From Monovariants to Zeckendorf Decompositions and Games

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Joint with Katherine Cordwell, Max Hlavacek, Chi Huynh, Carsten Peterson, and Yen Nhi Truong Vu, and Alyssa Epstein and Kristen Flint

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Summand Minimality
with Cordwell, Hlavacek, Huynh, Peterson, Vu

Introduction

Fibonaccis: $F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

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1, 2, 3, 5, 8, 13...

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

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*The Zeckendorf decomposition is **summand minimal**.*

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Overall Question

What other recurrences are summand minimal?

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence $\{a_n\}$ with

$$a_n \ := \ c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \ldots, c_t)$ the **signature of the sequence.**

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Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature $(c_1, c_2, \ldots, c_t)$, the Generalized Zeckendorf Decompositions are summand minimal if and only if

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of N, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

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- Move to $\mathcal{D}'$ by
  - ◇ $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - ◇ $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

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- Monovariant: Note $\mathrm{Ind}(\mathcal{D}') \leq \mathrm{Ind}(\mathcal{D})$.
  - ◇ $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k - 1$.
  - ◇ $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$.

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  - ◇ $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$.

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: $\mathrm{Ind}'(\mathcal{D}) = b_1\sqrt{1} + \cdots + b_n\sqrt{n}$.

The Zeckendorf Game
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Rules

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  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
2	0	1	1	0
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Games end

Theorem

All games end in finitely many moves.

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Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0$.
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0$.

Games Lengths: I

Upper bound: At most $n \log_{\phi}\left(n\sqrt{5} + 1/2\right)$ moves.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

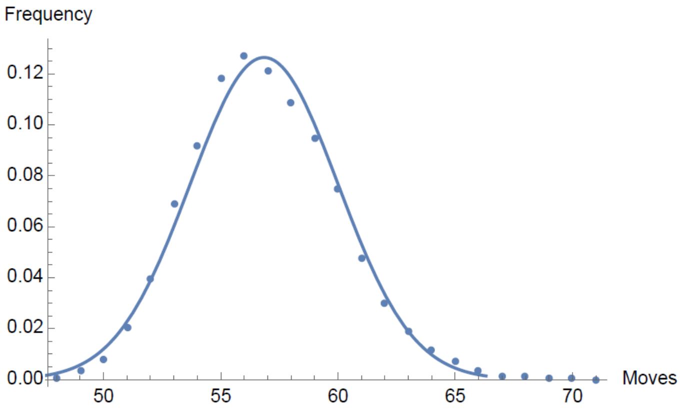


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

Winning Strategy

Theorem

Payer Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

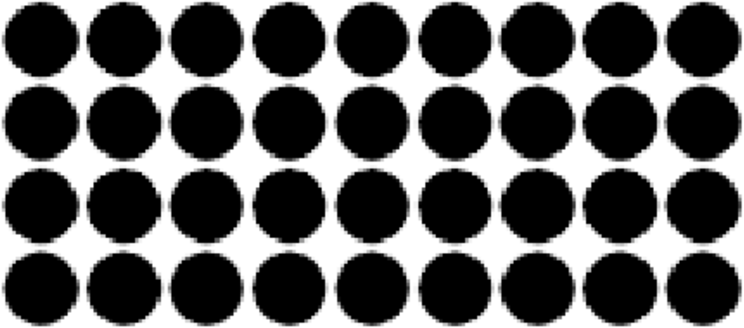
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.



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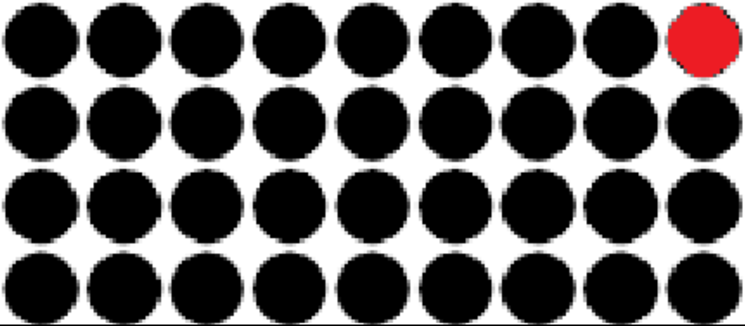

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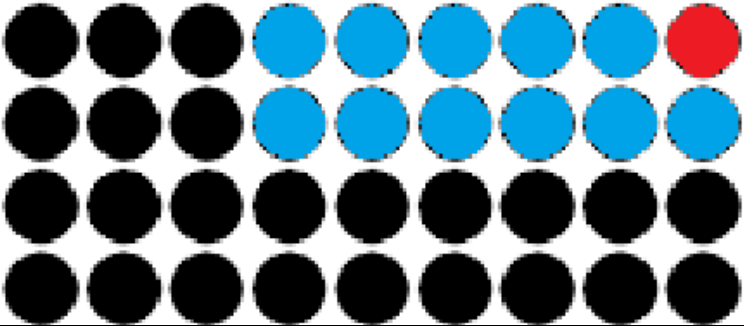

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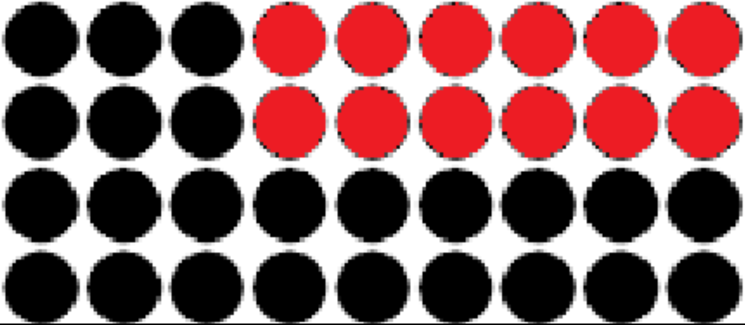

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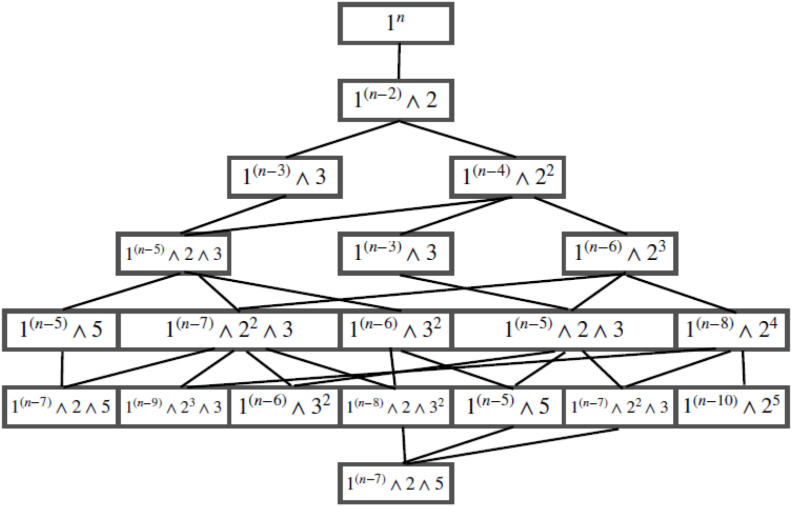

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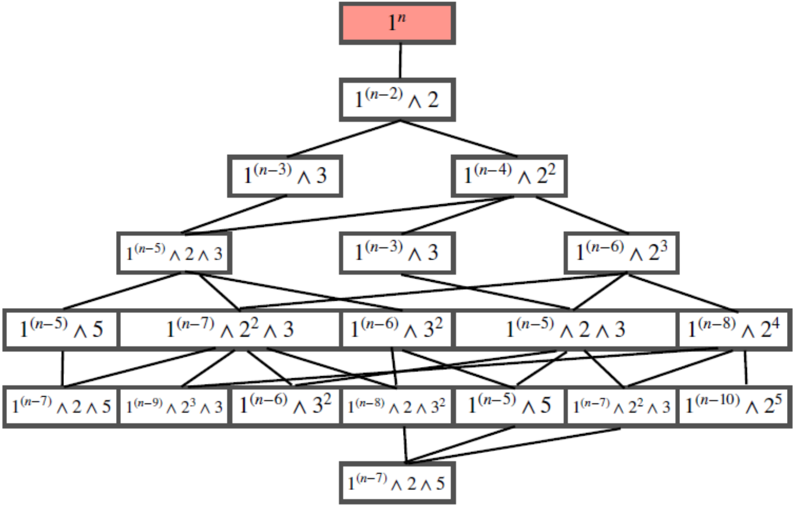

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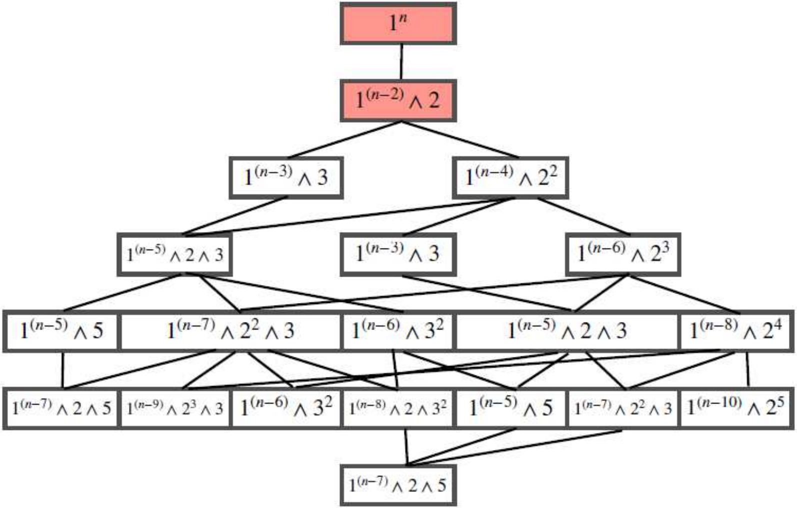

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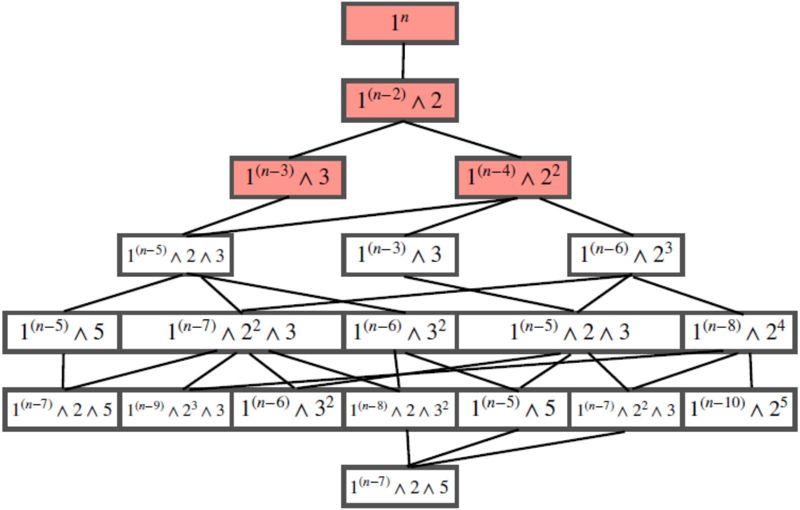

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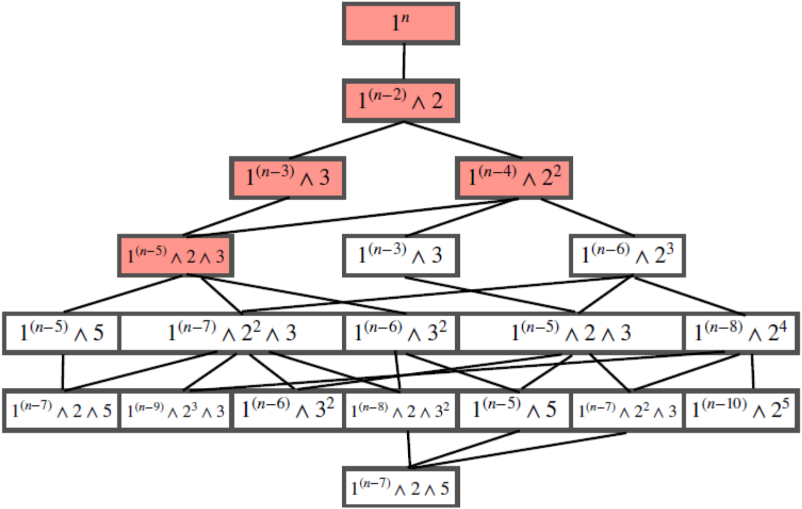

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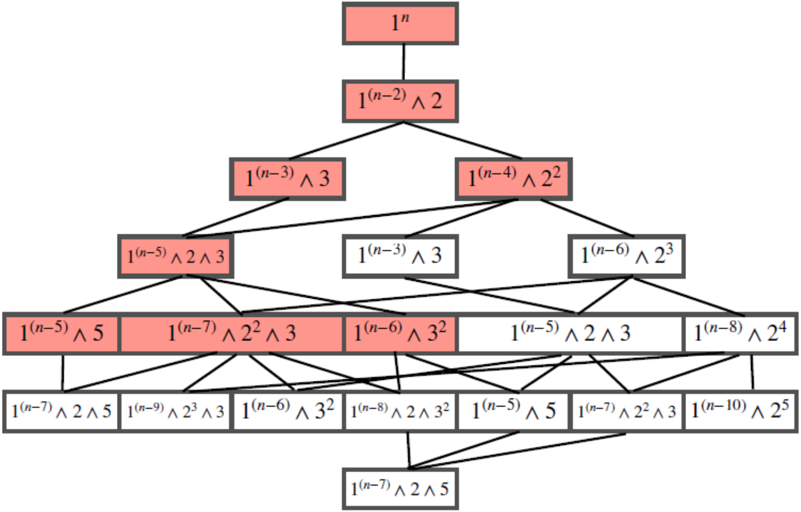

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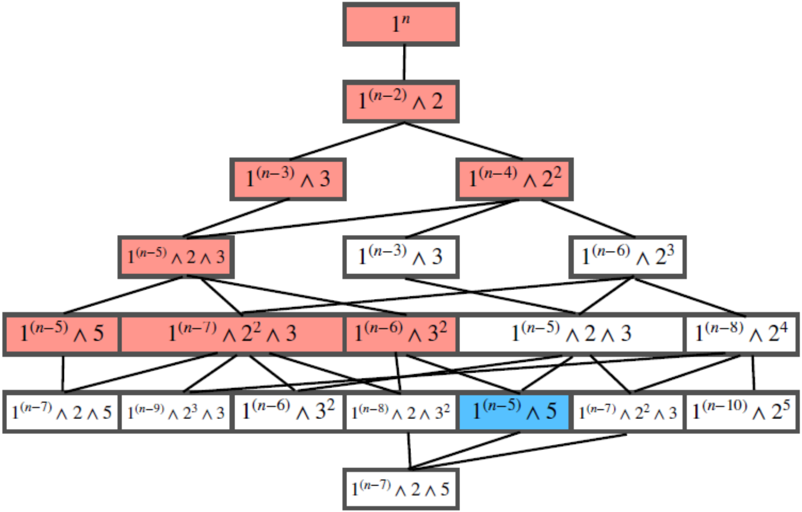

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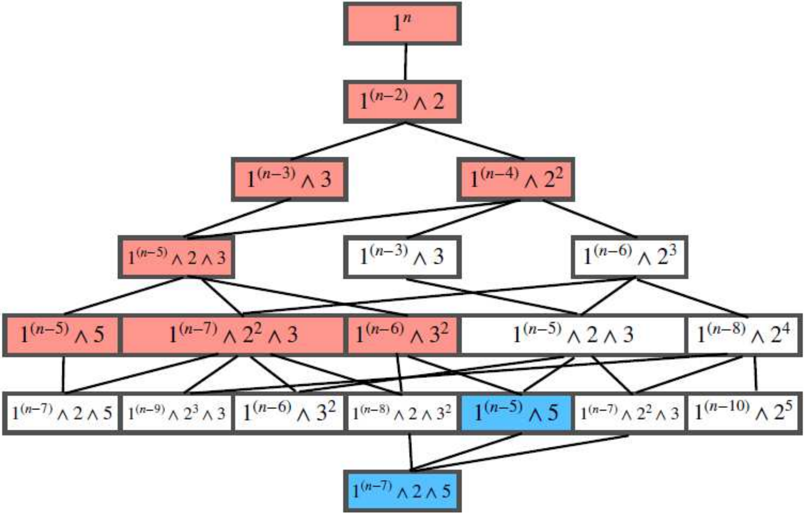

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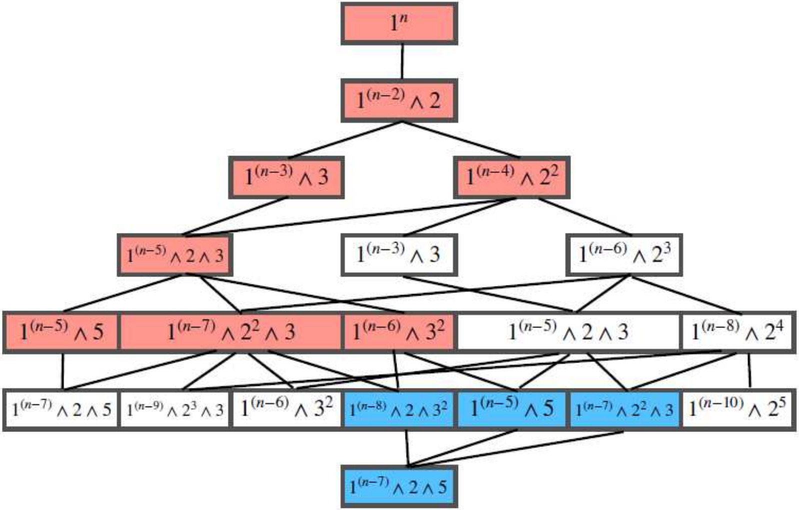

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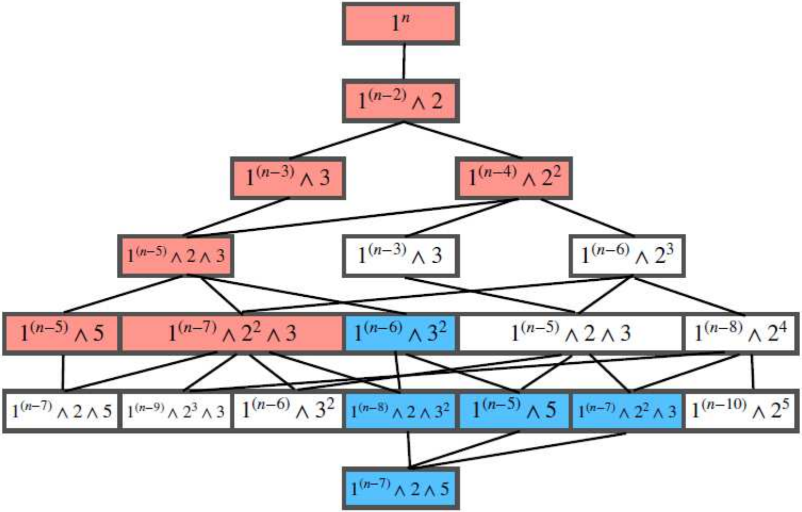

Future Work

- What if $p \geq 3$ people play the Fibonacci game?
- Does the number of moves in random games converge to a Gaussian?
- Define k-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?

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