Mind the Gap: Distribution of Gaps in Generalized Zeckendorf Decompositions

Steven J. Miller (sjm1@williams.edu)

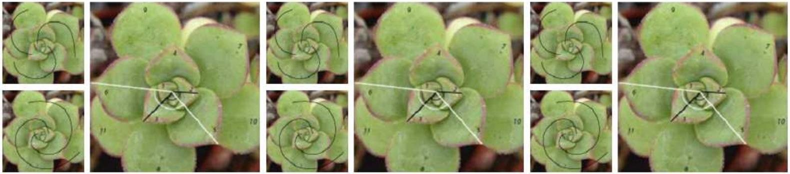
`http://www.williams.edu/Mathematics/sjmiller/public_html`

16th International Conference on Fibonacci Numbers and their Applications, Rochester, NY, July 25, 2014

Introduction

Goals of the Talk

- Generalize Zeckendorf decompositions
- Analyze gaps (in the bulk and longest)
- Power of generating functions
- Some open problems (if time permits)



Collaborators and Thanks

Collaborators: **Gaps (Bulk, Individual, Longest):** Joint with Olivia Beckwith, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li, Philip Tosteson; **Current work (Kentucky Sequence, Fibonacci Quilt):** Joint with Minerva Catral, Pari Ford, Pamela Harris & Dawn Nelson.

Supported by: NSF Grants DMS1265673, DMS0970067, DMS1347804 and DMS0850577, AIM and Williams College.

Collaborators and Thanks

Collaborators: **Gaps (Bulk, Individual, Longest):** Joint with Olivia Beckwith, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li, Philip Tosteson; **Current work (Kentucky Sequence, Fibonacci Quilt):** Joint with Minerva Catral, Pari Ford, Pamela Harris & Dawn Nelson.

Supported by: NSF Grants DMS1265673, DMS0970067, DMS1347804 and DMS0850577, AIM and Williams College.

Not supported by:

US AIRWAYS

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example:
$2014 = 1597 + 377 + 34 + 5 + 1 = F_{16} + F_{13} + F_8 + F_4 + F_1$.

Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Old Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of number of summands in Zeckendorf decomposition for $m \in [F_n, F_{n+1})$ is Gaussian.

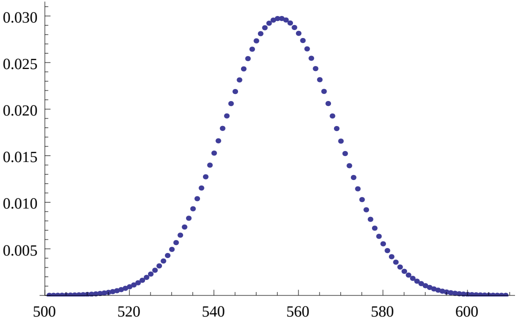


Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m \;=\; \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) \;=\; \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

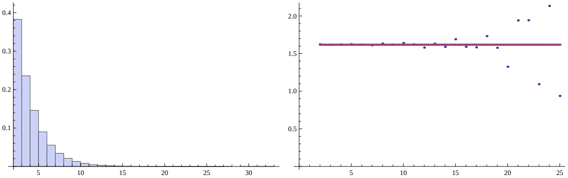


Figure: Distribution of gaps in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results: Longest Gap

Fair coin: largest gap tightly concentrated around $\log n/\log 2$.

Theorem (Longest Gap)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\text{Prob}\left(L_n(m) \le f(n)\right) \;\approx\; e^{-e^{\log n - f(n)\cdot \log\phi}}$$

- $\mu_n = \frac{\log\left(\frac{\phi^2}{\phi^2+1)}n\right)}{\log\phi} + \frac{\gamma}{\log\phi} - \frac12 +$ Small Error.

- If $f(n)$ grows ***slower*** (resp. ***faster***) than $\log n/\log\phi$, then $\text{Prob}(L_n(m) \le f(n))$ goes to **0** (resp. ***1***).

Gaps in the Bulk

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(g)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length g.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(g)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length g.

What is $P(g) = \lim_{n\to\infty} P_n(g)$?

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

Let $P_n(g)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length g.

What is $P(g) = \lim_{n\to\infty} P_n(g)$?

Can ask similar questions about binary or other expansions: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.

Main Result

Theorem (Distribution of Bulk Gaps (SMALL 2012))

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ *be a positive linear recurrence of length* L *where* $c_i \geq 1$ *for all* $1 \leq i \leq L$. *Then*

$$P(g) = \begin{cases} 1 - (\frac{a_1}{C_{Lek}})(2\lambda_1^{-1} + a_1^{-1} - 3) & : g = 0 \\ \lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1 - 2a_1) + a_1) & : g = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-g} & : g \geq 2. \end{cases}$$

Special Cases

Theorem (Base B Gap Distribution (SMALL 2011))

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $g \geq 1$, $P(g) = c_B B^{-g}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(g) = 1/\phi^g$ for $g \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.

Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

Let $X_{i,g} = \#\{m \in [F_n, F_{n+1})$: decomposition of m includes F_i, F_{i+g}, but not F_q for $i < q < i+g\}$.

Proof of Bulk Gaps for Fibonacci Sequence

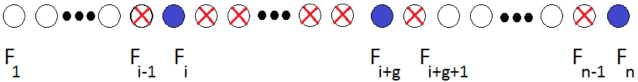
Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

Let $X_{i,g} = \#\{m \in [F_n, F_{n+1})$: decomposition of m includes F_i, F_{i+g}, but not F_q for $i < q < i+g\}$.

$$P(g) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-g} X_{i,g}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Calculating $X_{i,g}$

How many decompositions contain a gap from F_i to F_{i+g}?



For the indices less than i: F_{i-1} choices. Why? Have F_i as largest summand and follows by Zeckendorf: $\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.

For the indices greater than $i+g$: $F_{n-g-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \dots, F_{n-g-i+1}\}$ with $F_1, F_{n-g-i+1}$ chosen. Decompositions with largest summand $F_{n-g-i+1}$ minus decompositions with largest summand F_{n-g-i}.

So total choices number of choices is $F_{n-g-2-i}F_{i-1}$.

Determining $P(g)$

Recall

$$P(g) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-g} X_{i,g}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Use Binet's formula. Sums of geometric series: $P(g) = 1/\phi^g$.

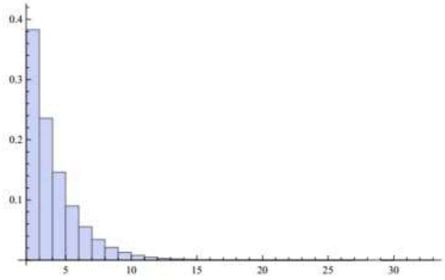


Figure: Distribution of summands in $[F_{1000}, F_{1001})$.

Individual Gaps

Main Result

- Decomposition:

$$m \;=\; \sum_{j=1}^{k(m)} F_{i_j}.$$

Main Result

- Decomposition:

$$m = \sum_{j=1}^{k(m)} F_{i_j}.$$

- Individual gap measure:

$$\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Main Result

- Decomposition:

$$m = \sum_{j=1}^{k(m)} F_{i_j}.$$

- Individual gap measure:

$$\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Distribution of Individual Gaps (SMALL 2012))

For almost all m, gap measures $\nu_{m;n}$ converge to average gap measure.

Proof Sketch of Individual Gap Measures

- $\widehat{\nu_{m;n}}(t)$: Characteristic fn of $\nu_{m;n}(x)$.
- $\widehat{\nu}(t)$: Characteristic fn of average gap distribution $\nu(x)$.
- $\mathbb{E}_m[\cdots]$: Expected value wrt uniform measure:

$$\mathbb{E}_m[X] \ := \ \frac{1}{F_{n+1} - F_n} \sum_{m=F_n}^{F_{n+1}-1} X(m).$$

Key ideas:

(1) $\mathbb{E}_m[\widehat{\nu_{m;n}}(t)] = \widehat{\nu}(t)$ (use $k(m) \sim \mu_n$ by Gaussianity).

(2) $\lim_{n\to\infty} \mathbb{E}_m \left[\left(\widehat{\nu_{m;n}}(t) - \widehat{\nu}(t)\right)^2 \right] = 0$ (use X_{i,g_1,j,g_2}).

Longest Gap

Fibonacci Case Generating Function

$G_{n,k,f}$ be the number of $m \in [F_n, F_{n+1})$ with k nonzero summands and all gaps less than $f(n)$.

Fibonacci Case Generating Function

$G_{n,k,f}$ be the number of $m \in [F_n, F_{n+1})$ with k nonzero summands and all gaps less than $f(n)$.

$G_{n,k,f}$ is the coefficient of x^n for the generating function

$$\frac{1}{1-x}\left[\sum_{j=2}^{f(n)-2} x^j\right]^{k-1}.$$

Fibonacci Case Generating Function

$G_{n,k,f}$ be the number of $m \in [F_n, F_{n+1})$ with k nonzero summands and all gaps less than $f(n)$.

$G_{n,k,f}$ is the coefficient of x^n for the generating function

$$\frac{1}{1-x}\left[\sum_{j=2}^{f(n)-2} x^j\right]^{k-1}.$$

Let $m = F_n + F_{n-g_1} + F_{n-g_1-g_2} + \cdots + F_{n-g_1-\cdots-g_{n-1}}$, then

- Each gap is ≥ 2.
- Each gap is $< f(n)$.
- The sum of the gaps of x is $\leq n$.

Gaps uniquely identify m by Zeckendorf's Theorem.

The Combinatorics

$G_{n,k,f}$ is the n^{th} coefficient of

$$\frac{1}{1-x}\left[x^2+\cdots+x^{f(n)-2}\right]^{k-1} = \frac{x^{2(k-1)}}{1-x}\left(\frac{1-x^{f(n)-3}}{1-x}\right)^{k-1}.$$

The Combinatorics

$G_{n,k,f}$ is the n^{th} coefficient of

$$\frac{1}{1-x}\left[x^2+\cdots+x^{f(n)-2}\right]^{k-1} = \frac{x^{2(k-1)}}{1-x}\left(\frac{1-x^{f(n)-3}}{1-x}\right)^{k-1}.$$

For fixed k hard to analyze, but only care about sum over k.

The Generating Function

Sum over k gives number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$, call it $G_{n,f}$.

It's the n^{th} coefficient (up to potentially small algebra errors!) of

$$F(x) = \frac{1}{1-x}\sum_{k=1}^{\infty}\left(\frac{x^2 - x^{f-2}}{1-x}\right)^{k-1} = \frac{x}{1-x-x^2+x^{f(n)}}.$$

The Generating Function

Sum over k gives number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$, call it $G_{n,f}$.

It's the n^{th} coefficient (up to potentially small algebra errors!) of

$$F(x) = \frac{1}{1-x}\sum_{k=1}^{\infty}\left(\frac{x^2 - x^{f-2}}{1-x}\right)^{k-1} = \frac{x}{1-x-x^2+x^{f(n)}}.$$

Use **partial fractions** and **Rouché's Theorem** to find CDF.

Partial Fractions

Write the roots of $x^f - x^2 - x - 1$ as $\{\alpha_i\}_{i=1}^f$, generating function is

$$F(x) = \frac{x}{1 - x - x^2 + x^{f(n)}} = \sum_{i=1}^{f(n)} \frac{-\alpha_i}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \sum_{j=1}^{\infty} \left(\frac{x}{\alpha_i}\right)^j.$$

Partial Fractions

Write the roots of $x^f - x^2 - x - 1$ as $\{\alpha_i\}_{i=1}^f$, generating function is

$$F(x) = \frac{x}{1 - x - x^2 + x^{f(n)}} = \sum_{i=1}^{f(n)} \frac{-\alpha_i}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \sum_{j=1}^{\infty} \left(\frac{x}{\alpha_i}\right)^j.$$

Take the n^{th} coefficient to find the number of m with gaps less than $f(n)$.

Partial Fractions

Divide the number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$ by the number of m, which is

$$F_{n+1} - F_n \;=\; F_{n-1} \;=\; 5^{-1/2}\left(\phi^{n-1} - (1/\phi)^{n-1}\right).$$

Theorem

The proportion of $m \in [F_n, F_{n+1})$ with $L(x) < f(n)$ is exactly

$$\sum_{i=1}^{f(n)} \frac{-\sqrt{5}(\alpha_i)}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i}\left(\frac{1}{\alpha_i}\right)^{n+1}\frac{1}{(\phi^n - (-1/\phi)^n)}$$

Now study the roots of $x^f - x^2 - x + 1$.

Rouché and Roots

When $f(n)$ is large, $z^{f(n)}$ is very small for $|z| < 1$. Thus, by Rouché's theorem:

Lemma

For $f \in \mathbb{N}$ and $f \geq 4$, the polynomial $p_f(z) = z^f - z^2 - z + 1$ has exactly one root z_f with $|z_f| < .9$. Further, $z_f \in \mathbb{R}$ and $z_f = \frac{1}{\phi} + \left|\frac{z_f^f}{z_f+\phi}\right|$, so as $f \to \infty$, z_f converges to $\frac{1}{\phi}$.

We only care about the **smallest root**.

Getting the CDF

As f grows, only one root goes to $1/\phi$. The other roots don't matter. So,

Getting the CDF

As f grows, only one root goes to $1/\phi$. The other roots don't matter. So,

Theorem

If $\lim_{n\to\infty} f(n) = \infty$, the proportion of m with $L(m) < f(n)$ is, as $n \to \infty$

$$\lim_{n\to\infty} (\phi z_f)^{-n} = \lim_{n\to\infty} \left(1 + \left|\frac{\phi z_f^{f(n)}}{\phi + z_f}\right|\right)^{-n}.$$

If $f(n)$ is bounded, then $P_f = 0$.

Take logarithms, Taylor expand, result follows from algebra.

Algebra increases greatly for general recurrence.

Current Work
with Minerva Catral, Pari Ford, Pamela Harris & Dawn Nelson

Kentucky Sequence

Rule: (s, b)-Sequence: Bins of length b, and:

- cannot take two elements from the same bin, and
- if have an element from a bin, cannot take anything from the first s bins to the left or the first s to the right.

Kentucky Sequence

Rule: (s, b)-Sequence: Bins of length b, and:

- cannot take two elements from the same bin, and
- if have an element from a bin, cannot take anything from the first s bins to the left or the first s to the right.

Fibonaccis: These are $(s, b) = (1, 1)$.

Kentucky: These are $(s, b) = (1, 2)$.

$[1,\ 2],\ [3,\ 4],\ [5,\ 8],\ [11,\ 16],\ [21,\ 32],\ [43,\ 64],\ [85,\ 128].$

Kentucky Sequence

Rule: (s, b)-Sequence: Bins of length b, and:

- cannot take two elements from the same bin, and
- if have an element from a bin, cannot take anything from the first s bins to the left or the first s to the right.

Fibonaccis: These are $(s, b) = (1, 1)$.

Kentucky: These are $(s, b) = (1, 2)$.

$[1,\ 2],\ [3,\ 4],\ [5,\ 8],\ [11,\ 16],\ [21,\ 32],\ [43,\ 64],\ [85,\ 128].$

- $a_{2n} = 2^n$ and $a_{2n+1} = \frac{1}{3}(2^{2+n} - (-1)^n)$: $a_{n+1} = a_{n-1} + 2a_{n-3}, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4.$

Kentucky Sequence

Rule: (s, b)-Sequence: Bins of length b, and:

- cannot take two elements from the same bin, and
- if have an element from a bin, cannot take anything from the first s bins to the left or the first s to the right.

Fibonaccis: These are $(s, b) = (1, 1)$.

Kentucky: These are $(s, b) = (1, 2)$.

$[1,\ 2],\ [3,\ 4],\ [5,\ 8],\ [11,\ 16],\ [21,\ 32],\ [43,\ 64],\ [85,\ 128].$

- $a_{2n} = 2^n$ and $a_{2n+1} = \frac{1}{3}(2^{2+n} - (-1)^n)$: $a_{n+1} = a_{n-1} + 2a_{n-3}, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4.$
- $a_{n+1} = a_{n-1} + 2a_{n-3}$: New as leading term 0.

Gaussian Behavior

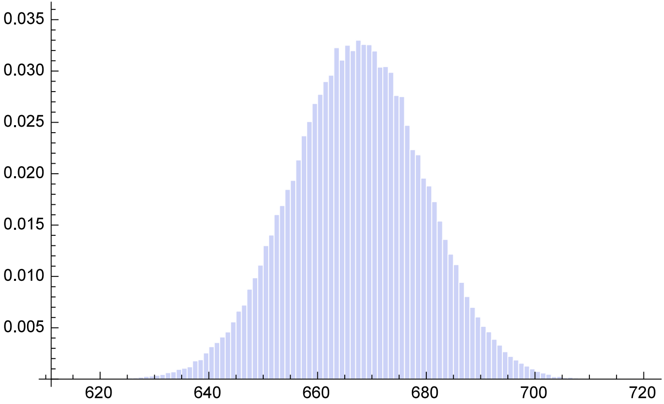


Figure: Plot of the distribution of the number of summands for 100,000 randomly chosen $m \in [1, a_{4000}) = [1, 2^{2000})$ (so m has on the order of 602 digits).

Gaps

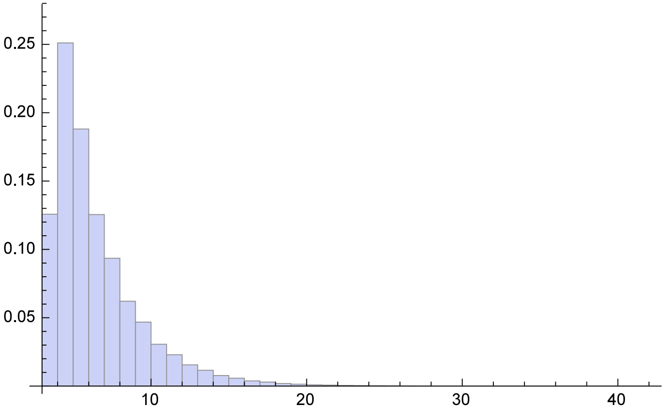


Figure: Plot of the distribution of gaps for 10,000 randomly chosen $m \in [1, a_{400}) = [1, 2^{200})$ (so m has on the order of 60 digits).

Gaps

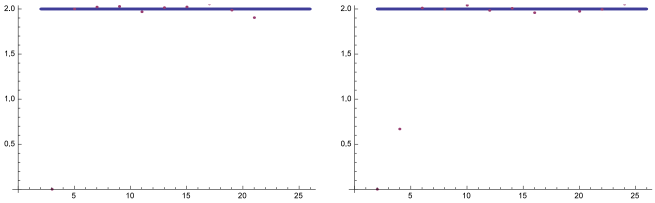
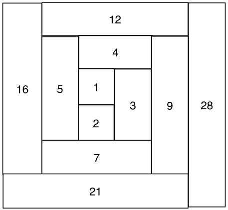


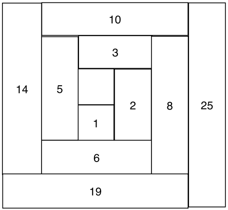
Figure: Plot of the distribution of gaps for 10,000 randomly chosen $m \in [1, a_{400}) = [1, 2^{200})$ (so m has on the order of 60 digits). Left (resp. right): ratio of adjacent even (resp odd) gap probabilities.

Again find geometric decay, but parity issues so break into even and odd gaps.

The Fibonacci (or Log Cabin) Quilt: Work in Progress



1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, ...



1, 2, 3, 5, 6, 8, 10, 14, 19, 25, 33, ...

- $a_{n+1} = a_{n-1} + a_{n-2}$, non-uniqueness (average number of decompositions grows exponentially).
- In process of investigating Gaussianity, Gaps, $K_{\min}, K_{\text{ave}}, K_{\max}, K_{\text{greedy}}$.

References

References

References

- Bower, Insoft, Li, Miller and Tosteson, *The Distribution of Gaps between Summands in Generalized Zeckendorf Decompositions*, preprint. http://arxiv.org/pdf/1402.3912.
- Beckwith, Bower, Gaudet, Insoft, Li, Miller and Tosteson: The Average Gap Distribution for Generalized Zeckendorf Decompositions: The Fibonacci Quarterly **51** (2013), 13–27. http://arxiv.org/abs/1208.5820
- Kologlu, Kopp, Miller and Wang: On the number of summands in Zeckendorf decompositions: Fibonacci Quarterly **49** (2011), no. 2, 116–130. http://arxiv.org/pdf/1008.3204
- Miller and Wang: From Fibonacci numbers to Central Limit Type Theorems: Journal of Combinatorial Theory, Series A **119** (2012), no. 7, 1398–1413. http://arxiv.org/pdf/1008.3202
- Miller and Wang: Survey: Gaussian Behavior in Generalized Zeckendorf Decompositions: To appear in CANT 2011 Proceedings. http://arxiv.org/pdf/1107.2718

Generalizations

Positive Linear Recurrence Sequences

This method can be greatly generalized to **Positive Linear Recurrence Sequences**: linear recurrences with non-negative coefficients:

$$H_{n+1} = c_1 H_{n-(j_1=0)} + c_2 H_{n-j_2} + \cdots + c_L H_{n-j_L}.$$

Theorem (Zeckendorf's Theorem for *PLRS* recurrences)

*Any $b \in \mathbb{N}$ has a unique **legal** decomposition into sums of H_n,*
$b = a_1 H_{i_1} + \cdots + a_{i_k} H_{i_k}$.

Here **legal** reduces to non-adjacency of summands in the Fibonacci case.

Messier Combinatorics

The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of x^{n-s} in the generating function:

Messier Combinatorics

The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of x^{n-s} in the generating function:

$$\frac{1}{1-x}\left(c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L}\right) \times$$

$$\times \sum_{k \geq 0} \Bigg[\left((c_1 - 1)x^{t_1} + \cdots + (c_L - 1)x^{t_L}\right) \left(\frac{x^{s+1} - x^f}{1-x}\right) +$$

$$+x^{t_1}\left(\frac{x^{s+t_2-t_1+1} - x^f}{1-x}\right) + \cdots + x^{t_{L-1}}\left(\frac{x^{s+t_L-t_{L-1}} + 1 - x^f}{1-x}\right)\Bigg]^k.$$

Messier Combinatorics

The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of x^{n-s} in the generating function:

$$\frac{1}{1-x}\left(c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L}\right) \times$$

$$\times \sum_{k \geq 0} \left[\left((c_1 - 1)x^{t_1} + \cdots + (c_L - 1)x^{t_L}\right) \left(\frac{x^{s+1} - x^f}{1-x}\right) + \right.$$

$$\left. +x^{t_1}\left(\frac{x^{s+t_2-t_1+1} - x^f}{1-x}\right) + \cdots + x^{t_{L-1}}\left(\frac{x^{s+t_L-t_{L-1}+1} - x^f}{1-x}\right)\right]^k.$$

A geometric series!

Generalized Generating Function

Let $f > j_L$. The number of $x \in [H_n, H_{n+1})$, with longest gap $< f$ is given by **the coefficient of** s^n in the generating function

$$F(s) = \frac{1 - s^{j_L}}{\mathcal{M}(s) + s^f \mathcal{R}(s)},$$

where

$$\mathcal{M}(s) = 1 - c_1 s - c_2 s^{j_2+1} - \cdots - c_L s^{j_L+1},$$

and

$$\mathcal{R}(s) = c_{j_1+1} s^{j_1} + c_{j_2+1} s^{j_2} + \cdots + (c_{j_L+1} - 1) s^{j_L}.$$

and c_i and j_i are defined **as above**.

What are the extra obstructions?

The **coefficients** in the **partial fraction** expansion might blow up from multiple roots.

What are the extra obstructions?

The **coefficients** in the **partial fraction** expansion might blow up from multiple roots.

Theorem (Mean and Variance for "Most Recurrences")

For x in the interval $[H_n, H_{n+1})$, the mean longest gap μ_n and the variance of the longest gap σ_n^2 are given by

$$\mu_n = \frac{\log\left(\frac{\mathcal{R}(\frac{1}{\lambda_1})}{\mathcal{G}(\frac{1}{\lambda_1})}n\right)}{\log \lambda_1} + \frac{\gamma}{\log \lambda_1} - \frac{1}{2} + \textit{Small Error} + \epsilon_1(n),$$

and

$$\sigma_n^2 = \frac{\pi^2}{6 \log \lambda_1} - \frac{1}{12} + \textit{Small Error} + \epsilon_2(n),$$

where $\epsilon_i(n)$ tends to zero in the limit, and Small Error comes from the Euler-Maclaurin Formula.