Mind the Gap: Distribution of Gaps in Generalized Zeckendorf Decompositions

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Introduction
Goals of the Talk

- Combinatorial perspective.
- Asking for help: completing elementary proof.
- New results on longest gap.
- Techniques: Generating fns, partial fractions, Rouche.

Joint with Olivia Beckwith, Iddo Ben-Ari, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li, Philip Tosteson.
**Previous Results**

**Fibonacci Numbers**: \( F_{n+1} = F_n + F_{n-1} \);
\( F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots \).

**Zeckendorf’s Theorem**

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

**Example**: \( 2013 = 1597 + 377 + 34 + 5 = F_{16} + F_{13} + F_8 + F_4 \).

**Lekkerkerkerker’s Theorem (1952)**

The average number of summands in the Zeckendorf decomposition for integers in \([F_n, F_{n+1})\) tends to \( \frac{n}{\varphi^2 + 1} \approx .276n \), where \( \varphi = \frac{1 + \sqrt{5}}{2} \) is the golden mean.
Old Results

Central Limit Type Theorem

As \( n \to \infty \), the distribution of the number of summands in the Zeckendorf decomposition for integers in \( [F_n, F_{n+1}) \) is Gaussian.

\[ \text{Figure: Number of summands in } [F_{2010}, F_{2011}); F_{2010} \approx 10^{420}. \]
New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m = \sum_{j=1}^{k(m)=n} F_i, \quad \nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta \left( x - (i_j - i_{j-1}) \right).$$

**Theorem (Zeckendorf Gap Distribution)**

Gap measures $\nu_{m;n}$ converge to average gap measure where $P(k) = \frac{1}{\phi^k}$ for $k \geq 2$.

![Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.](image)
New Results: Longest Gap

Fair coin: largest gap tightly concentrated around $\log n / \log 2$.

**Theorem (Longest Gap)**

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$
\text{Prob}(L_n(m) \leq f(n)) \approx e^{-e^{\log n - f(n) \cdot \log \phi}}
$$

- $\mu_n = \frac{\log \left( \frac{\phi^2}{\phi^2 + 1} n \right)}{\log \phi} + \frac{\gamma}{\log \phi} - \frac{1}{2} + \text{Small Error}$.

- If $f(n)$ grows **slower** (resp. **faster**) than $\log n / \log \phi$, then $\text{Prob}(L_n(m) \leq f(n))$ goes to 0 (resp. 1).
Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is
\[
\binom{C+P-1}{P-1}.
\]

Let $p_{n,k} = \# \left\{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \right\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is $F_n$.

\[
N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,
\]

where $1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n$, $i_j - i_{j-1} \geq 2$.

\[
d_1 := i_1 - 1, \quad d_j := i_j - i_{j-1} - 2 \quad (j > 1).
\]

\[
d_1 + d_2 + \cdots + d_k = n - 2k + 1, \quad d_j \geq 0.
\]

Cookie counting $\Rightarrow p_{n,k} = \binom{n - 2k + 1 + k - 1}{k-1} = \binom{n-k}{k-1}$. 
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Gaps in the Bulk
For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

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Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $k$. 
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Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $k$.

What is $P(k) = \lim_{n \to \infty} P_n(k)$?
Distribution of Gaps

For \( F_{i_1} + F_{i_2} + \cdots + F_{i_n} \), the gaps are the differences \( i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1 \).

Example: For \( F_1 + F_8 + F_{18} \), the gaps are 7 and 10.

Let \( P_n(k) \) be the probability that a gap for a decomposition in \([F_n, F_{n+1}]\) is of length \( k \).

What is \( P(k) = \lim_{n \to \infty} P_n(k) \)?

Can ask similar questions about binary or other expansions: \( 2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2 \).
Main Result

Theorem (Distribution of Bulk Gaps (SMALL 2012))

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ be a positive linear recurrence of length $L$ where $c_i \geq 1$ for all $1 \leq i \leq L$. Then

$$P(j) = \begin{cases} 
1 - \left( \frac{a_1}{C_{Lek}} \right) \left( 2\lambda_1^{-1} + a_1^{-1} - 3 \right) & : j = 0 \\
\lambda_1^{-1} \left( \frac{1}{C_{Lek}} \right) \left( \lambda_1 \left( 1 - 2a_1 \right) + a_1 \right) & : j = 1 \\
\left( \lambda_1 - 1 \right)^2 \left( \frac{a_1}{C_{Lek}} \right) \lambda_1^{-j} & : j \geq 2.
\end{cases}$$
Theorem (Base $B$ Gap Distribution (SMALL 2011))

For base $B$ decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(k) = 1/\phi^k$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.
Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1} \frac{n}{\phi^2 + 1}$. 
Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker ⇒ total number of gaps $\sim F_{n-1} \frac{n}{\phi^2 + 1}$.

Let $X_{i,j} = \# \{ m \in [F_n, F_{n+1}) : \text{decomposition of } m \text{ includes } F_i, F_j, \text{ but not } F_q \text{ for } i < q < j \}$. 
Proof of Bulk Gaps for Fibonacci Sequence

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Let $X_{i,j} = \#\{m \in [F_n, F_{n+1}) \; : \; \text{decomposition of } m \text{ includes } F_i, F_j, \text{ but not } F_q \text{ for } i < q < j\}$.

$$P(k) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1} \frac{n}{\phi^2 + 1}}.$$
Calculating $X_{i,j+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?
**Calculating** $X_{i,i+k}$

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For the indices less than $i$: $F_{i-1}$ choices. Why? Have $F_i$ as largest summand and follows by Zeckendorf:

$$
\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}.
$$
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$$

For the indices greater than $i+k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \ldots, F_{n-k-i+1}\}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand $F_{n-k-i}$. 

\[\includegraphics[width=\textwidth]{diagram.png}\]
Calculating $X_{i, i+k}$

How many decompositions contain a gap from $F_i$ to $F_{i+k}$?

For the indices less than $i$: $F_{i-1}$ choices. Why? Have $F_i$ as largest summand and follows by Zeckendorf:

$$\# [F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}. \)$$

For the indices greater than $i + k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{ F_1, \ldots, F_{n-k-i+1} \}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand $F_{n-k-i}$. 

So total choices number of choices is $F_{n-k-2-i}F_{i-1}$. 
Determining $P(k)$

Recall

$$P(k) = \lim_{n \to \infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1} \frac{n}{\phi^2 + 1}}.$$ 

Use Binet’s formula. Sums of geometric series: $P(k) = 1 / \phi^k$. 

**Figure:** Distribution of summands in $[F_{1000}, F_{1001})$. 
Individual Gaps
Main Result

- Decomposition: \( m = \sum_{j=1}^{k(m)} F_{ij} \).
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- **Decomposition:** \( m = \sum_{j=1}^{k(m)} F_{ij} \).

- **Individual gap measure:**
  \[
  \nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta \left( x - (i_j - i_{j-1}) \right).
  \]
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  \]

**Theorem (Distribution of Individual Gaps (SMALL 2012))**

Gap measures \( \nu_{m;n} \) converge to average gap measure.
Proof Sketch of Individual Gap Measures

\[ \mu_{m,n}(t) = \int x^t d\nu_{m,n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t. \]
Proof Sketch of Individual Gap Measures

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- Show \( E_m[\mu_{m,n}(t)] \) equals average gap moments, \( \mu(t) \).
Proof Sketch of Individual Gap Measures

- \( \mu_{m,n}(t) = \int x^t d\nu_{m,n}(x) = \frac{1}{\binom{k(m)}{2}} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t. \)

- Show \( \mathbb{E}_m[\mu_{m,n}(t)] \) equals average gap moments, \( \mu(t) \).

- Show \( \mathbb{E}_m[(\mu_{m,n}(t) - \mu(t))^2] \) and \( \mathbb{E}_m[(\mu_{m,n}(t) - \mu(t))^4] \) tend to zero.
Proof Sketch of Individual Gap Measures

- $\mu_{m,n}(t) = \int x^t d\nu_{m,n}(x) = \frac{1}{k(m) - 1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t$.

- Show $\mathbb{E}_m[\mu_{m,n}(t)]$ equals average gap moments, $\mu(t)$.

- Show $\mathbb{E}_m[(\mu_{m,n}(t) - \mu(t))^2]$ and $\mathbb{E}_m[(\mu_{m,n}(t) - \mu(t))^4]$ tend to zero.

Key ideas:
(1) Replace $k(m)$ with average (Gaussianity);
(2) use $X_{i,i+g_1,j,j+g_2}$. 
Future Research

- Finish elementary proof of convergence of individual gap measures (maybe probabilities instead of moments). Email sjm1@williams.edu if interested.

- Extend to recurrences with coefficients that can be zero: SMALL ’13.

- Generalize to signed decompositions, $\ell$ largest gaps, .... SMALL ’13.
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Fibonacci Case Generating Function

$G_{n,k,f}$ be the number of $m \in [F_n, F_{n+1})$ with $k$ nonzero summands and all gaps less than $f(n)$. 
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$G_{n,k,f}$ is the coefficient of $x^n$ for the generating function

$$\frac{1}{1 - x} \left[ \sum_{j=2}^{f(n)-2} x^j \right]^{k-1}.$$
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$$\frac{1}{1 - x} \left[ \sum_{j=2}^{f(n)-2} x^j \right]^{k-1}.$$

Let $m = F_n + F_{n-g_1} + F_{n-g_1-g_2} + \cdots + F_{n-g_1-\cdots-g_{n-1}}$, then

- Each gap is $\geq 2$.
- Each gap is $< f(n)$.
- The sum of the gaps of $x$ is $\leq n$.

Gaps uniquely identify $m$ by Zeckendorf’s Theorem.
The Combinatorics

\( G_{n,k,f} \) is the \( n^{th} \) coefficient of

\[
\frac{1}{1-x} \left[ x^2 + \cdots + x^{f(n)-2} \right]^{k-1} = \frac{x^{2(k-1)}}{1-x} \left( \frac{1 - x^{f(n)-3}}{1-x} \right)^{k-1}.
\]
The Combinatorics

$G_{n,k,f}$ is the $n^{th}$ coefficient of

$$\frac{1}{1-x} \left[ x^2 + \cdots + x^{f(n)-2} \right]^{k-1} = \frac{x^{2(k-1)}}{1-x} \left( \frac{1 - x^{f(n)-3}}{1-x} \right)^{k-1}.$$  

For fixed $k$ hard to analyze, but only care about sum over $k$. 
The Generating Function

Sum over $k$ gives number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$, call it $G_{n,f}$.

It’s the $n^{th}$ coefficient (up to potentially small algebra errors!) of

$$F(x) = \frac{1}{1-x} \sum_{k=1}^{\infty} \left( \frac{x^2 - x^{f-2}}{1-x} \right)^{k-1} = \frac{x}{1 - x - x^2 + x^{f(n)}}.$$

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Use partial fractions and Rouché’s Theorem to find CDF.
Partial Fractions

Write the roots of $x^f - x^2 - x - 1$ as $\{\alpha_i\}_{i=1}^{f}$, generating function is

$$F(x) = \frac{x}{1 - x - x^2 + x^f} = \frac{f(n)}{f(n)\alpha_i^f(n)} - \frac{-\alpha_i}{\alpha_i^f(n) - 2\alpha_i^2 - \alpha_i} \sum_{j=1}^{\infty} \left( \frac{x}{\alpha_i} \right)^j.$$
Partial Fractions

Write the roots of $x^f - x^2 - x - 1$ as $\{\alpha_i\}_{i=1}^f$, generating function is

$$F(x) = \frac{x}{1 - x - x^2 + x^{f(n)}} = \sum_{i=1}^{f(n)} \frac{-\alpha_i}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \sum_{j=1}^{\infty} \left(\frac{x}{\alpha_i}\right)^j.$$

Take the $n^{th}$ coefficient to find the number of $m$ with gaps less than $f(n)$. 
Partial Fractions

Divide the number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$ by the number of $m$, which is

$$F_{n+1} - F_n = F_{n-1} = 5^{-1/2} \left( \phi^{n-1} - (1/\phi)^{n-1} \right).$$

**Theorem**

*The proportion of $m \in [F_n, F_{n+1})$ with $L(x) < f(n)$ is exactly*

$$\sum_{i=1}^{f(n)} \frac{-\sqrt{5}(\alpha_i)}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \left( \frac{1}{\alpha_i} \right)^{n+1} \frac{1}{(\phi^n - (-1/\phi)^n)}$$

Now study the roots of $x^f - x^2 - x + 1$. 
Rouché and Roots

When \( f(n) \) is large, \( z^{f(n)} \) is very small for \( |z| < 1 \). Thus, by Rouché’s theorem:

**Lemma**

For \( f \in \mathbb{N} \) and \( f \geq 4 \), the polynomial \( p_f(z) = z^f - z^2 - z + 1 \) has exactly one root \( z_f \) with \( |z_f| < .9 \). Further, \( z_f \in \mathbb{R} \) and

\[
z_f = \frac{1}{\phi} + \left| \frac{z_f}{z_f + \phi} \right|,
\]

so as \( f \to \infty \), \( z_f \) converges to \( \frac{1}{\phi} \).

We only care about the **smallest root**.
Getting the CDF

As $f$ grows, only one root goes to $1/\phi$. The other roots don’t matter. So,
Getting the CDF

As $f$ grows, only one root goes to $1/\phi$. The other roots don’t matter. So,

**Theorem**

If $\lim_{n \to \infty} f(n) = \infty$, the proportion of $m$ with $L(m) < f(n)$ is, as $n \to \infty$

$$\lim_{n \to \infty} (\phi Z_f)^{-n} = \lim_{n \to \infty} \left( 1 + \left| \frac{\phi Z_f^{f(n)}}{\phi + Z_f} \right| \right)^{-n}.$$

If $f(n)$ is bounded, then $P_f = 0$.

Take logarithms, Taylor expand, result follows from algebra.

Algebra increases greatly for general recurrence.
References
References


Generalizations
Positive Linear Recurrence Sequences

This method can be greatly generalized to **Positive Linear Recurrence Sequences**: linear recurrences with non-negative coefficients:

\[ H_{n+1} = c_1 H_{n-j_1} + c_2 H_{n-j_2} + \cdots + c_L H_{n-j_L}. \]

**Theorem (Zeckendorf’s Theorem for PLRS recurrences)**

*Any* \( b \in \mathbb{N} \) *has a unique legal decomposition into sums of* \( H_n \),

\[ b = a_1 H_{i_1} + \cdots + a_k H_{i_k}. \]

Here *legal* reduces to non-adjacency of summands in the Fibonacci case.
Messier Combinatorics

The number of \( b \in [H_n, H_{n+1}) \), with longest gap \(< f \) is the coefficient of \( x^{n-s} \) in the generating function:
The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of $x^{n-s}$ in the generating function:

$$
\frac{1}{1-x} \left( c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L} \right) \times \sum_{k \geq 0} \left[ \left( (c_1 - 1) x^{t_1} + \cdots + (c_L - 1) x^{t_L} \right) \left( \frac{x^{s+1} - x^f}{1 - x} \right) + x^{t_1} \left( \frac{x^{s+t_2-t_1+1} - x^f}{1 - x} \right) + \cdots + x^{t_L-1} \left( \frac{x^{s+t_L-t_{L-1}+1} - x^f}{1 - x} \right) \right]^k.
$$
Messier Combinatorics

The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of $x^{n-s}$ in the generating function:

\[
\frac{1}{1-x} \left( c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L} \right) \times
\]

\[
\times \sum_{k \geq 0} \left[ ((c_1 - 1)x^{t_1} + \cdots + (c_L - 1)x^{t_L}) \left( \frac{x^{s+1} - x^f}{1-x} \right) + \right.
\]

\[
+ x^{t_1} \left( \frac{x^{s+t_2-t_1+1} - x^f}{1-x} \right) + \cdots + x^{t_{L-1}} \left( \frac{x^{s+t_L-t_{L-1}+1} + 1 - x^f}{1-x} \right) \right]^k.
\]

**A geometric series!**
Let $f > j_L$. The number of $x \in [H_n, H_{n+1})$, with longest gap $< f$ is given by the coefficient of $s^n$ in the generating function

$$F(s) = \frac{1 - s^j}{\mathcal{M}(s) + sf\mathcal{R}(s)},$$

where

$$\mathcal{M}(s) = 1 - c_1 s - c_2 s^{j_2 + 1} - \cdots - c_L s^{j_L + 1},$$

and

$$\mathcal{R}(s) = c_{j_1 + 1} s^{j_1} + c_{j_2 + 1} s^{j_2} + \cdots + (c_{j_L + 1} - 1) s^{j_L}.$$ 

and $c_i$ and $j_i$ are defined as above.
The **coefficients** in the **partial fraction** expansion might **blow up** from multiple roots.
The coefficients in the partial fraction expansion might blow up from multiple roots.

**Theorem (Mean and Variance for "Most Recurrences")**

For $x$ in the interval $[H_n, H_{n+1})$, the mean longest gap $\mu_n$ and the variance of the longest gap $\sigma_n^2$ are given by

$$
\mu_n = \frac{\log \left( \frac{\mathcal{R}(\frac{1}{\lambda_1})}{\mathcal{G}(\frac{1}{\lambda_1})} n \right)}{\log \lambda_1} + \frac{\gamma}{\log \lambda_1} - \frac{1}{2} + \text{Small Error} + \epsilon_1(n),
$$

and

$$
\sigma_n^2 = \frac{\pi^2}{6 \log \lambda_1} - \frac{1}{12} + \text{Small Error} + \epsilon_2(n),
$$

where $\epsilon_i(n)$ tends to zero in the limit, and Small Error comes from the Euler-Maclaurin Formula.