# From the Manhattan Project to Elliptic Curves

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# Introduction

## Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

**Functional Equation:**

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

**Riemann Hypothesis (RH):**

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

## General $L$-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

**Functional Equation:**

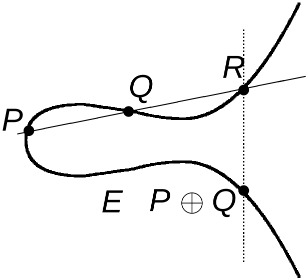
$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

**Generalized Riemann Hypothesis (GRH):**

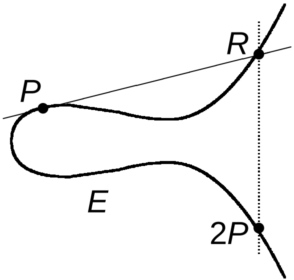
All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

## Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.




Addition of distinct points $P$ and $Q$




Adding a point $P$ to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

## Elliptic curve $L$-function

$E : y^2 = x^3 + ax + b$, associate $L$-function

$$L(s,E) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)/\sqrt{n}}{n^s} \;=\; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

**Birch and Swinnerton-Dyer Conjecture**

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

## One parameter family

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T), \ \ A(T), B(T) \in \mathbb{Z}[T].$$

### Silverman's Specialization Theorem

Assume (geometric) rank of $\mathcal{E}/\mathbb{Q}(T)$ is $r$. Then for all $t \in \mathbb{Z}$ sufficiently large, each $E_t : y^2 = x^3 + A(t)x + B(t)$ has (geometric) rank at least $r$.

### Average rank conjecture

For a generic one-parameter family of rank $r$ over $\mathbb{Q}(T)$, expect in the limit half the specialized curves have rank $r$ and half have rank $r + 1$.

## Measures of Spacings: $n$-Level Density and Families

Let $\phi_i$ be even Schwartz functions whose Fourier Transform is compactly supported, $L(s,f)$ an $L$-function with zeros $\frac{1}{2}+\mathrm{i}\gamma_f$ and conductor $Q_f$:

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots\phi_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

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- Properties of *n*-level density:
  - ◇ Individual zeros contribute in limit.

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- Properties of *n*-level density:
  - ◇ Individual zeros contribute in limit.
  - ◇ Most of contribution is from low zeros.
  - ◇ Average over similar *L*-functions (family).

## $n$-Level Density

$n$-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of $L$-functions ordered by conductors, $\phi_k$ an even Schwartz function: $D_{n,\mathcal{F}}(\phi) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots\phi_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, $n$-level density converges to

$$\int \phi(\vec{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\vec{x})d\vec{x} \quad = \quad \int \widehat{\phi}(\vec{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\vec{u})d\vec{u}.$$

**Conjecture (Katz-Sarnak)**

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

## Correspondences

Similarities between *L*-Functions and Nuclei:

$$\text{Zeros} \quad \longleftrightarrow \quad \text{Energy Levels}$$

$$\text{Schwartz test function} \quad \longrightarrow \quad \text{Neutron}$$

$$\text{Support of test function} \quad \longleftrightarrow \quad \text{Neutron Energy.}$$

## Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank $r$ one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r+2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

# Theory and Models

## Orthogonal Random Matrix Models

RMT: $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} \ = \ \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model: Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$: $1 \ \leq \ j,k \ \leq \ N-r$:

$$d\varepsilon_{2r}(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

## Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Indep):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

## Comparing the RMT Models

### Theorem: M– '04

For small support, one-param family of rank $r$ over $\mathbb{Q}(T)$:

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) \\
= &\int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)
\end{aligned}$$

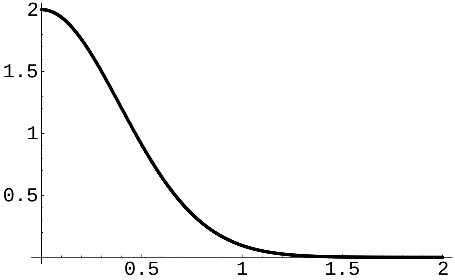
where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd.} \end{cases}$$

**Supports Katz-Sarnak, B-SD, and Independent model in limit.**

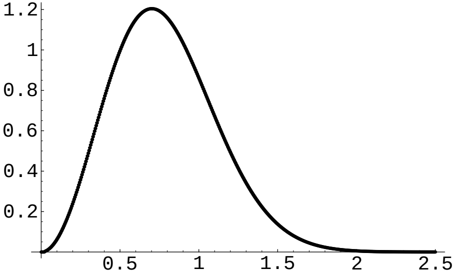
# Data

## RMT: Theoretical Results ($N \to \infty$)




1st normalized evalue above 1: SO(even)

## RMT: Theoretical Results ($N \to \infty$)




1st normalized evalue above 1: SO(odd)

# Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

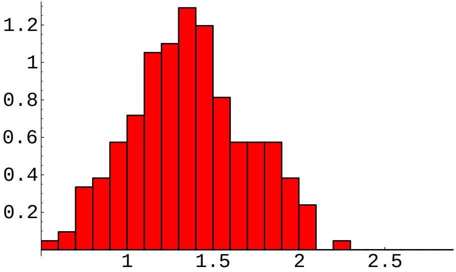



Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

## Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

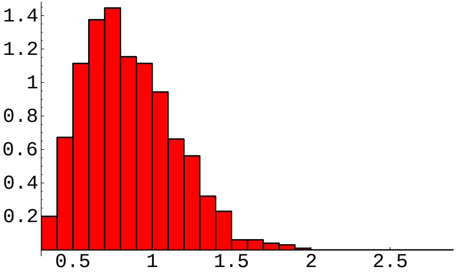



Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

## Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

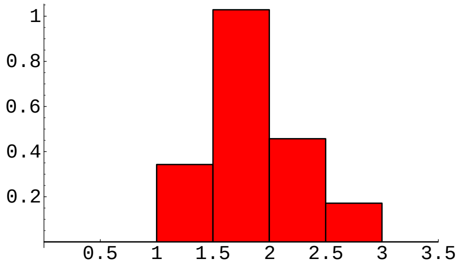



Figure 5*a*: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

## Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

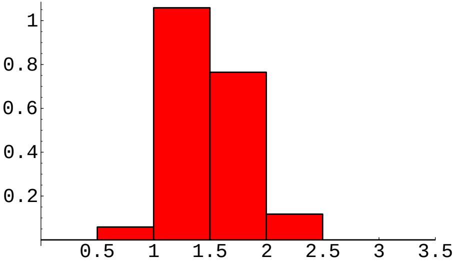



Figure 5$b$: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

## Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of $j^{\text{th}}$ normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

| | **863 Rank 0 Curves** | **701 Rank 2 Curves** | **t-Statistic** |
|---|---|---|---|
| **Median** $z_2 - z_1$ | 1.28 | 1.30 | |
| **Mean** $z_2 - z_1$ | 1.30 | 1.34 | -1.60 |
| **StDev** $z_2 - z_1$ | 0.49 | 0.51 | |
| **Median** $z_3 - z_2$ | 1.22 | 1.19 | |
| **Mean** $z_3 - z_2$ | 1.24 | 1.22 | 0.80 |
| **StDev** $z_3 - z_2$ | 0.52 | 0.47 | |
| **Median** $z_3 - z_1$ | 2.54 | 2.56 | |
| **Mean** $z_3 - z_1$ | 2.55 | 2.56 | -0.38 |
| **StDev** $z_3 - z_1$ | 0.52 | 0.52 | |

# Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- $z_j$ = imaginary part of the $j^{\text{th}}$ norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

| | 64 Rank 2 Curves | 23 Rank 4 Curves | t-Statistic |
|---|---|---|---|
| **Median** $z_2 - z_1$ | 1.26 | 1.27 | |
| **Mean** $z_2 - z_1$ | 1.36 | 1.29 | 0.59 |
| **StDev** $z_2 - z_1$ | 0.50 | 0.42 | |
| **Median** $z_3 - z_2$ | 1.22 | 1.08 | |
| **Mean** $z_3 - z_2$ | 1.29 | 1.14 | 1.35 |
| **StDev** $z_3 - z_2$ | 0.49 | 0.35 | |
| **Median** $z_3 - z_1$ | 2.66 | 2.46 | |
| **Mean** $z_3 - z_1$ | 2.65 | 2.43 | 2.05 |
| **StDev** $z_3 - z_1$ | 0.44 | 0.42 | |

# Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- $z_j$ = imaginary part of the $j^{\text{th}}$ norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

| | **701 Rank 2 Curves** | **64 Rank 2 Curves** | **t-Statistic** |
|---|---|---|---|
| **Median** $z_2 - z_1$ | 1.30 | 1.26 | |
| **Mean** $z_2 - z_1$ | 1.34 | 1.36 | 0.69 |
| **StDev** $z_2 - z_1$ | 0.51 | 0.50 | |
| **Median** $z_3 - z_2$ | 1.19 | 1.22 | |
| **Mean** $z_3 - z_2$ | 1.22 | 1.29 | 1.39 |
| **StDev** $z_3 - z_2$ | 0.47 | 0.49 | |
| **Median** $z_3 - z_1$ | 2.56 | 2.66 | |
| **Mean** $z_3 - z_1$ | 2.56 | 2.65 | 1.93 |
| **StDev** $z_3 - z_1$ | 0.52 | 0.44 | |

## Summary of Data

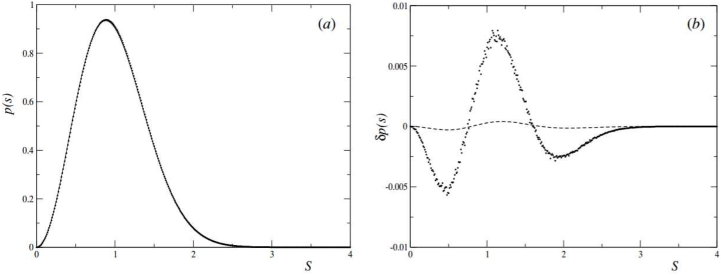
- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i. e., shifted by the same amount).

## Convergence to the RMT limit: What's the right matrix size?

- RMT + Katz-Sarnak: *Limiting behavior* for random matrices as $N \to \infty$ and $L$-functions as conductors tend to infinity agree.

- How well do the classical matrix groups model local statistics of $L$-functions *outside* the scaling limit? (Arithmetic enters!)

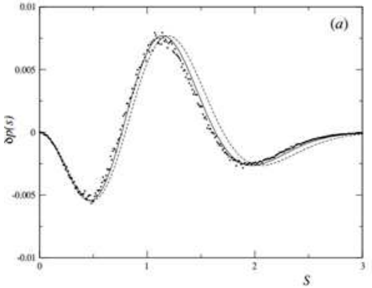
# Convergence to the RMT limit




L: 70 million $\zeta(s)$ nearest-neighbor spacings (Odlyzko).

R: Difference b/w $\zeta(s)$ and asymptotic CUE curve (dots) compared to difference b/w CUE of size $N_0$ and asymptotic curve (dashed line) (from Bogomolny et. al.).

# Convergence to the RMT limit: Incorporating Finite Matrix Size



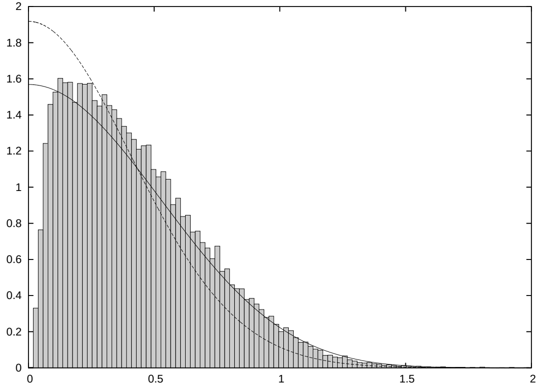

Difference b/w nearest-neighbor spacing of $\zeta(s)$ zeros and asymptotic CUE for a billion zeros in window near $2.504 \times 10^{15}$ (dots) compared to theory that takes into account arithmetic of lower order terms (full line) (from Bogomolny et. al.).

New model should incorporate finite matrix size....

## New Model for Finite Conductors

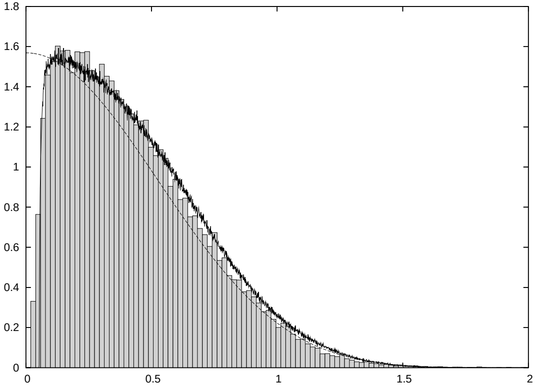
- Replace conductor $N$ with $N_{\text{effective}}$.
  - ⋄ Arithmetic info, predict with $L$-function Ratios Conj.
  - ⋄ Do the number theory computation.
- Excised Orthogonal Ensembles.
  - ⋄ $L(1/2, E)$ discretized.
  - ⋄ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.
- Painlevé VI differential equation solver.
  - ⋄ Use explicit formulas for densities of Jacobi ensembles.
  - ⋄ Key input: Selberg-Aomoto integral for initial conditions.

# Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$




Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_{\rm eff}$ (solid), standard $N_0$ (dashed).

## Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$




Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): $N_{\text{eff}}$ = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

# Ratio’s Conjecture

## History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of $L$-functions over families:

$$R_{\mathcal{F}} \;=\; \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

## Uses of the Ratios Conjecture

- Applications:
  - ◇ $n$-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point;

- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ predicts lower order terms, often to square-root level.

## Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) = \sum_{m \le x} \frac{a_m}{m^s} + \epsilon \mathbb{X}_L(s) \sum_{n \le y} \frac{a_n}{n^{1-s}};$$

◇ $\epsilon$ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

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◇ $\epsilon$ sign of the functional equation,
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- Explicit Formula: $g$ Schwartz test function,

$$\sum_{f\in\mathcal{F}}\omega_f\sum_{\gamma}g\left(\gamma\frac{\log N_f}{2\pi}\right) \;=\; \frac{1}{2\pi i}\int_{(c)}-\int_{(1-c)}R'_{\mathcal{F}}(\cdots)g\left(\cdots\right)$$

◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha}R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

## Procedure (Recipe)

- Use approximate functional equation to expand numerator.

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$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Extend the $m$ and $n$ sums to infinity (complete the products).
- Differentiate with respect to the parameters.

## Procedure ('Illegal Steps')

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- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the $m$ and $n$ sums to infinity (complete the products).
- Differentiate with respect to the parameters.

# 1-Level Prediction from Ratio's Conjecture

$$
\begin{aligned}
& A_E(\alpha,\gamma) \\
= \quad & Y_E^{-1}(\alpha,\gamma) \times \prod_{p \mid M} \left( \sum_{m=0}^{\infty} \left( \frac{\lambda(p^m)\omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)}{p^{1/2+\gamma}} \frac{\lambda(p^m)\omega_E^{m+1}}{p^{m(1/2+\alpha)}} \right) \right) \times \\
& \prod_{p \nmid M} \left( 1 + \frac{p}{p+1} \left( \sum_{m=1}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}} \right.\right. \\
& \quad \left.\left. + \frac{1}{p^{1+2\gamma}} \sum_{m=0}^{\infty} \frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} \right) \right)
\end{aligned}
$$

where

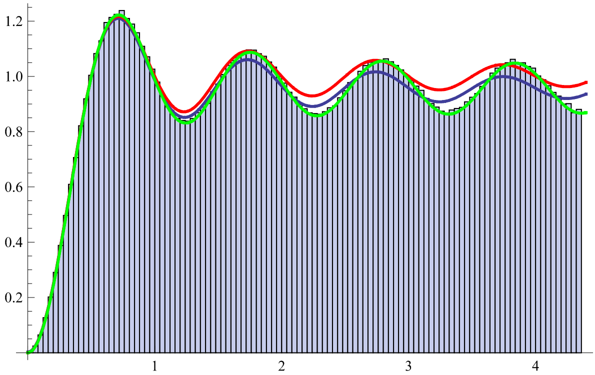
$$
Y_E(\alpha,\gamma) = \frac{\zeta(1+2\gamma) L_E(\mathrm{sym}^2, 1+2\alpha)}{\zeta(1+\alpha+\gamma) L_E(\mathrm{sym}^2, 1+\alpha+\gamma)}.
$$

Huynh, Morrison and Miller confirmed Ratios' prediction, which is

# 1-Level Prediction from Ratio's Conjecture

$$
\begin{aligned}
&\frac{1}{X^*}\sum_{d\in\mathcal{F}(X)}\sum_{\gamma_d} g\left(\frac{\gamma_d L}{\pi}\right)\\
&= \frac{1}{2LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)+\frac{\Gamma'}{\Gamma}\left(1+\frac{i\pi\tau}{L}\right)+\frac{\Gamma'}{\Gamma}\left(1-\frac{i\pi\tau}{L}\right)\right]d\tau\\
&+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\left(-\frac{\zeta'}{\zeta}\left(1+\frac{2\pi i\tau}{L}\right)+\frac{L_E'}{L_E}\left(\mathrm{sym}^2,1+\frac{2\pi i\tau}{L}\right)-\sum_{\ell=1}^{\infty}\frac{(M^\ell-1)\log M}{M^{\left(2+\frac{2i\pi\tau}{L}\right)\ell}}\right)d\tau\\
&-\frac{1}{L}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g(\tau)\frac{\log M}{M^{(k+1)(1+\frac{\pi i\tau}{L})}}d\tau+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\sum_{p\nmid M}\frac{\log p}{(p+1)}\sum_{k=0}^{\infty}\frac{\lambda(p^{2k+2})-\lambda(p^{2k})}{p^{(k+1)(1+\frac{2\pi i\tau}{L})}}d\tau\\
&-\frac{1}{LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2i\pi\tau/L}\frac{\Gamma(1-\frac{i\pi\tau}{L})}{\Gamma(1+\frac{i\pi\tau}{L})}\frac{\zeta(1+\frac{2i\pi\tau}{L})L_E(\mathrm{sym}^2,1-\frac{2i\pi\tau}{L})}{L_E(\mathrm{sym}^2,1)}\right.\\
&\left.\times A_E\left(-\frac{i\pi\tau}{L},\frac{i\pi\tau}{L}\right)\right]d\tau+O(X^{-1/2+\varepsilon});
\end{aligned}
$$

## Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$




Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.

$\diamond$ Green: all lower order terms.

# Excised Orthogonal Ensembles

## Excised Orthogonal Ensemble: Preliminaries

Characteristic polynomial of $A \in \mathrm{SO}(2N)$ is

$$\Lambda_A(e^{i\theta}, N) := \det(I - Ae^{-i\theta}) = \prod_{k=1}^{N}(1 - e^{i(\theta_k - \theta)})(1 - e^{i(-\theta_k - \theta)}),$$

with $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N}$ the eigenvalues of $A$.

Motivated by the arithmetical size constraint on the central values of the $L$-functions, consider Excised Orthogonal Ensemble $T_{\mathcal{X}}$: $A \in \mathrm{SO}(2N)$ with $|\Lambda_A(1, N)| \geq \exp(\mathcal{X})$.

## One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.

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where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases. The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) := C_{\mathcal{X}} \cdot N \int_0^\pi \cdots \int_0^\pi H(\log|\Lambda_A(1,N)| - \mathcal{X}) \times \\ \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N,$$

Here $H(x)$ denotes the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0, \end{cases}$$

and $C_{\mathcal{X}}$ is a normalization constant

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where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.
The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) = \frac{C_{\mathcal{X}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{Nr} \frac{\exp(-r\mathcal{X})}{r} R_1^{J_N}(\theta_1; r - 1/2, -1/2) dr$$

where $C_{\mathcal{X}}$ is a normalization constant and

$$\begin{aligned} R_1^{J_N}(\theta_1; r - 1/2, -1/2) = N \int_0^{\pi} \cdots \int_0^{\pi} & \prod_{j=1}^{N} w^{(r-1/2,-1/2)}(\cos\theta_j) \\ & \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N \end{aligned}$$

is the one-level density for the Jacobi ensemble $J_N$ with weight function

$$w^{(\alpha,\beta)}(\cos\theta) = (1-\cos\theta)^{\alpha+1/2}(1+\cos\theta)^{\beta+1/2}, \qquad \alpha = r - 1/2 \text{ and } \beta = -1/2.$$

## Results

- With $C_{\mathcal{X}}$ normalization constant and $P(N,r,\theta)$ defined in terms of Jacobi polynomials,

$$R_1^{T_{\mathcal{X}}}(\theta) = \frac{C_{\mathcal{X}}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(-r\mathcal{X})}{r} 2^{N^2+2Nr-N} \times$$
$$\times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)} \times$$
$$\times (1-\cos\theta)^r \frac{2^{1-r}}{2N+r-1} \frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)} P(N,r,\theta)\, dr.$$

- Residue calculus implies $R_1^{T_{\mathcal{X}}}(\theta) = 0$ for $d(\theta,\mathcal{X}) < 0$ and

$$R_1^{T_{\mathcal{X}}}(\theta) = R_1^{\mathrm{SO}(2N)}(\theta) + C_{\mathcal{X}} \sum_{k=0}^{\infty} b_k \exp((k+1/2)\mathcal{X}) \quad \text{for } d(\theta,\mathcal{X}) \geq 0,$$

where $d(\theta,\mathcal{X}) := (2N-1)\log 2 + \log(1-\cos\theta) - \mathcal{X}$ and $b_k$ are coefficients arising from the residues. As $\mathcal{X} \to -\infty$, $\theta$ fixed, $R_1^{T_{\mathcal{X}}}(\theta) \to R_1^{\mathrm{SO}(2N)}(\theta)$.

## Numerical check

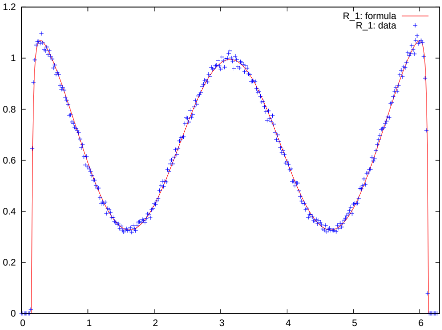


**Figure:** One-level density of excized $SO(2N), N = 2$ with cut-off $|\Lambda_A(1, N)| \geq 0.1$. The red curve uses our formula. The blue crosses give the empirical one-level density of 200,000 numerically generated matrices.

## Theory vs Experiment

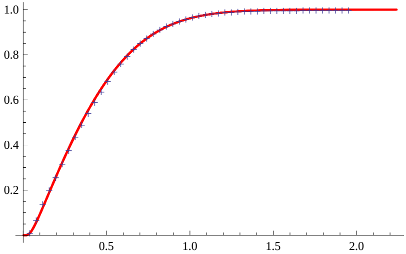


**Figure:** Cumulative probability density of the first eigenvalue from $3 \times 10^6$ numerically generated matrices $A \in SO(2N_{\rm std})$ with $|\Lambda_A(1, N_{\rm std})| \geq 2.188 \times \exp(-N_{\rm std}/2)$ and $N_{\rm std} = 12$ red dots compared with the first zero of even quadratic twists $L_{E_{11}}(s, \chi_d)$ with prime fundamental discriminants $0 < d \leq 400{,}000$ blue crosses. The random matrix data is scaled so that the means of the two distributions agree.

# Conclusion and References

## Conclusion and Future Work

- In the limit: Birch and Swinnerton-Dyer, Katz-Sarnak appear true.
- Finite conductors: model with Excised Ensembles (cut-off on characteristic polynomials due to discretization at central point).
- Future Work: Joint with Owen Barrett and Nathan Ryan (and possibly some of his students): looking at other GL2 families (and hopefully higher) to study the relationship between repulsion at finite conductors and central values (effect of weight, level).

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