Finite conductor models for zeros near the central point of elliptic curve L-functions

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Introduction

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

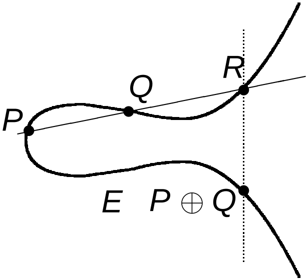
$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

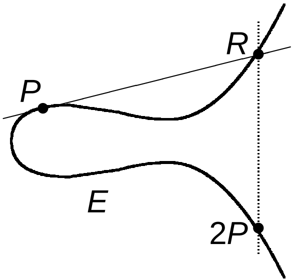
All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

One parameter family

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T), A(T), B(T) \in \mathbb{Z}[T].$$

Silverman's Specialization Theorem

Assume (geometric) rank of $\mathcal{E}/\mathbb{Q}(T)$ is r. Then for all $t \in \mathbb{Z}$ sufficiently large, each $E_t : y^2 = x^3 + A(t)x + B(t)$ has (geometric) rank at least r.

Average rank conjecture

For a generic one-parameter family of rank r over $\mathbb{Q}(T)$, expect in the limit half the specialized curves have rank r and half have rank $r + 1$.

Measures of Spacings: *n*-Level Correlations

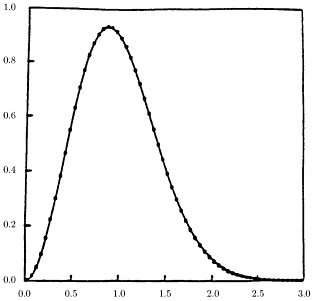
$\{\alpha_j\}$ increasing sequence of numbers, $B \subset \mathbb{R}^{n-1}$ a compact box. Define the *n*-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k\right\}}{N}$$

Instead of using a box, can use a smooth test function.

Measures of Spacings: *n*-Level Correlations

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)



70 million spacings between adjacent normalized zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko)

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5. insensitive to any finite set of zeros

Measures of Spacings: n-Level Density and Families

Let g_i be even Schwartz functions whose Fourier Transform is compactly supported, $L(s,f)$ an L-function with zeros $\frac{1}{2}+\mathrm{i}\gamma_f$ and conductor Q_f:

$$D_{n,f}(g) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

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- Properties of *n*-level density:
  - ◇ Individual zeros contribute in limit

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- Properties of n-level density:
  - ◇ Individual zeros contribute in limit
  - ◇ Most of contribution is from low zeros
  - ◇ Average over similar L-functions (family)

n-Level Density

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, g_k an even Schwartz function: $D_{n,\mathcal{F}}(g) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, n-level density converges to

$$\int g(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} \;=\; \int \widehat{g}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r+2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Theory and Models

Orthogonal Random Matrix Models

RMT: $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model: Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$: $1 \leq j,k \leq N-r$:

$$d\varepsilon_{2r}(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Indep):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) \\ = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0) \end{aligned}$$

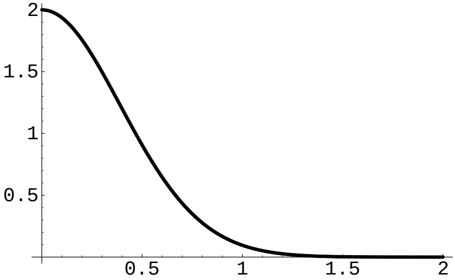
where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

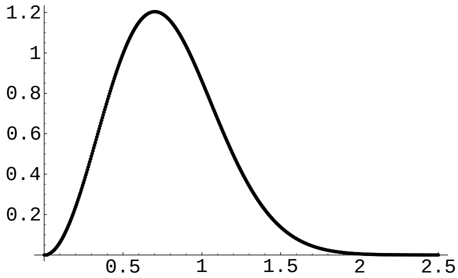
Data

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

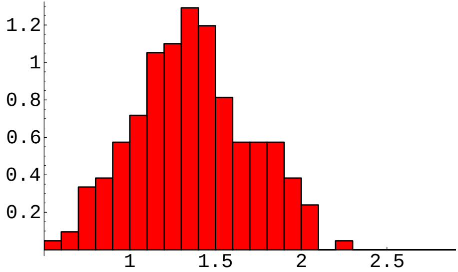


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

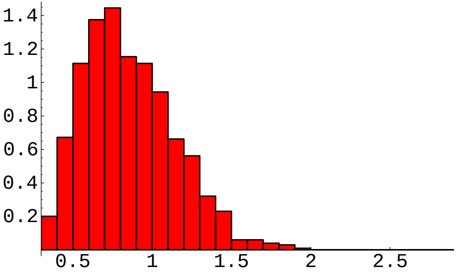


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	701 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	64 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

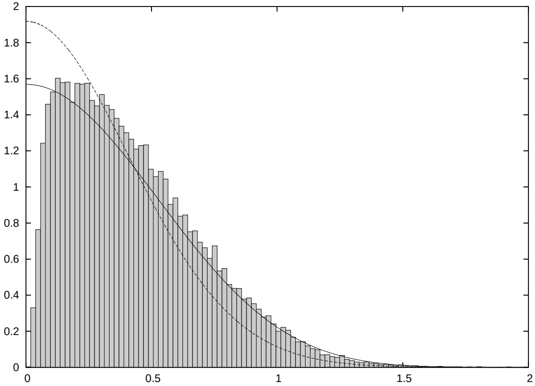
Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i. e., shifted by the same amount).

New Model for Finite Conductors

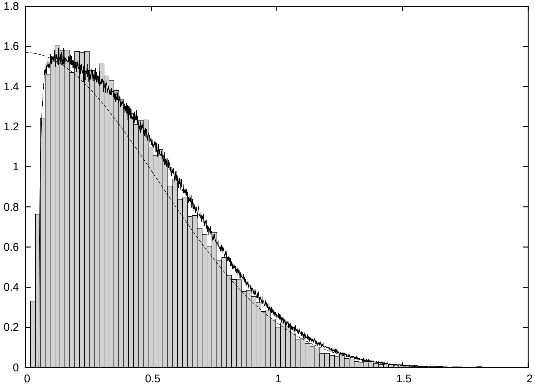
- Replace conductor N with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with L-function Ratios Conj.
  - ◇ Do the number theory computation.
- Excised Orthogonal Ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Study matrices in $\mathrm{SO}(2N_{\mathit{eff}})$ with $|\Lambda_A(1)| \geq ce^N$.
- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): $N_{\rm eff}$ = 2 (solid) with discretisation, and $N_{\rm eff} = 2.32$ (dashed) without discretisation.

Ratio's Conjecture

History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} = \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha,f\right)}{L\left(\frac{1}{2}+\gamma,f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  ◇ *n*-level correlations and densities;
  ◇ mollifiers;
  ◇ moments;
  ◇ vanishing at the central point;

- Advantages:
  ◇ RMT models often add arithmetic ad hoc;
  ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) = \sum_{m \leq x} \frac{a_m}{m^s} + \epsilon \mathbb{X}_L(s) \sum_{n \leq y} \frac{a_n}{n^{1-s}};$$

◇ ϵ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

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- Explicit Formula: g Schwartz test function,

$$\sum_{f \in \mathcal{F}} \omega_f \sum_{\gamma} g\left(\gamma \frac{\log N_f}{2\pi}\right) \ = \ \frac{1}{2\pi i} \int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\cdots) g(\cdots)$$

◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial \alpha} R_{\mathcal{F}}(\alpha, \gamma)\Big|_{\alpha=\gamma=r}$.

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- Use approximate functional equation to expand numerator.

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$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Differentiate with respect to the parameters.

Procedure ('Illegal Steps')

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- Differentiate with respect to the parameters.

1-Level Prediction from Ratio's Conjecture

$$\begin{aligned} & A_E(\alpha,\gamma) \\ = \quad & Y_E^{-1}(\alpha,\gamma) \times \prod_{p|M}\left(\sum_{m=0}^{\infty}\left(\frac{\lambda(p^m)\omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)}{p^{1/2+\gamma}}\frac{\lambda(p^m)\omega_E^{m+1}}{p^{m(1/2+\alpha)}}\right)\right) \times \\ & \prod_{p\nmid M}\left(1+\frac{p}{p+1}\left(\sum_{m=1}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}}\right.\right. \\ & \quad \left.\left. + \frac{1}{p^{1+2\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}}\right)\right) \end{aligned}$$

where

$$Y_E(\alpha,\gamma) = \frac{\zeta(1+2\gamma)L_E(\text{sym}^2, 1+2\alpha)}{\zeta(1+\alpha+\gamma)L_E(\text{sym}^2, 1+\alpha+\gamma)}.$$

Huynh, Morrison and Miller confirmed Ratios' prediction, which is

1-Level Prediction from Ratio's Conjecture

$$\frac{1}{X^*}\sum_{d\in\mathcal{F}(X)}\sum_{\gamma_d} g\left(\frac{\gamma_d L}{\pi}\right)$$

$$= \frac{1}{2LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)+\frac{\Gamma'}{\Gamma}\left(1+\frac{i\pi\tau}{L}\right)+\frac{\Gamma'}{\Gamma}\left(1-\frac{i\pi\tau}{L}\right)\right]d\tau$$

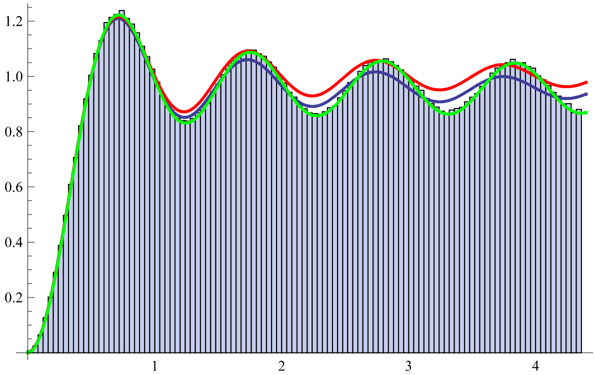
$$+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\left(-\frac{\zeta'}{\zeta}\left(1+\frac{2\pi i\tau}{L}\right)+\frac{L_E'}{L_E}\left(\mathrm{sym}^2, 1+\frac{2\pi i\tau}{L}\right)-\sum_{\ell=1}^{\infty}\frac{(M^\ell-1)\log M}{M^{\left(2+\frac{2i\pi\tau}{L}\right)\ell}}\right)d\tau$$

$$-\frac{1}{L}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g(\tau)\frac{\log M}{M^{(k+1)(1+\frac{\pi i\tau}{L})}}d\tau+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\sum_{p\nmid M}\frac{\log p}{(p+1)}\sum_{k=0}^{\infty}\frac{\lambda(p^{2k+2})-\lambda(p^{2k})}{p^{(k+1)(1+\frac{2\pi i\tau}{L})}}d\tau$$

$$-\frac{1}{LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2i\pi\tau/L}\frac{\Gamma(1-\frac{i\pi\tau}{L})}{\Gamma(1+\frac{i\pi\tau}{L})}\frac{\zeta(1+\frac{2i\pi\tau}{L})L_E(\mathrm{sym}^2, 1-\frac{2i\pi\tau}{L})}{L_E(\mathrm{sym}^2, 1)}\right.$$

$$\left.\times A_E\left(-\frac{i\pi\tau}{L}, \frac{i\pi\tau}{L}\right)\right]d\tau+O(X^{-1/2+\varepsilon});$$

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.

$\diamond$ Green: all lower order terms.

Excised Orthogonal Ensembles

Excised Orthogonal Ensemble: Preliminaries

Characteristic polynomial of $A \in \mathrm{SO}(2N)$ is

$$\Lambda_A(e^{i\theta}, N) := \det(I - Ae^{-i\theta}) = \prod_{k=1}^{N}(1 - e^{i(\theta_k - \theta)})(1 - e^{i(-\theta_k - \theta)}),$$

with $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N}$ the eigenvalues of A.

Motivated by the arithmetical size constraint on the central values of the L-functions, consider Excised Orthogonal Ensemble $T_{\mathcal{X}}$: $A \in \mathrm{SO}(2N)$ with $|\Lambda_A(1, N)| \geq \exp(\mathcal{X})$.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.

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The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) := C_{\mathcal{X}} \cdot N \int_0^{\pi} \cdots \int_0^{\pi} H(\log|\Lambda_A(1,N)| - \mathcal{X}) \times \\ \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N,$$

Here $H(x)$ denotes the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0, \end{cases}$$

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The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) = \frac{C_{\mathcal{X}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{Nr} \frac{\exp(-r\mathcal{X})}{r} R_1^{J_N}(\theta_1; r-1/2, -1/2) dr$$

where $C_{\mathcal{X}}$ is a normalization constant and

$$R_1^{J_N}(\theta_1; r-1/2, -1/2) = N \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} w^{(r-1/2,-1/2)}(\cos\theta_j) \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N$$

is the one-level density for the Jacobi ensemble J_N with weight function

$$w^{(\alpha,\beta)}(\cos\theta) = (1-\cos\theta)^{\alpha+1/2}(1+\cos\theta)^{\beta+1/2}, \qquad \alpha = r - 1/2 \text{ and } \beta = -1/2.$$

Results

- With $C_{\mathcal{X}}$ normalization constant and $P(N, r, \theta)$ defined in terms of Jacobi polynomials,

$$R_1^{T_{\mathcal{X}}}(\theta) = \frac{C_{\mathcal{X}}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(-r\mathcal{X})}{r} 2^{N^2+2Nr-N} \times$$
$$\times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)} \times$$
$$\times (1-\cos\theta)^r \frac{2^{1-r}}{2N+r-1} \frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)} P(N, r, \theta)\, dr.$$

- Residue calculus implies $R_1^{T_{\mathcal{X}}}(\theta) = 0$ for $d(\theta, \mathcal{X}) < 0$ and

$$R_1^{T_{\mathcal{X}}}(\theta) = R_1^{\mathrm{SO}(2N)}(\theta) + C_{\mathcal{X}} \sum_{k=0}^{\infty} b_k \exp((k+1/2)\mathcal{X}) \quad \text{for } d(\theta, \mathcal{X}) \geq 0,$$

where $d(\theta, \mathcal{X}) := (2N-1)\log 2 + \log(1-\cos\theta) - \mathcal{X}$ and b_k are coefficients arising from the residues. As $\mathcal{X} \to -\infty$, θ fixed, $R_1^{T_{\mathcal{X}}}(\theta) \to R_1^{\mathrm{SO}(2N)}(\theta)$.

Numerical check

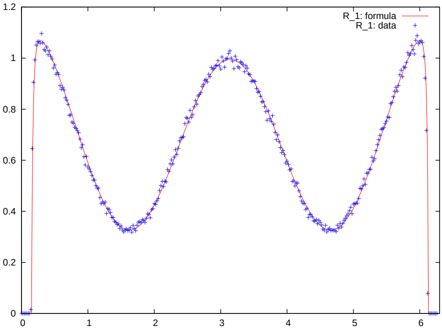

Figure: One-level density of excized $SO(2N), N = 2$ with cut-off $|\Lambda_A(1, N)| \geq 0.1$. The red curve uses our formula. The blue crosses give the empirical one-level density of 200,000 numerically generated matrices.

Theory vs Experiment

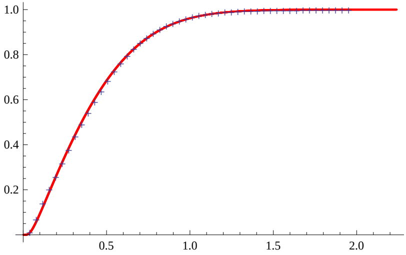

Figure: Cumulative probability density of the first eigenvalue from 3×10^6 numerically generated matrices $A \in SO(2N_{\rm std})$ with $|\Lambda_A(1, N_{\rm std})| \geq 2.188 \times \exp(-N_{\rm std}/2)$ and $N_{\rm std} = 12$ red dots compared with the first zero of even quadratic twists $L_{E_{11}}(s, \chi_d)$ with prime fundamental discriminants $0 < d \leq 400{,}000$ blue crosses. The random matrix data is scaled so that the means of the two distributions agree.

Another Explanation
(This section is by Simon Marshall)

Waldspurger's special value formula

Can explain observed repulsion using Waldspurger's formula and some complex analysis.

Waldspurger's formula

$L(1/2, E \times \chi_d) = \kappa_E c_E(|d|)^2 |d|^{-1/2}$, where $c_E(|d|) \in \mathbb{Z}$.

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$\Rightarrow$ If $L(1/2, E \times \chi_d) \neq 0$, then $L(1/2, E \times \chi_d) \gg_E |d|^{-1/2}$.
By combining this with Jensen's formula obtain

Theorem: Marshall, '11

Define γ_d to be the height of the lowest nonreal zero of $L(s, E \times \chi_d)$. If $L(1/2, E \times \chi_d) \neq 0$ then we have

$$\frac{\ln |\gamma_d|}{\ln |d|} \geq -1/4 + O_{E,\epsilon}(\ln \ln |d|^{-1+\epsilon}).$$

Connection with large-scale zero repulsion

- Repulsion $|\gamma_d| \gg d^{-1/4+o(1)}$ on a much smaller scale than the mean zero spacing $(\ln|d|)^{-1}$. Why should it imply anything about the rescaled limiting density of zeros?

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- Repulsion $|\gamma_d| \gg d^{-1/4+o(1)}$ on a much smaller scale than the mean zero spacing $(\ln|d|)^{-1}$. Why should it imply anything about the rescaled limiting density of zeros?
- The answer is because it holds for every member of a (conjecturally) large family of L-functions.

The family of nonvanishing quadratic twists

Define $\mathcal{D}^0(E)$ and $\mathcal{L}_D$ by

$$\begin{aligned} \mathcal{D}^0(E) &= \{d \mid d \text{ fundamental}, L(1/2, E \times \chi_d) \neq 0\}, \\ \mathcal{L}_D &= \{|\gamma_d| \ln|d| \ : \ d \in \mathcal{D}^0(E), D/2 \leq |d| \leq D\}. \end{aligned}$$

By 'minimal rank conjecture', expect $\mathcal{D}^0(E)$ to have $\gg D$ elements of size at most D, so $|\mathcal{L}_D| \gg D$.

The limiting distribution vanishes at the origin

Suppose $\mathcal{L}_D$ has limiting distribution of the form $\rho(x)dx$, ρ a smooth function on $[0, \infty)$. If ρ vanishes to order r at the origin, then for sufficiently large D we have

$$\text{Prob}(\exists x \in \mathcal{L}_D : x \leq D^{-1/(r+1)}) \geq \delta > 0.$$

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- Disagrees with the Katz-Sarnak heuristics, as they predict a limiting distribution which does not vanish at the origin.
- Similar results hold in the case of first order vanishing, using the Gross-Zagier formula in place of Waldspurger. One again finds a disagreement with Katz-Sarnak.