B}\right) \;=\; 0$$

**Theorem (JKKKM)**

- *If conditions hold, as $n \to \infty$ the distribution of leading digits of $X_n$ tends to Benford's law.*
- *The error is a nice function of the Mellin transforms: if $Y_n = \log_B X_n$, then*

$$\left|\mathrm{Prob}(Y_n \bmod 1 \in [a,b]) - (b+a)\right| \le \left|(b-a) \cdot \sum_{\substack{\ell=-\infty \\ \ell \neq 0}}^{\infty} \prod_{m=1}^{n} \left(\mathcal{M} f_{\mathcal{D}_{p(m)}(1)}\right) \left(1 - \frac{2\pi i \ell}{\log B}\right)\right|.$$

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Follow Mellin transform proof of CLT mod 1 for product independent random variables.