Power Sums of Primes in Arithmetic Progression

Zhangze Li*, Muhammet Boran, John Byun
Mentor: Steven J. Miller, Stephanie Reyes

Polymath 2023

01/06/2024

Introduction

Definition 1.1

- For $k \in \mathbb{R}$, define

$$\pi(x) := \sum_{p \leq x} 1 \qquad \text{and} \qquad \pi_k(x) := \sum_{p \leq x} p^k,$$

where p denotes a prime number here and throughout; $\pi(x)$ is the *prime-counting function*.

Introduction

Definition 1.1

- For $k \in \mathbb{R}$, define

$$\pi(x) := \sum_{p \leq x} 1 \qquad \text{and} \qquad \pi_k(x) := \sum_{p \leq x} p^k,$$

where p denotes a prime number here and throughout; $\pi(x)$ is the *prime-counting function*.

Definition 1.2

- For real $x \geq 0$, define the *logarithmic integral*

$$\text{li}(x) := \int_0^x \frac{dt}{\log t}.$$

Introduction

Definition 1.3

- Let f and g be two functions defined on $\mathbb{R}_{>0}$, and $g(x)$ be strictly positive for all large enough values of x. We say

$$f(x) = O(g(x)) \quad \text{as} \quad x \to \infty$$

if there exist constants M and x_0 such that

$$|f(x)| \leq M|g(x)| \quad \text{for all} \quad x \geq x_0.$$

Introduction

Definition 1.4

- Let f and g be two functions defined on $\mathbb{R}_{>0}$, and $g(x)$ be strictly positive for all large enough values of x. We say

$$f(x) \sim g(x) \qquad \text{as} \qquad x \to \infty,$$

read $f(x)$ *is asymptotic to* $g(x)$, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \;=\; 1.$$

Historical Context

- In 1896, Vallée-Poussin proved that

$$\pi(x) \sim \mathrm{li}(x) \qquad \text{as} \qquad x \to \infty,$$

which is known as the *prime number theorem*.

- Vallée-Poussin also estimated the error term in the prime number theorem by proving that

$$\pi(x) \;=\; \mathrm{li}(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right) \quad (x \geq 2)$$

where c is a positive constant.

Historical Context

- In 1896, Vallée-Poussin proved that

$$\pi(x) \sim \text{li}(x) \qquad \text{as} \qquad x \to \infty,$$

which is known as the *prime number theorem*.

- Vallée-Poussin also estimated the error term in the prime number theorem by proving that

$$\pi(x) \;=\; \text{li}(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right) \quad (x \geq 2)$$

where c is a positive constant.

- In 1958, Korobov and Vinogradov proved

$$\pi(x) = \text{li}(x) + O\left(x \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right) \quad (x \geq 2).$$

Historical Context

- In 2016, Gerard and Washington proved that $\pi_k(x)$ is asymptotic to $\pi(x^{k+1})$:

$$\pi_k(x) - \pi(x^{k+1})$$
$$= \begin{cases} O\left(x^{k+1}\exp\left(-A(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right) & \text{if } k \in (0,\infty), \\ O\left(x^{k+1}\exp\left(-(k+1)^{3/5}A(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right) & \text{if } k \in (-1,0), \end{cases}$$

where $A = .2098$.

- This result greatly contributes to approximate sums of powers of primes in arithmetic progression in our research.

Statement of Main Results

Definition 1.5

- For $k \in \mathbb{R}$, define

$$\pi(x; m, n) = \sum_{\substack{p \le x \\ p \equiv n \pmod{m}}} 1 \quad \text{and} \quad \pi_k(x; m, n) = \sum_{\substack{p \le x \\ p \equiv n \pmod{m}}} p^k,$$

where $m, n \in \mathbb{Z}_{>0}$ are coprime, with n a unit in $\mathbb{Z}/m\mathbb{Z}$.

Statement of Main Results

Definition 1.5

- For $k \in \mathbb{R}$, define

$$\pi(x;m,n) \;=\; \sum_{\substack{p \le x \\ p \equiv n \pmod{m}}} 1 \qquad \text{and} \qquad \pi_k(x;m,n) \;=\; \sum_{\substack{p \le x \\ p \equiv n \pmod{m}}} p^k,$$

where $m, n \in \mathbb{Z}_{>0}$ are coprime, with n a unit in $\mathbb{Z}/m\mathbb{Z}$.

- Also, recall the definition of Euler's totient function:

$$\varphi(m) \;=\; \#\{x \in \mathbb{N} \;:\; x \le m \text{ and } \gcd(x,m) = 1\},$$

which counts the number of positive integers up to m which are coprime to m.

Statement of Main Results

- Vallée-Poussin also proved the *prime number theorem on arithmetic progressions*, concluding that

$$\pi(x; m, n) \sim \frac{\pi(x)}{\varphi(m)} \sim \frac{\mathrm{li}(x)}{\varphi(m)} \qquad \text{as} \qquad x \to \infty,$$

where $\gcd(m, n) = 1$.

Statement of Main Results

- Vallée-Poussin also proved the *prime number theorem on arithmetic progressions*, concluding that

$$\pi(x; m, n) \sim \frac{\pi(x)}{\varphi(m)} \sim \frac{\mathrm{li}(x)}{\varphi(m)} \qquad \text{as} \qquad x \to \infty,$$

where $\gcd(m, n) = 1$.

- In 1935, Page and Siegel separately proved that

$$\pi(x; m, n) \quad = \quad \frac{\mathrm{li}(x)}{\varphi(m)} + O\left(x \exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) \ (x \geq 2)$$

where α is a positive constant.

- We use the above version of the prime number theorem on arithmetic progressions to obtain our main result.

Statement of Main Results

Theorem 2.1 (Boran, Byun, Li, Miller, Reyes 2023)

- Fix a real number $k > -1$ and positive integers $m, n \in \mathbb{Z}_{>0}$ such that $\gcd(m, n) = 1$. Then we can approximate the number of primes $p \equiv n \pmod{m}$ less than x^{k+1} by the sum of k-powers of primes $p \equiv n \pmod{m}$ less than a real number x:

$$\pi_k(x; m, n) - \pi(x^{k+1}; m, n)$$
$$= \begin{cases} O\left(x^{k+1} \exp\left(-\frac{1}{2}\alpha\sqrt{\log x}\right)\right) & \text{if } k > 0, \\ O\left(x^{k+1} \exp\left(-\frac{1}{2}\alpha\sqrt{(k+1)\log x}\right)\right) & \text{if } -1 < k < 0, \end{cases}$$

where α is a positive constant.

Proof: Riemann–Stieltjes integral.

Statement of Main Results

Theorem 2.2 (Boran, Byun, Li, Miller, Reyes 2023)

- Fix a real number $k > -1$ and positive integers $m, n \in \mathbb{Z}_{>0}$ such that $\gcd(m, n) = 1$. Then

$$\int_1^\infty \frac{\pi_k(t; m, n) - \pi(t^{k+1}; m, n)}{t^{k+2}}\, dt \;=\; -\frac{\log(k+1)}{(k+1)\varphi(m)}.$$

- Further,

$$\begin{aligned} -\frac{\log(k+1)}{(k+1)\varphi(m)} \;&<\; 0 \qquad (k > 0) \\ -\frac{\log(k+1)}{(k+1)\varphi(m)} \;&>\; 0 \qquad (-1 < k < 0). \end{aligned}$$

Proof: Integration by parts.

Discussion

Question 1

- **Theorem 2.1** indicates that $\pi(x^{k+1}; m, n)$ can be well approximated by $\pi_k(x; m, n)$.
- Can we quantify the magnitude of the error for some specific data examples?

Discussion

Question 2

- **Theorem 2.2** suggests that

$$\int_1^\infty \frac{\pi_k(t; m, n) - \pi(t^{k+1}; m, n)}{t^{k+2}}\, dt \;=\; -\frac{\log(k+1)}{(k+1)\varphi(m)}.$$

- Therefore, it seems there is a trend that

$$\pi_k(x; m, n) - \pi(x^{k+1}; m, n)$$

  tends to be negative when $k > 0$ and tends to be positive that when $-1 < k < 0$.

- Can we confirm this trend through data analysis?

Discussion

Remark

- We will see how closely $\pi_k(x^{1/(k+1)}; m, n)$ approximates $\pi(x; m, n)$ through data. Utilizing High Performance Computing, we provide calculations of various examples for different values of k, m, and n.
- In the following examples,

$$\text{Error } (x, k; m, n) \;:=\; \frac{\pi(x; m, n) - \pi_k(x^{1/(k+1)}; m, n)}{\pi(x; m, n)}.$$

- Remember that

$$\int_1^{\infty} \frac{\pi_k(t; m, n) - \pi(t^{k+1}; m, n)}{t^{k+2}}\, dt \;=\; -\frac{\log(k+1)}{(k+1)\varphi(m)}.$$

Discussion

x	$\pi(x;4,1)$	$\pi_1(x^{1/2};4,1)$	Error %
1×10^4	609	515	15.43514%
5×10^4	2549	2025	20.55708%
1×10^5	4783	4418	7.63119%
5×10^5	20731	19668	5.12759%
1×10^6	39175	36628	6.50160%
5×10^6	174193	165373	5.06335%
1×10^7	332180	323048	2.74911%
5×10^7	1500452	1475230	1.68096%
1×10^8	2880504	2863281	0.59792%

Figure: $\pi(x;4,1)$ and $\pi_1(x^{1/2};4,1)$.

Discussion

x	$\pi(x;4,1)$	$\pi_{1/2}(x^{2/3};4,1)$	Error %
1×10^4	609	617.62512	−1.41628%
5×10^4	2549	2477.64505	2.79933%
1×10^5	4783	4659.83812	2.57499%
5×10^5	20731	20125.89212	2.91886%
1×10^6	39175	38904.00140	0.69176%
5×10^6	174193	173246.23939	0.54351%
1×10^7	332180	329252.45078	0.88131%
5×10^7	1500452	1492885.30185	0.50429%
1×10^8	2880504	2873027.62482	0.25955%

Figure: $\pi(x;4,1)$ and $\pi_{1/2}(x^{2/3};4,1)$.

Discussion

x	$\pi(x;4,1)$	$\pi_{-1/10}(x^{10/9};4,1)$	Error %
1×10^4	609	613.50169	−0.73919%
5×10^4	2549	2562.89963	−0.54530%
1×10^5	4783	4788.03485	−0.10527%
5×10^5	20731	20771.18437	−0.19384%
1×10^6	39175	39266.51644	−0.23361%
5×10^6	174193	174232.64634	−0.02276%
1×10^7	332180	332314.25320	−0.04042%
5×10^7	1500452	1500545.39963	−0.00622%
1×10^8	2880504	2880813.47274	−0.01074%

Figure: $\pi(x;4,1)$ and $\pi_{-1/10}(x^{10/9};4,1)$.

Discussion

x	$\pi(x;5,1)$	$\pi_{-1/10}(x^{10/9};5,1)$	Error %
1×10^4	306	306.84917	−0.27750%
5×10^4	1274	1284.72538	−0.84187%
1×10^5	2387	2397.35493	−0.43380%
5×10^5	10386	10381.05165	−0.04764%
1×10^6	19617	19624.30360	−0.03723%
5×10^6	87062	87119.41013	−0.06594%
1×10^7	166104	166161.85950	−0.03483%
5×10^7	750340	750274.35740	0.00875%
1×10^8	1440298	1440380.00374	−0.00569%

Figure: $\pi(x;5,1)$ and $\pi_{-1/10}(x^{10/9};5,1)$.

Discussion

Remark

- $$\int_1^\infty \frac{\pi_k(t;m,n) - \pi(t^{k+1};m,n)}{t^{k+2}}\,dt \;=\; -\frac{\log(k+1)}{(k+1)\varphi(m)}.$$
- The integral computation in **Theorem 2.2** confirms the observations from our data that
$$\pi_k(x;m,n) - \pi(x^{k+1};m,n)$$
tends to be negative when $k > 0$, and tends to be positive when $-1 < k < 0$ for large values of x.
- Note that this does not definitively specify the sign of the error for given large values of x. We can only conclude that the "net" sign will either be negative or positive for specific k.

Future Direction

Remark

- A more refined approach to evaluating $\pi_k(x; m, n) - \pi(x^{k+1}; m, n)$ when x is sufficiently large necessitates the utilization of the Riemann ζ-function and the Riemann Hypothesis to analyze the bias between $\pi(x; m, n)$ and $\mathrm{li}(x)$.

Future Direction

Remark

- A more refined approach to evaluating $\pi_k(x; m, n) - \pi(x^{k+1}; m, n)$ when x is sufficiently large necessitates the utilization of the Riemann ζ-function and the Riemann Hypothesis to analyze the bias between $\pi(x; m, n)$ and $\mathrm{li}(x)$.

Riemann Hypothesis

- Riemann ζ-function: $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$.
- The Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $1/2$.

Future Direction

Theorem 3.1 (possible)

There exists $M > 0$ such that for every $0 < k \leq M$, the followings are equivalent:

1. Riemann hypothesis.
2. $$\int_1^x \pi_k(t; m, n) - \pi(t^{k+1}; m, n)\, dt < 0 \quad \text{for all } x \text{ sufficiently large.}$$

Acknowledgement

- Our work was partially supported by NSF award DMS-2113535.
- High Performance Computing technique is supported by the Great Lakes Slurm cluster at the University of Michigan, where Zhangze Li is currently enrolled as an undergraduate student, and the National Science Foundation under Grant No. DMS-1929284 while Stephanie Reyes was in residence at the Institute for Computational and Experimental Research in Mathematics during the Summer@ICERM program.
- Also, thanks the opportunity provided by Polymath 2023 program!

References

J. Gerard, L. C. Washington, **Sums of powers of primes**, *Ramanujan J.* **45**, (2018), 171–180.

L. C. Washington, **Sums of powers of primes II**, ArXiv: 2209.12845v1 (2022).

Muhammet Boran, John Byun, Zhangze Li, Steven J. Miller, and Stephanie Reyes, **Sums of Powers of Primes in Arithmetic Progression**, ArXiv: 2309.16007 (2023).