Analytic Number Theory Seminar

Random Matrix Theory & L-Functions: II

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Review

- Similar behavior in different systems.
- Find correct scale.
- Average over similar elements.
- Need an Explicit Formula.
- Different statistics tell different stories.

n-Level Density and Families

Let $f(x) = \prod_i f_i(x_i)$, f_i even Schwartz functions whose Fourier Transforms are compactly supported.

$$D_{n,E}(f) \;=\; \sum_{\substack{j_1,\dots,j_n \\ distinct}} f_1\left(L_E\gamma_E^{(j_1)}\right)\cdots f_n\left(L_E\gamma_E^{(j_n)}\right)$$

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

To any geometric family, Katz-Sarnak predict the n-level density depends only on a symmetry group attached to the family.

Elliptic Curves

Consider $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $a_i \in \mathbb{Q}$.

Often can write as $y^2 = x^3 + Ax + B$.

Let N_p be the number of solns mod p:

$$N_p = \sum_{x(p)} \left[1 + \left(\frac{x^3 + Ax + B}{p}\right)\right] = p + \sum_{x(p)} \left(\frac{x^3 + Ax + B}{p}\right)$$

Local data: $a_p = p - N_p$. Use to build the L-function.

One-parameter families:

$$y^2 = x^3 + A(t)x + B(t), \quad A(t), B(t) \in \mathbb{Z}(t).$$

Elliptic Curves (cont)

Modularity Theorem [Wiles]: $L(s,E) = L(s,f)$ for a weight 2 cuspidal newform of level N_E.

$$\Lambda(s,E) = (2\pi)^{-s} N^{s/2} \Gamma(s+\frac{1}{2}) L(s,E) = \epsilon_E \Lambda(1-s,E)$$

By GRH: All zeros on the critical line. Makes sense to talk about spacings between zeros.

Rational solutions a group: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$.

Birch and Swinnerton-Dyer Conjecture: Geometric rank equals the analytic rank.

Normalization of Zeros

Local (hard) vs Global (easy).

Hope: for f a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} D_{n,E}(f) = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i f_i\left(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int f(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx$$

$$= \int\cdots\int \widehat{f}(u)\widehat{W_{n,\mathcal{G}(\mathcal{F})}}(u)du.$$

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,O^+}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,O}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,O^-}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and $\eta(u)$ is 1, $\frac{1}{2}$, and 0 for $|u|$ less than 1, 1, and greater than 1.

2-Level Densities

Let $c(\mathcal{G}) = 0$, $\frac{1}{2}$ or 1 for $\mathcal{G} = SO(\text{even})$, O, and $SO(\text{odd})$. For $\mathcal{G}$ one of these three groups we have

$$\begin{aligned}\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,\mathcal{G}}}(u)du_1du_2 &= \left[\widehat{f_1}(0) + \frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{1}{2}f_2(0)\right] \\ &\quad + 2\int |u|\widehat{f_1}(u)\widehat{f_2}(u)du - 2\widehat{f_1f_2}(0) \\ &\quad -f_1(0)f_2(0) \\ &\quad + c(\mathcal{G})f_1(0)f_2(0).\end{aligned}$$

For $\mathcal{G} = U$ we have

$$\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,U}}(u)du_1du_2 = \widehat{f_1}(0)\widehat{f_2}(0) + \int |u|\widehat{f_1}(u)\widehat{f_2}(u)du - \widehat{f_1f_2}(0),$$

and for $\mathcal{G} = Sp$, we have

$$\begin{aligned}\int\int \widehat{f_1}(u_1)\widehat{f_2}(u_2)\widehat{W_{2,\mathcal{G}}}(u)du_1du_2 &= \left[\widehat{f_1}(0) + \frac{1}{2}f_1(0)\right]\left[\widehat{f_2}(0) + \frac{1}{2}f_2(0)\right] \\ &\quad + 2\int |u|\widehat{f_1}(u)\widehat{f_2}(u)du - 2\widehat{f_1f_2}(0) \\ &\quad - f_1(0)f_2(0) \\ &\quad -f_1(0)\widehat{f_2}(0) - \widehat{f_1}(0)f_2(0) + 2f_1(0)f_2(0).\end{aligned}$$

These densities are all distinguishable for functions with arbitrarily small support.

For the orthogonal groups, the densities (in this range) depend solely on the distribution of sign.

2-Level Density: Orthogonal Groups

For small support, the difference due to distribution of signs.

Subtract off $j_1 = \pm j_2$ terms.

Let $\rho = 1 + i\gamma_E^{(j)}$ be a zero.

Even functional equation, label the zeros by

$$\cdots \le \gamma_E^{(-2)} \le \gamma_E^{(-1)} \le 0 \le \gamma_E^{(1)} \le \gamma_E^{(2)} \le \cdots, \gamma_E^{(-k)} = -\gamma_E^{(k)},$$

Odd functional equation, label the zeros by

$$\cdots \le \gamma_E^{(-1)} \le 0 \le \gamma_E^{(0)} = 0 \le \gamma_E^{(1)} \le \cdots, \gamma_E^{(-k)} = -\gamma_E^{(k)}.$$

Explicit Formula

Relates sums of test functions over zeros to sums over primes.

$$\begin{aligned}
\sum_{\gamma_E^{(j)}} G\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) \;&=\; \widehat{G}(0) + G(0) \\
&\quad - 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p}\widehat{G}\left(\frac{\log p}{\log N_E}\right)a_E(p) \\
&\quad - 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{G}\left(\frac{2\log p}{\log N_E}\right)a_E^2(p) \\
&\quad + O\left(\frac{\log\log N_E}{\log N_E}\right).
\end{aligned}$$

Proof: Complex Analysis (Contour Integration)

Comments on Previous Results

Families from Rubinstein, Iwaniec-Luo-Sarnak, Miller and Hughes-Rudnick

- good averaging formulas for the family;
- conductors easy to control (constant or monotone)

Elliptic curves, let $\Delta(t)$ be the discriminant. Conductor $C(t)$ is

$$C(t) \;=\; \prod_{p|\Delta(t)} p^{f_p(t)}$$

Two t close could yield $\Delta(t)$'s with wildly differing factorization, hence the conductors can fluctuate greatly.

1-Level Expansion

$$D_{1,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{j} f\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right)$$

$$= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\widehat{f}(0) + f_i(0)$$
$$- \frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p}\widehat{f}\left(\frac{\log p}{\log N_E}\right)a_E(p)$$
$$- \frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{f}\left(2\frac{\log p}{\log N_E}\right)a_E^2(p)$$
$$+ O\left(\frac{\log\log N_E}{\log N_E}\right)$$

Want to move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$

Leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t(p)} a_t^r(p), \quad r = 1 \text{ or } 2.$$

2-Level Expansion

Need to evaluate terms like

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\prod_{i=1}^{2}\frac{1}{p_i^{r_i}}g_i\left(\frac{\log p_i}{\log N_E}\right)a_E^{r_i}(p_i).$$

Analogue of Petersson / Orthogonality: If $p_1,\dots,p_n$ are distinct primes

$$\sum_{t(p_1\cdots p_n)} a_{t_1}^{r_1}(p_1)\cdots a_{t_n}^{r_n}(p_n) = A_{r_1,\mathcal{F}}(p_1)\cdots A_{r_n,\mathcal{F}}(p_n).$$

Needed Input

For many families

$$\begin{aligned}(1): A_{1,\mathcal{F}}(p) &= -rp + O(1)\\(2): A_{2,\mathcal{F}}(p) &= p^2 + O(p^{3/2})\end{aligned}$$

Rational Elliptic Surfaces (Silverman and Rosen):

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\le X} -A_{\mathcal{E}}(p)\log p = r$$

Surfaces with $j(t)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p^2 + O\left(p^{3/2}\right).$$

New Results

Rational Surfaces Density Theorem: *Consider a* 1*-parameter family of elliptic curves of rank* r *over* $\mathbb{Q}(t)$ *that is a rational surface. Assume GRH,* $j(t)$ *non-constant, and the ABC (or Sq-Free Sieve) conjecture if* $\Delta(t)$ *has an irreducible polynomial factor of degree* ≥ 4*. Let* $m =$ *deg* $C(t)$ *and* f_i *be an even Schwartz function of small support* σ_i *(*$\sigma_1 < \min(\frac{1}{2}, \frac{2}{3m})$ *for the* 1*-level density,* $\sigma_1 + \sigma_2 < \frac{1}{3m}$ *for the* 2*-level density). Possibly after passing to a subsequence, we observe two pieces. The first equals the expected contribution from* r *zeros at the critical point (agreeing with what B-SD suggests). The second is*

$$\begin{aligned} D_{1,\mathcal{F}}^{(r)}(f_1) &= \widehat{f_1}(0) + \frac{1}{2} f_1(0) \\ D_{2,\mathcal{F}}^{(r)}(f) &= \prod_{i=1}^{2} \left[\widehat{f_i}(0) + \frac{1}{2} f_i(0) \right] + 2 \int_{-\infty}^{\infty} |u| \widehat{f_1}(u) \widehat{f_2}(u) du \\ &\quad -2\widehat{f_1 f_2}(0) - f_1(0) f_2(0) + (f_1 f_2)(0) N(\mathcal{F}, -1) \end{aligned}$$

where $N(\mathcal{F}, -1)$ *is the percent of curves with odd sign.*

1 and 2-level densities confirm Katz-Sarnak predictions for small support.

Sieving

$$\sum_{\substack{t=N \\ D(t) \\ sqfree}}^{2N} S(t) = \sum_{d=1}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2) \\ t\in[N,2N]}} S(t)$$

$$= \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2) \\ t\in[N,2N]}} S(t) + \sum_{d\geq \log^l N}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2) \\ t\in[N,2N]}} S(t).$$

Handle first piece by progressions (need progressions to evaluate sums of $a_t(p)$).

Handle second piece by Cauchy-Schwartz: The number of t in the second sum (by ABC or SqFree Sieve Conj) is $o(N)$. Can show $\sum_{t=N}^{2N} S^2(t) = O(N)$. Then

$$\sum_{t\in\mathcal{T}} S(t) \ll \left(\sum_{t\in\mathcal{T}} S^2(t)\right)^{\frac{1}{2}} \cdot \left(\sum_{t\in\mathcal{T}} 1\right)^{\frac{1}{2}}$$

$$\ll \left(\sum_{t\in[N,2N]} S^2(t)\right)^{\frac{1}{2}} \cdot o\left(\sqrt{N}\right).$$

Partial Summation

Notation: $\widetilde{a}_{d,i,p}(t') = a_{t(d,i,t')}(p)$, $G_{d,i,P}(u)$ is related to the test functions, d and i from progressions.

Applying Partial Summation

$$
\begin{aligned}
S(d,i,r,p) &= \sum_{t'=0}^{[N/d^2]} \widetilde{a}^{r}_{d,i,p}(t') G_{d,i,p}(t') \\
&= \left(\frac{[N/d^2]}{p} A_{r,\mathcal{F}}(p) + O\Big(p^R\Big) \right) G_{d,i,p}([N/d^2]) \\
&\quad - \sum_{u=0}^{[N/d^2]-1} \left(\frac{u}{p} A_{r,\mathcal{F}}(p) + O\Big(p^R\Big) \right) \\
&\quad \cdot \Big(G_{d,i,p}(u) - G_{d,i,p}(u+1) \Big)
\end{aligned}
$$

$O(p^R)$ is the error from using Hasse to bound the partial sums: $p^R = p^{1+\frac{r}{2}}$.

Difficult Piece

$$\frac{1}{N}\sum_{p}\frac{1}{p^r}\sum_{d,i}\sum_{u=0}^{[N/d^2]-1} O(p^{1+\frac{r}{2}})\cdot\Big(G_{d,i,p}(u)-G_{d,i,p}(u+1)\Big)$$

Taylor Expansion not enough.

Use Bounded Variation: conductors must be monotone.

$$\begin{aligned}&\sum_{u=0}^{[N/d^2]-1}\left|G_{d,i,p}(u)-G_{d,i,p}(u+1)\right|\\ =&\sum_{u=0}^{[N/d^2]-1}\left|g\left(\frac{\log p}{\log C(t_i(d)+ud^2)}\right)-g\left(\frac{\log p}{\log C(t_i(d)+(u+1)d^2)}\right)\right|\end{aligned}$$

Handling the Conductors

$$
\begin{aligned}
C(t) &= \prod_{p|\Delta(t)} p^{f_p(t)} \\
D_1(t) &= \text{primitive irred. poly. factors } \Delta(t) \text{ and } c_4(t) \text{ share} \\
D_2(t) &= \text{remaining primitive irred. poly. factors of } \Delta(t) \\
D(t) &= D_1(t)D_2(t)
\end{aligned}
$$

$D(t)$ square-free, $C(t)$ like $D_1^2(t)D_2(t)$ except for a finite set of bad primes.

Let P be the product of the bad primes.

By Tate's Algorithm, can determine $f_p(t)$, which depends on the coefficients $a_i(t)$ mod powers of p.

Apply Tate's Algorithm to E_{t_1} to determine $f_p(t_1)$ for the bad primes. m large, $f_p(\tau) = f_p(P^m t + t_1) = f_p(t_1)$ for $p|P$.

m enormous, for bad primes, the order of p dividing $D(P^m t + t_1)$ is independent of t. So can find integers st $C(\tau) = c_{bad} \frac{D_1^2(\tau)}{c_1} \frac{D_2(\tau)}{c_2}$, $D(\tau)$ square-free.

Application: Bounding Excess Rank

$$D_{1,\mathcal{F}}(f_1) = \widehat{f_1}(0) + \frac{1}{2}f_1(0) + rf_1(0).$$

To estimate the percent with rank at least $r + R$, P_R, we get

$$Rf_1(0)P_R \leq \widehat{f_1}(0) + \frac{1}{2}f_1(0), \ R > 1.$$

Note the family rank r has been cancelled from both sides.

The 2-level density gives *squares* of the rank on the left, get a cross term rR.

The disadvantage is our support is smaller.

Once R is large, the 2-level density yields better results.

Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).

- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Will review Orthogonal case and talk about extensions.

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc = 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \quad f(\gamma z) = (cz+d)^k f(z).$$

Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$.

Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{g(z)} y^{k-2} dx dy$.

Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}} \frac{1}{||f||} a_f(n)$$

Petersson Formula

Let $B_k(N)$ be an ONB for weight k level N.

Define

$$\Delta_{k,N}(m,n) \;=\; \sum_{f\in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Then

$$\Delta_{k,N}(m,n) \;=\; 2\pi i^k \sum_{c\equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi\frac{\sqrt{mn}}{c}\right) \\ +\; \delta(m,n).$$

Fourier Coefficient Review

$$\begin{aligned}\lambda_f(n) &= a_f(n)n^{\frac{k-1}{2}}\\ \lambda_f(m)\lambda_f(n) &= \sum_{\substack{d|(m,n)\\(d,M)=1}} \lambda_f\left(\frac{mn}{d}\right).\end{aligned}$$

For a newform of level N, $\lambda_f(N)$ is trivially related to the sign of the form:

$$\epsilon_f = i^k\mu(N)\lambda_f(N)\sqrt{N}.$$

The above will allow us to split into even and odd families:

$$1 \pm \epsilon_f$$

Limited Support

Estimate Kloosterman Sums trivially.

Use Fourier Coefficients to split by sign: N fixed, consider

$$\pm \sum_f \lambda_f(N) * (\cdots)$$

Works for support up to 1.

Increasing Support

Using Dirichlet Characters, handle Kloosterman terms.

Works for support up to 2.

Have terms like

$$\int_0^\infty J_{k-1}\left(4\pi\frac{\sqrt{m^2yN}}{c}\right)\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{\sqrt{y}}$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables:

$$\begin{aligned} u_2 &= x_1x_2 & \qquad x_2 &= \frac{u_2}{u_1} \\ u_1 &= x_1 & \qquad x_1 &= u_1 \end{aligned}$$

The Jacobean is

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{array}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log(u_2/u_1)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}.$$

2-Level Density (cont)

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}\left(w_1\right) \widehat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) \;=\; \int_{w_1=-\infty}^{\infty} \widehat{\phi}\left(w_1\right) \widehat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right) \Psi_2\left(\frac{\log u_2}{\log R}\right) \frac{du_2}{\sqrt{u_2}}$$

3-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\int_{x_3=2}^{R^\sigma} \widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)\widehat{\phi}\left(\frac{\log x_3}{\log R}\right)$$

$$* \quad J_{k-1}\left(4\pi\frac{\sqrt{m^2 x_1 x_2 x_3 N}}{c}\right)\frac{dx_1 dx_2 dx_3}{\sqrt{x_1 x_2 x_3}}$$

Change variables:

$$\begin{array}{rclcrcl} u_3 & = & x_1x_2x_3 & \qquad & x_3 & = & \frac{u_3}{u_2} \\ u_2 & = & x_1x_2 & & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

The Jacobean is

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} & 0 \\ 0 & -\frac{u_3}{u_2^2} & \frac{1}{u_2} \end{array}\right| = \frac{1}{u_1u_2}.$$

Support for n-Level Density

Careful book-keeping gives

$$\sigma_n < \frac{2}{2n-1}.$$

n-Level trivial for $\sigma_n < \frac{1}{n}$.

Is non-trivial.

Expected $\frac{2}{n}$. Obstruction from partial summation on primes.

Summary

- More support for RMT Conjectures.
- Control of Conductors.
- Averaging Formulas.