Finite conductor models for zeros near the central point of elliptic curve L-functions

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Introduction

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha \bmod 1$.
- Spacings b/w Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

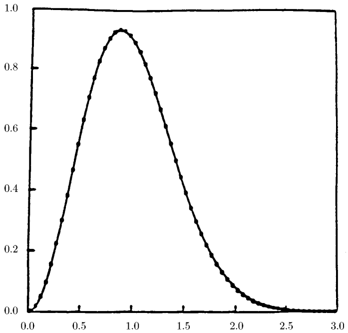
$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f) L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \quad = \quad -\frac{d}{ds}\log\zeta(s) \; = \; -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p\log\left(1-p^{-s}\right)$$

$$= \sum_p\frac{\log p\cdot p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds \quad \text{vs} \quad \sum_p\log p\int\left(\frac{x}{p}\right)^s\frac{ds}{s}.$$

Explicit Formula (Contour Integration)

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$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.
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Explicit Formula (Contour Integration)

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&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)e^{-\sigma\log p}e^{-it\log p}ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cuspidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned}
\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi)\\
&\quad + O\left(\frac{\log\log R}{\log R}\right)\\
P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.
\end{aligned}$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1} - \alpha_{j_2}, \dots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: *n*-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: *n*-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

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Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1}\sum_{f\in\mathcal{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\begin{aligned}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) &= \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n \\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right) \\ &\to \int\cdots\int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx.\end{aligned}$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Correspondences

Similarities between *L*-Functions and Nuclei:

$$\text{Zeros} \longleftrightarrow \text{Energy Levels}$$

$$\text{Schwartz test function} \longrightarrow \text{Neutron}$$

$$\text{Support of test function} \longleftrightarrow \text{Neutron Energy.}$$

Some Number Theory Results

- Orthogonal: Iwaniec-Luo-Sarnak, Ricotta-Royer: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).
- Symplectic: Rubinstein, Gao, Levinson-Miller, and Entin, Roddity-Gershon and Rudnick: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.
- Unitary: Fiorilli-Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.
- Orthogonal: Miller, Young: One and two-parameter families of elliptic curves.

Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.

2. Explicit Formula: Relates sums over zeros to sums over primes.

3. Averaging Formulas: Petersson formula in Iwaniec-Luo-Sarnak, Orthogonality of characters in Fiorilli-Miller, Gao, Hughes-Rudnick, Levinson-Miller, Rubinstein.

Applications of *n*-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Applications of n-level density

One application: bounding the order of vanishing at the central point.
$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.
Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

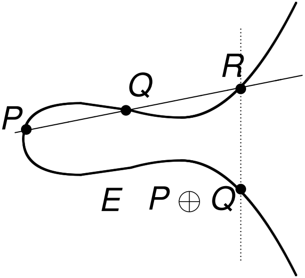
Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

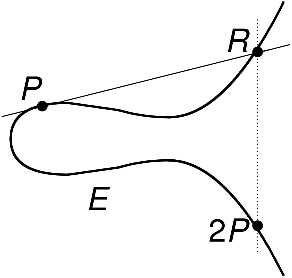
Elliptic Curves

Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)/\sqrt{n}}{n^s} \;=\; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

One parameter family

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T), \;\; A(T), B(T) \in \mathbb{Z}[T].$$

Silverman's Specialization Theorem

Assume (geometric) rank of $\mathcal{E}/\mathbb{Q}(T)$ is r. Then for all $t \in \mathbb{Z}$ sufficiently large, each $E_t : y^2 = x^3 + A(t)x + B(t)$ has (geometric) rank at least r.

Average rank conjecture

For a generic one-parameter family of rank r over $\mathbb{Q}(T)$, expect in the limit half the specialized curves have rank r and half have rank $r + 1$.

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r + 2$.
2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Theory and Models

Orthogonal Random Matrix Models

RMT: $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} \ = \ \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model: Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$: $1 \ \leq \ j,k \ \leq \ N-r$:

$$d\varepsilon_{2r}(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Indep):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right)$$

$$= \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

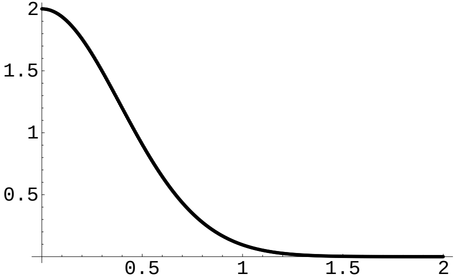
where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

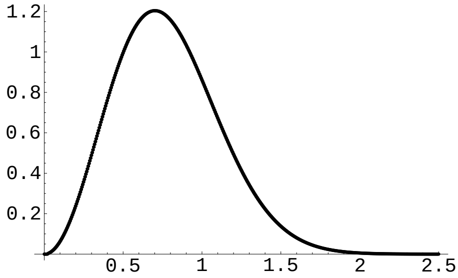
Data

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

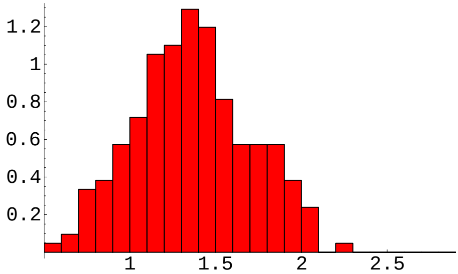


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

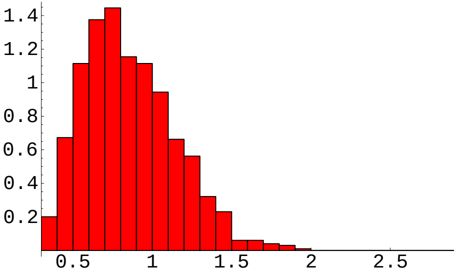


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

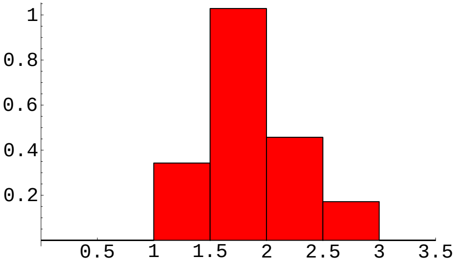


Figure 5*a*: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

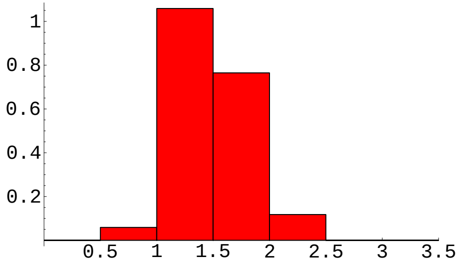


Figure 5*b*: 34 curves, $\log(\mathrm{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	701 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Summary of Data

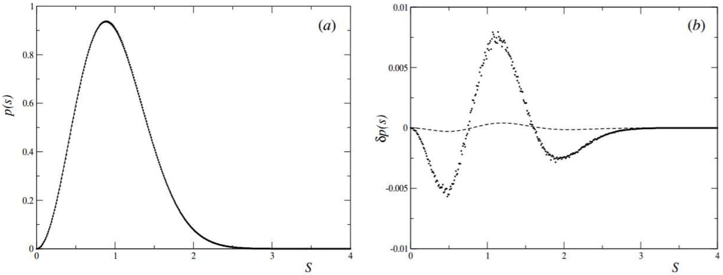
- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i. e., shifted by the same amount).

Convergence to the RMT limit: What's the right matrix size?

- RMT + Katz-Sarnak: *Limiting behavior* for random matrices as $N \to \infty$ and L-functions as conductors tend to infinity agree.

- How well do the classical matrix groups model local statistics of L-functions *outside* the scaling limit? (Arithmetic enters!)

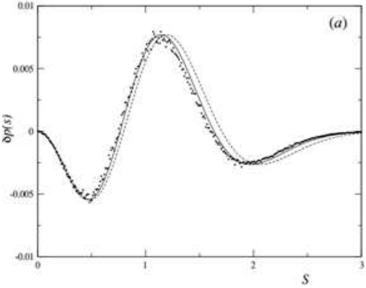
Convergence to the RMT limit



L: 70 million $\zeta(s)$ nearest-neighbor spacings (Odlyzko).

R: Difference b/w $\zeta(s)$ and asymptotic CUE curve (dots) compared to difference b/w CUE of size N_0 and asymptotic curve (dashed line) (from Bogomolny et. al.).

Convergence to the RMT limit: Incorporating Finite Matrix Size



Difference b/w nearest-neighbor spacing of $\zeta(s)$ zeros and asymptotic CUE for a billion zeros in window near 2.504×10^{15} (dots) compared to theory that takes into account arithmetic of lower order terms (full line) (from Bogomolny et. al.).

New model should incorporate finite matrix size....

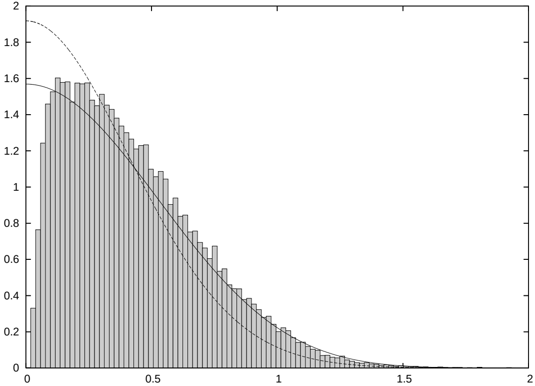
New Model for Finite Conductors

- Replace conductor N with $N_{\text{effective}}$.
  - ⋄ Arithmetic info, predict with L-function Ratios Conj.
  - ⋄ Do the number theory computation.

- Excised Orthogonal Ensembles.
  - ⋄ $L(1/2, E)$ discretized.
  - ⋄ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.

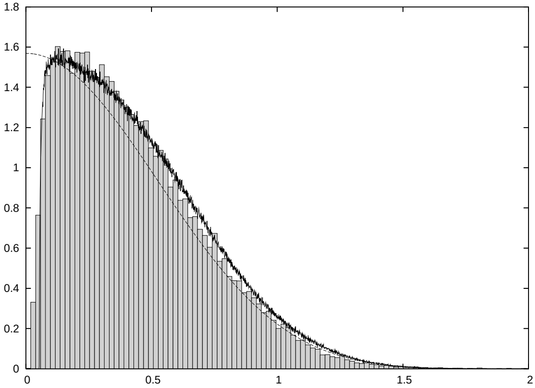
- Painlevé VI differential equation solver.
  - ⋄ Use explicit formulas for densities of Jacobi ensembles.
  - ⋄ Key input: Selberg-Aomoto integral for initial conditions.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_{\rm eff}$ (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Ratio's Conjecture

History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} \;=\; \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ◇ n-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point;
- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) \;=\; \sum_{m\le x}\frac{a_m}{m^s} + \epsilon\mathbb{X}_L(s)\sum_{n\le y}\frac{a_n}{n^{1-s}};$$

◇ ϵ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

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◇ ϵ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}}\omega_f\sum_{\gamma}g\left(\gamma\frac{\log N_f}{2\pi}\right) \;=\; \frac{1}{2\pi i}\int_{(c)}-\int_{(1-c)}R'_{\mathcal{F}}(\cdots)g\left(\cdots\right)$$

◇ $R'_{\mathcal{F}}(r)=\frac{\partial}{\partial\alpha}R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}.$

Procedure (Recipe)

- Use approximate functional equation to expand numerator.

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- Expand denominator by generalized Mobius function: cusp form

$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Procedure ('Illegal Steps')

- Use approximate functional equation to expand numerator.
- Expand denominator by generalized Mobius function: cusp form

$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

  where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.
- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

1-Level Prediction from Ratio's Conjecture

$$
\begin{aligned}
& A_E(\alpha,\gamma) \\
= \quad & Y_E^{-1}(\alpha,\gamma) \times \prod_{p|M}\left(\sum_{m=0}^{\infty}\left(\frac{\lambda(p^m)\omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)}{p^{1/2+\gamma}}\frac{\lambda(p^m)\omega_E^{m+1}}{p^{m(1/2+\alpha)}}\right)\right) \times \\
& \prod_{p\nmid M}\left(1+\frac{p}{p+1}\left(\sum_{m=1}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}}\right.\right. \\
& \left.\left. + \frac{1}{p^{1+2\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}}\right)\right)
\end{aligned}
$$

where

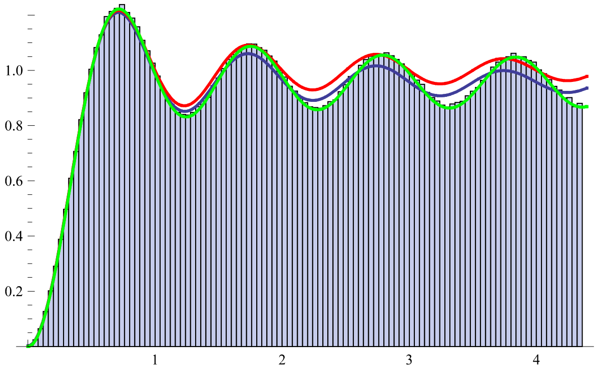
$$
Y_E(\alpha,\gamma) = \frac{\zeta(1+2\gamma)L_E(\mathrm{sym}^2, 1+2\alpha)}{\zeta(1+\alpha+\gamma)L_E(\mathrm{sym}^2, 1+\alpha+\gamma)}.
$$

Huynh, Morrison and Miller confirmed Ratios' prediction, which is

1-Level Prediction from Ratio's Conjecture

$$
\begin{aligned}
&\frac{1}{X^*}\sum_{d\in\mathcal{F}(X)}\sum_{\gamma_d} g\left(\frac{\gamma_d L}{\pi}\right)\\
&= \frac{1}{2LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)+\frac{\Gamma'}{\Gamma}\left(1+\frac{i\pi\tau}{L}\right)+\frac{\Gamma'}{\Gamma}\left(1-\frac{i\pi\tau}{L}\right)\right]d\tau\\
&+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\left(-\frac{\zeta'}{\zeta}\left(1+\frac{2\pi i\tau}{L}\right)+\frac{L_E'}{L_E}\left(\mathrm{sym}^2,1+\frac{2\pi i\tau}{L}\right)-\sum_{\ell=1}^{\infty}\frac{(M^\ell-1)\log M}{M^{\left(2+\frac{2i\pi\tau}{L}\right)\ell}}\right)d\tau\\
&\quad-\frac{1}{L}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g(\tau)\frac{\log M}{M^{(k+1)(1+\frac{\pi i\tau}{L})}}d\tau+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\sum_{p\nmid M}\frac{\log p}{(p+1)}\sum_{k=0}^{\infty}\frac{\lambda(p^{2k+2})-\lambda(p^{2k})}{p^{(k+1)(1+\frac{2\pi i\tau}{L})}}d\tau\\
&-\frac{1}{LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2i\pi\tau/L}\frac{\Gamma(1-\frac{i\pi\tau}{L})}{\Gamma(1+\frac{i\pi\tau}{L})}\frac{\zeta(1+\frac{2i\pi\tau}{L})L_E(\mathrm{sym}^2,1-\frac{2i\pi\tau}{L})}{L_E(\mathrm{sym}^2,1)}\right.\\
&\left.\times A_E\left(-\frac{i\pi\tau}{L},\frac{i\pi\tau}{L}\right)\right]d\tau+O(X^{-1/2+\varepsilon});
\end{aligned}
$$

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

◇ Red: main term. ◇ Blue: includes $O(1/\log X)$ terms.

◇ Green: all lower order terms.

Excised Orthogonal Ensembles

Excised Orthogonal Ensemble: Preliminaries

Characteristic polynomial of $A \in SO(2N)$ is

$$\Lambda_A(e^{i\theta}, N) := \det(I - Ae^{-i\theta}) = \prod_{k=1}^{N}(1 - e^{i(\theta_k - \theta)})(1 - e^{i(-\theta_k - \theta)}),$$

with $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_N}$ the eigenvalues of A.

Motivated by the arithmetical size constraint on the central values of the L-functions, consider Excised Orthogonal Ensemble $T_{\mathcal{X}}$: $A \in SO(2N)$ with $|\Lambda_A(1, N)| \geq \exp(\mathcal{X})$.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.

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where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.
The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) := C_{\mathcal{X}} \cdot N \int_0^{\pi} \cdots \int_0^{\pi} H(\log |\Lambda_A(1, N)| - \mathcal{X}) \times \\ \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N,$$

Here $H(x)$ denotes the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0, \end{cases}$$

and $C_{\mathcal{X}}$ is a normalization constant

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where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.
The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) = \frac{C_{\mathcal{X}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{Nr} \frac{\exp(-r\mathcal{X})}{r} R_1^{J_N}(\theta_1; r-1/2, -1/2) dr$$

where $C_{\mathcal{X}}$ is a normalization constant and

$$R_1^{J_N}(\theta_1; r-1/2, -1/2) = N \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} w^{(r-1/2,-1/2)}(\cos\theta_j) \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N$$

is the one-level density for the Jacobi ensemble J_N with weight function

$$w^{(\alpha,\beta)}(\cos\theta) = (1-\cos\theta)^{\alpha+1/2}(1+\cos\theta)^{\beta+1/2}, \qquad \alpha = r-1/2 \text{ and } \beta = -1/2.$$

Results

- With $C_{\mathcal{X}}$ normalization constant and $P(N,r,\theta)$ defined in terms of Jacobi polynomials,

$$R_1^{T_{\mathcal{X}}}(\theta) = \frac{C_{\mathcal{X}}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(-r\mathcal{X})}{r} 2^{N^2+2Nr-N} \times$$
$$\times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)} \times$$
$$\times (1-\cos\theta)^r \frac{2^{1-r}}{2N+r-1} \frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)} P(N,r,\theta)\,dr.$$

- Residue calculus implies $R_1^{T_{\mathcal{X}}}(\theta) = 0$ for $d(\theta,\mathcal{X}) < 0$ and

$$R_1^{T_{\mathcal{X}}}(\theta) = R_1^{\mathrm{SO}(2N)}(\theta) + C_{\mathcal{X}} \sum_{k=0}^{\infty} b_k \exp((k+1/2)\mathcal{X}) \quad \text{for } d(\theta,\mathcal{X}) \geq 0,$$

where $d(\theta,\mathcal{X}) := (2N-1)\log 2 + \log(1-\cos\theta) - \mathcal{X}$ and b_k are coefficients arising from the residues. As $\mathcal{X} \to -\infty$, θ fixed, $R_1^{T_{\mathcal{X}}}(\theta) \to R_1^{\mathrm{SO}(2N)}(\theta)$.

Numerical check

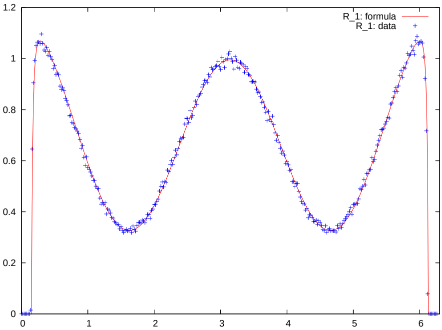

Figure: One-level density of excized SO($2N$), $N = 2$ with cut-off $|\Lambda_A(1, N)| \geq 0.1$. The red curve uses our formula. The blue crosses give the empirical one-level density of 200,000 numerically generated matrices.

Theory vs Experiment

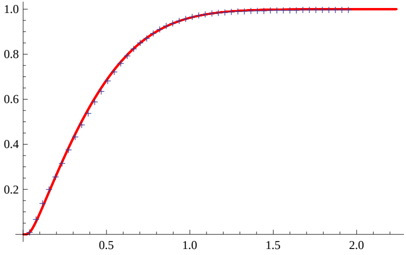

Figure: Cumulative probability density of the first eigenvalue from 3×10^6 numerically generated matrices $A \in SO(2N_{\rm std})$ with $|\Lambda_A(1, N_{\rm std})| \geq 2.188 \times \exp(-N_{\rm std}/2)$ and $N_{\rm std} = 12$ red dots compared with the first zero of even quadratic twists $L_{E_{11}}(s, \chi_d)$ with prime fundamental discriminants $0 < d \leq 400{,}000$ blue crosses. The random matrix data is scaled so that the means of the two distributions agree.

Conclusion and References

Conclusion and Future Work

- In the limit: Birch and Swinnerton-Dyer, Katz-Sarnak appear true.
- Finite conductors: model with Excised Ensembles (cut-off on characteristic polynomials due to discretization at central point).
- Future Work: Joint with Owen Barrett and Nathan Ryan (and possibly some of his students): looking at other GL2 families (and hopefully higher) to study the relationship between repulsion at finite conductors and central values (effect of weight, level).

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Limiting Behavior
(joint with John Goes)

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Supports Independent and not Interaction model in the limit.

Previous Results on low-lying zeros

Expect zeros near central point of size $\frac{1}{\log N_E}$.

Mestre: zero with imaginary part at most $\frac{B}{\log \log N_E}$.

Goal: bound (from above and below) number of zeros in a neighborhood of size $\frac{1}{\log N_E}$ near the central point in a family.

One-Level Density: Sketch of result

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\tilde{\gamma}_{j,f}}\phi(\tilde{\gamma}_{j,f}) = \left(r+\frac{1}{2}\right)\phi(0)+\hat{\phi}(0)+O\left(\frac{\log\log R}{\log R}\right)$$

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$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{|\tilde{\gamma}_{j,f}|\leq\tau}\phi(\tilde{\gamma}_{j,f}) \geq \left(r+\frac{1}{2}\right)\phi(0)+\hat{\phi}(0)+O\left(\frac{\log\log R}{\log R}\right)$$

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$$N_{ave}(\tau,R)\phi(0) \geq \left(r+\frac{1}{2}\right)\phi(0)+\hat{\phi}(0)+O\left(\frac{\log\log R}{\log R}\right)$$

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$$N_{ave}(\tau,R) \ge \left(r+\frac{1}{2}\right)+\frac{\widehat{\phi}(0)}{\phi(0)}+O\left(\frac{\log\log R}{\log R}\right)$$

Towards an average version of Birch and Swinnerton-Dyer

Theorem (Goes, M–)

Let $\tau = C(\phi)/\sigma$, where σ is the support of $\widehat{\phi}$, $C(\phi)$ is constant depending on the choice of test function, and $N_{\rm ave}(\tau, R)$ the average number of normalized zeros in $(-\tau, \tau)$ for $t \in [R, 2R]$. Then assuming GRH

$$N_{\rm ave}(\tau, R) \ \geq \ \left(r + \frac{1}{2}\right) + \frac{\widehat{\phi}(0)}{\phi(0)} + O\left(\frac{\log\log R}{\log R}\right).$$

Technical requirements for ϕ:

- ϕ even, positive in $(-\tau, \tau)$, negative elsewhere;
- ϕ monotonically decreasing on $(0, \tau)$;
- ϕ differentiable;
- $\widehat{\phi}$ compactly supported in $(-\sigma, \sigma)$.

Construction Preliminaries

- Convolution:

$$(A * B)(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt.$$

- Fourier Transform:

$$\begin{aligned} \widehat{A}(y) &= \int_{-\infty}^{\infty} A(x)e^{-2\pi ixy}dx \\ \widehat{A''}(y) &= -(2\pi y)^2\widehat{A}(y). \end{aligned}$$

- Lemma: $\widehat{(A * B)}(y) = \widehat{A}(y) \cdot \widehat{B}(y)$;
  in particular, $\widehat{(A * A)}(y) = \widehat{A}(y)^2 \geq 0$ if A is even.

Constructing good ϕ's

- Let h be supported in $(-1, 1)$.
- Let $f(x) = h(2x/\sigma)$, so f supported in $(-\sigma/2, \sigma/2)$.
- Let $g(x) = (f * f)(x)$, so g supported in $(-\sigma, \sigma)$.
  $\widehat{g}(y) = \widehat{f}(y)^2$.
- Let $\phi(y) := (\widehat{g + \beta^2 g''})(y) = \widehat{f}(y)^2 \left(1 - (2\pi\beta y)^2\right)$.
  For β sufficiently small above is non-negative.

Constructing good ϕ's (cont)

$N_{\rm ave}(\tau, R)$ is average number of zeros in $(-\tau, \tau)$, and

$$N_{\rm ave}(\tau, R) \ \geq \ \left(r + \frac{1}{2}\right) + \frac{\widehat{\phi}(0)}{\phi(0)} + O\left(\frac{\log\log R}{\log R}\right).$$

Want to maximize $\widehat{\phi}(0)/\phi(0)$, which is

$$\mathcal{P}_\beta \ := \ \frac{(\int_0^1 h(u)^2 du) + (\frac{2\beta}{\sigma})^2 (\int_0^1 h(u) h''(u) du)}{\sigma(\int_0^1 h(u) du)^2}.$$

Birch and Swinnerton-Dyer on "average"

Setting $\mathcal{P}_\beta = 0$ gives $\beta = \mathrm{C}(h)\sigma$ gives

Theorem (Goes, M–)

Assume GRH and let $\beta = \mathrm{C}(h)\sigma$ so that $\mathcal{P}_\beta = 0$. Then there are on average at least $r + \frac{1}{2}$ normalized zeros within the band $\left(-\frac{1}{2\pi\mathrm{C}(h)\sigma}, \frac{1}{2\pi\mathrm{C}(h)\sigma}\right)$ for $t \in [R, 2R]$.

Using $h(x) = (1 - x^2)^2$ gives at least $r + \frac{1}{2}$ normalized zeros on average within the band $\approx \left(-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma}\right)$

Results for certain test functions

$h(x) = 0$ for $|x| > 1$, and

- Class: $h(x) = (1 - x^{2k})^{2j}$,$(j, k \in \mathbb{Z})$
  Optimum: $h(x) = (1 - x^2)^2$
  gives interval approximately $\left(-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma}\right)$.

- Class: $h(x) = \exp\left(-1/(1 - x^{2k})\right)$,$(k \in \mathbb{Z})$
  Optimum: $h(x) = \exp\left(-1/(1 - x^2)\right)$
  gives approximately $\left(-\frac{0.558415}{\sigma}, \frac{0.558415}{\sigma}\right)$.

- Class: $h(x) = \exp\left(-k/(1 - x^2)\right)$
  Optimum: $h(x) = \exp\left(-.754212/(1 - x^2)\right)$
  gives approximately $\left(-\frac{0.552978}{\sigma}, \frac{0.552978}{\sigma}\right)$.

Upper bounds

Theorem (Goes, M–)

For an elliptic curve with explicit formulas as above, the number of normalized zeros within $(-\tau, \tau)$ is bounded above by $(r+\frac{1}{2}) + \frac{(r+\frac{1}{2})(\psi(0)-\psi(\tau))+\hat{\psi}(0)}{\psi(\tau)}$, for all strictly positive, even test functions monotonically decreasing over $(0, \infty)$.