Ramsey Theory over Number Fields, Finite Fields and Quaternions

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With Nathan McNew and Megumi Asada, Andrew Best, Eva Fourakis, Eli Goldstein, Karen Huan, Gwyn Moreland, Jasmine Powell, Kimsy Tor, Maddie Weinstein.

CANT, May 26, 2017

History

In 1961: Rankin: subsets of $\mathbb{N}$ avoiding geometric progressions: $\{n, nr, nr^2\}$ and $r \in \mathbb{N} \setminus \{1\}$.

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Greedy construction asymptotic density approximately 0.71974.
, modification by McNew improve to about 0.72195.

Improved bounds (Riddell, Brown–Gordon, Beiglböck–Bergelson–Hindman–Strauss, Nathanson–O'Bryant, McNew) on the greatest possible upper density of such a set, between 0.81841 and 0.81922.

Goals of Talk

Generalize to quadratic number fields.

Study geometric-progression-free subsets of the algebraic integers (or ideals) and give bounds on the possible densities of such sets.

In an imaginary quadratic field with unique factorization we are able to consider the possible densities of sets of algebraic integers which avoid 3-term geometric progressions.

Consider similar cases in Function Fields and Quaternions (non-commutative!).

Definitions

The **density** of a set $A \subseteq \mathbb{N}$ is defined to be

$$d(A) = \lim_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}$$

if this limit exists.

The **upper density** of a set $A \subseteq \mathbb{N}$ is defined to be

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}.$$

Greedy Set over $\mathbb{Z}$

Joint work with Andrew Best, Karen Huan, Nathan McNew, Jasmine Powell, Kimsy Tor, Madeleine Weinstein: *Geometric-Progression-Free Sets over Quadratic Number Fields*. To appear in the Proceedings of the Royal Society of Edinburgh, Section A: Mathematics.

https://arxiv.org/pdf/1412.0999v1.pdf.

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1 2 3 ~~4~~ 5 6 7

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1 2 3 ~~4~~ 5 6 7 8

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1 2 3 ~~4~~ 5 6 7 8 ~~9~~ 10

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Rankin's Greedy Set

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$$\prod_p \frac{p-1}{p} \prod_{i=1}^{\infty}\left(1+\frac{1}{p^{3^i}}\right) = \frac{1}{\zeta(2)} \prod_{i=1}^{\infty} \frac{\zeta(3^i)}{\zeta(2\cdot 3^i)} \approx 0.72.$$

Greedy Set over $\mathbb{Z}[i]$

Onto the Gaussian Integers

Definition

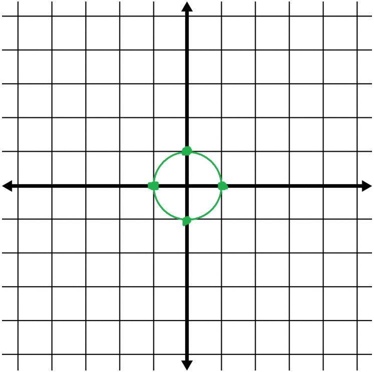
The **Gaussian integers** are defined to be the set of all $a + bi$, where a and b are integers.

Definition

The **norm** of a Gaussian integer $a + bi$ is defined to be

$$N(a + bi) \ = \ a^2 + b^2.$$

Defining the Greedy Set



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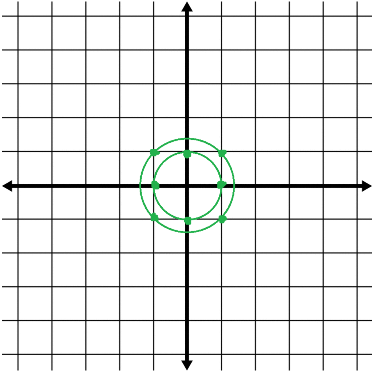

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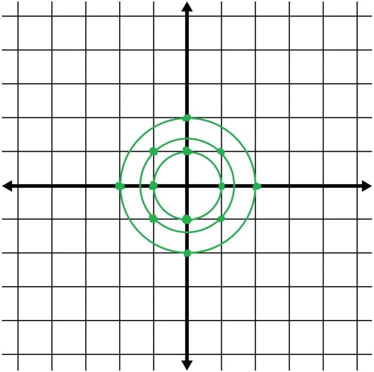

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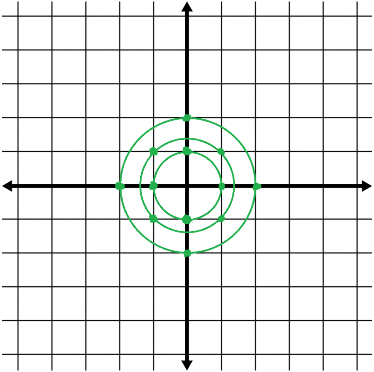
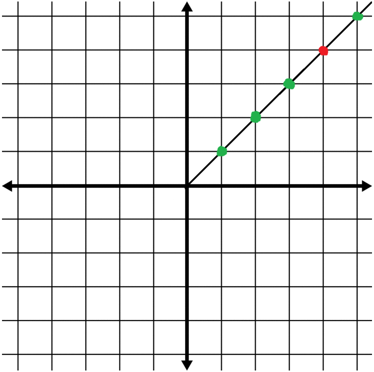


The greedy set is defined by consideration of “norm circles" whose radii increase.

Having defined it, we consider geometric progressions which avoid various kinds of ratios.

Avoiding Integral Ratios



This case can be thought of as a projection of the integral greedy set onto every line through the origin.

Depicted is the progression $1+i, 2+2i, 4+4i$.

Avoiding Integral Ratios

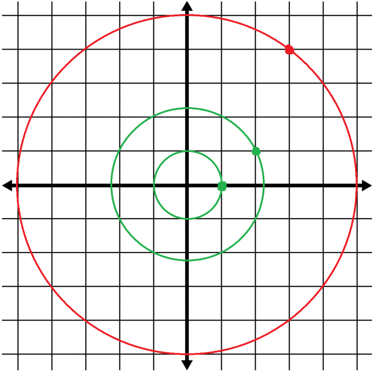
We exclude a Gaussian integer $a+bi$ exactly when it can be written in the form $k(c+di)$, where k is not in Rankin's set and $(c,d)=1$.

Theorem 1 [B,H,Mc,Mi,P,T,W '14]

The density of the greedy set of Gaussian integers that avoids integral ratios is

$$\prod_p \left(\frac{p^2-1}{p^2} \prod_{i=0}^{\infty} \left(1+\frac{1}{p^{2\cdot 3^i}}\right)\right) = \frac{1}{\zeta(4)} \prod_{i=1}^{\infty} \frac{\zeta(2\cdot 3^i)}{\zeta(4\cdot 3^i)} \approx 0.9397.$$

Avoiding Gaussian Ratios



We also consider sets that avoid progressions with Gaussian integer ratios.

Depicted is the progression $1, 2+i, 3+4i$.

Density of the Gaussian Greedy Set

We can determine the likelihood of a Gaussian integer being included by evaluating the primes in its prime factorization and whether each prime is raised to an appropriate power.

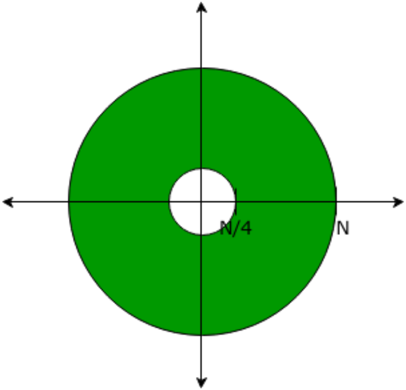
Theorem 2 **[B,H,Mc,Mi,P,T,W '14]**

$$\text{Let } f(x) = \left(1 - \frac{1}{x}\right) \prod_{i=0}^{\infty} \left(1 + \frac{1}{x^{3^i}}\right).$$

Then the density of the greedy set of Gaussian integers that avoids Gaussian integral ratios is

$$f(2) \left(\prod_{p \equiv 1 \bmod 4} f^2(p) \right) \left(\prod_{q \equiv 3 \bmod 4} f(q^2) \right) \approx 0.771.$$

A Lower Bound for Upper Density



Generalizing an argument by McNew, we see that if we take the Gaussian integers with norm between $N/4$ and N, no three of these elements will comprise a 3-term geometric progression.

A Lower Bound for Upper Density

Similarly, we can include integers with norm between $N/16$ and $N/8$ without introducing a progression, and continue in this fashion.

Theorem 4 [B,H,Mc,Mi,P,T,W '14]

A set of acceptable norms is

$$\left(\frac{N}{25}, \frac{N}{20}\right] \cup \left(\frac{N}{16}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

The density of the Gaussian integers that fall inside this set gives us a lower bound of 0.8225.

Overview of Bounds

- A lower bound for maximal density of sets of Gaussian integers avoiding integral ratios is 0.9397.
- A lower bound for maximal density of sets of Gaussian integers avoiding Gaussian ratios is 0.771.
- Bounds for upper density for sets S of Gaussian integers avoiding Gaussian ratios are $0.8225 < \overline{d}(S) < 0.857$.

Number Fields

Class Number 1

d	Density of the greedy set
-1	0.762340
-2	0.693857
-3	0.825534
-7	0.674713
-11	0.742670
-19	0.823728
-43	0.898250
-67	0.917371
-163	0.933580

Table: Density of the greedy set, $G_{K,3} \subset \mathcal{O}_K$, of algebraic integers which avoid 3-term geometric progressions with ratios in $\mathcal{O}_K$ for each of the class number 1 imaginary quadratic number fields $K = \mathbb{Q}\left(\sqrt{d}\right)$.

Arbitrary Class Number

Theorem 5 [B,H,Mc,Mi,P,T,W '14]

K a quadratic number field, $f : \mathbb{N} \to \mathbb{R}$ defined by

$$f(x) \ := \ \left(1 - \frac{1}{x}\right) \prod_{i=0}^{\infty} \left(1 + \frac{1}{x^{3^i}}\right).$$

Then the density of the greedy set of ideals, $G^*_{K,3}$, of $\mathcal{O}_K$ avoiding 3-term geometric progressions with ratio a (non-trivial) ideal of $\mathcal{O}_K$ is

$$\begin{aligned} d(G^*_{K,3}) &= \left(\prod_{\chi_K(p)=-1} f(p^2)\right)\left(\prod_{\chi_K(p)=1} [f(p)]^2\right)\left(\prod_{\chi_K(p)=0} f(p)\right) \\ &= \frac{1}{\zeta_K(2)} \prod_{i=1}^{\infty} \frac{\zeta_K(3^i)}{\zeta_K(2 \cdot 3^i)}. \end{aligned}$$

Upper bounds for the upper density

d	Upper Bound	Approximation
-1	6/7	0.857143
-2	6/7	0.857143
-3	25/26	0.961538
-7	6/7	0.857143
-11	25/26	0.961538
-19	62/63	0.984127
-43	62/63	0.984127
-67	62/63	0.984127
-163	62/63	0.984127

Table: Upper bounds for the upper density of 3-term geometric-progression-free subsets of the algebraic integers in the class number 1 imaginary quadratic number fields, $\mathbb{Q}\left(\sqrt{d}\right)$.

Improved Upper bounds for the upper density

d	Smallest Non-unit Norms	Upper Bound
-1	2, 5, 5	0.851090
-2	2, 3, 3	0.839699
-3	3, 4, 7	0.910089
-7	2, 2, 7	0.858880
-11	3, 3, 4	0.917581
-19	4, 5, 5	0.949862
-43	4, 9, 11	0.945676
-67	4, 9, 17	0.946772
-163	4, 9, 25	0.946682

Table: Improved upper bounds for the upper density of a set free of 3-term geometric progressions in each of the class number 1 imaginary quadratic number fields $\mathbb{Q}\left(\sqrt{d}\right)$.

Idea of Proof

Construct intervals T_M to avoid introducing progressions with norms in the given imaginary quadratic number field.

Example: $d = -43$:

$$(M/1472, M/1377] \cup (M/576, M/208] \cup (M/81, M/64] \cup (M/16, M],$$

giving a lower bound of 0.943897.

Finite Fields

Joint with Megumi Asada, Eva Fourakis, Sarah Manski, Gwyneth Moreland and Nathan McNew: *Subsets of $\mathbb{F}_q[x]$ free of 3-term geometric progressions*. To appear in Finite Fields and their Applications.

https://arxiv.org/pdf/1512.01932.pdf.

Preliminaries

Functon Field

We view $\mathbb{F}_q[x]$, with $q = p^n$, as the ring of all polynomials with coefficients in the finite field $\mathbb{F}_q$.

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Goal

Construct a Greedy Set of polynomials in $\mathbb{F}_q[x]$ free of geometric progressions.

The Greedy Set

- Rewrite any $f(x)$ as $f(x) = uP_1^{\alpha_1} \cdots P_k^{\alpha_k}$ where u is a unit, and each P_i is a monic irreducible polynomial.
- Exclude $f(x)$ with $\alpha_i \notin A_3^*(\mathbb{Z}) = \{0, 1, 3, 4, 9, 10, 12, 13, \ldots\}$.

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Greedy Set in $\mathbb{F}_q[x]$

The Greedy Set is exactly the set of all $f(x) \in \mathbb{F}_q[x]$ only with prime exponents in $A_3^*(\mathbb{Z})$

Asymptotic Density

The *asymptotic density of the greedy set* $G^*_{3,q} \subseteq \mathbb{F}_q[x]$ is

$$d(G^*_3) = \left(1 - \frac{1}{q}\right) \prod_{i=1}^{\infty} \prod_{n=1}^{\infty} \left(1 + q^{-n3^i}\right)^{m(n,q)},$$

where $m(n,q) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d$ gives the number of monic irreducibles over $\mathbb{F}_q[x]$.

Asymptotic Density

The *asymptotic density of the greedy set* $G^*_{3,q} \subseteq \mathbb{F}_q[x]$ is

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where $m(n,q) \;=\; \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)q^d$ gives the number of monic irreducibles over $\mathbb{F}_q[x]$.

Becomes a lower bound when truncated.

Lower Bound

Table: Lower Bound for Density of $G_3^*(\mathbb{F}_q[x])$.

q	$d(G_3^*)$ for $\mathbb{F}_q[x]$
2	.648361
3	.747027
4	.799231
5	.833069
7	.874948
8	.888862

q	$d(G_3^*)$ for $\mathbb{F}_q[x]$
9	.899985
25	.961538
27	.964286
49	.980000
125	.992063
343	.997093

Bounds on Upper Densities

Use similar combinatorial methods to McNew, Riddell, and Nathanson and O'Byrant to give lower and upper bounds for the upper density of a progression free set for specific values of q.

q	New Bound (q-smooth)	Old Bound	Lower Bound
2	.846435547	.857142857	.845397956
3	.921933009	.923076923	.921857532
4	.967684196	.96774193	.967680495
5	.967684196	.967741935	.967680495
7	.982448450	.982456140	.982447814

Table: New upper bounds (q-smooth) compared to the old upper bounds, as well as the lower bounds for the supremum of upper densities.

Quaternions

Joint with Megumi Asada, Eva Fourakis, Eli Goldstein, Sarah Manski, Gwyneth Moreland and Nathan McNew: *Avoiding 3-Term Geometric Progressions in Non-Commutative Settings*, preprint. Available at: https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/Ramsey_NonComm2015SMALLv10.pdf.

Question

Previous work in commutative settings. How does non-commutivity affect the problem in, say, free groups or the Hurwitz quaternions $\mathcal{H}$? How does the lack of unique factorization affect the problem in $\mathcal{H}$?

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Previous work in commutative settings. How does non-commutivity affect the problem in, say, free groups or the Hurwitz quaternions $\mathcal{H}$? How does the lack of unique factorization affect the problem in $\mathcal{H}$?

Building on methods of McNew, SMALL '14, and Rankin, we construct large subsets of $\mathcal{H}$ that avoid 3-term geometric progressions.

Types of Quaternions

Definition

Quaternions constitute the algebra over the reals generated by units i, j, and k such that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions can be written as $a + bi + cj + dk$ for $a, b, c, d \in \mathbb{R}$.

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Definition

We say that $a + bi + cj + dk$ is in the Hurwitz Order, $\mathcal{H}$, if a, b, c, d are all integers or halves of odd integers.

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Definition

The Norm of a quaternion $Q = a + bi + cj + dk$ is given by Norm$[Q] = a^2 + b^2 + c^2 + d^2$.

Counting Quaternions

The number of Hurwitz Quaternions below a given norm is given by the corresponding number of lattice points in a 4-dimensional hypersphere.

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Fact

The number of Hurwitz quaternions of norm N is

$$S(\{N\}) = 24 \sum_{2 \nmid d | N} d,$$

the sum of the odd divisors of N multiplied by 24.

Units and Factorization

Fact

The Hurwitz Order contains 24 units, namely

$$\pm 1, \pm i, \pm j, \pm k \text{ and } \pm \frac{1}{2} \pm \frac{1}{2} i \pm \frac{1}{2} j \pm \frac{1}{2} k.$$

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Fact

Let Q be a Hurwitz quaternion of norm q. For any factorization of q into a product $p_0 p_1 \cdots p_k$ of integer primes, there is a factorization

$$Q = P_0 P_1 \cdots P_k$$

where P_i is a Hurwitz prime of norm p_i.

The Goal

Goal

Construct and bound Greedy and maximally sized sets of quaternions of the Hurwitz Order free of three-term geometric progressions. For definiteness, we exclude progressions of the form

$$Q,\ QR,\ QR^2$$

where $Q, R \in \mathcal{H}$ and $\mathrm{Norm}[R] \neq 1$.

Difficulty: Hurwitz Quaternions lack unique factorization.

We can consider the set of Hurwitz Quaternions with norm in $G_3^*(\mathbb{Z})$, which is 3-term progression-free.

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Want: formula for the proportion of quaternions whose norm is divisible by p^n and not p^{n+1}. We study the proportion of (Hurwitz) quaternions up to norm N whose norm is exactly divisible by p^n.

$$\frac{\text{Quats with norm div by } p^n \text{ - Quats with norm div by } p^{n+1}}{\text{Quats with norm} \leq N}$$

$$= \frac{(\text{Quats with norm } p^n)(\text{Quats with norm} \leq N/p^n)}{24 \cdot (\text{Quats with norm} \leq N)} - \frac{(\text{Quats with norm } p^{n+1})(\text{Quats with norm} \leq N/p^{n+1})}{24 \cdot (\text{Quats with norm} \leq N)}$$

$$= \frac{\left(\sum_{2\nmid d | p^n} d\right) V_4(\sqrt{N/p^n}) - \left(\sum_{2\nmid d | p^{n+1}} d\right) V_4(\sqrt{N/p^{n+1}})}{V_4(\sqrt{N})} + \text{error},$$

where $V_4(M)$ denotes the volume of a 4-dimensional sphere of radius M. For p odd

$$\sum_{2\nmid d | p^n} d = 1 + \cdots + p^n = (p^{n+1} - 1)/(p-1).$$

For $p = 2$, the quantity is 1.

We sum up probabilities of having norm divisible by p^n to find the proportion of quaternions whose norm is exactly divisible by p^n for p fixed, $n \in A_3^*(\mathbb{Z})$:

$$\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{p^2(p-1)p^{2n}}.$$

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To find the density of $\{q \in \mathcal{H} : \text{Norm}[q] \in G_3^*(\mathbb{Z})\}$, we multiply these terms to get all norms with prime powers in $A_3^*(\mathbb{Z})$, i.e., norms in $G_3^*(\mathbb{Z})$.

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$$\begin{aligned} d(\{q \in \mathcal{H} : \text{N}[q] \in G_3^*(\mathbb{Z})\}) &= \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{2^2 - 1}{2^2 2^{2n}}\right] \\ &\quad \times \prod_{p \text{ odd}} \left[\sum_{n \in A_3^*(\mathbb{Z})} \frac{p^{n+3} - p^{n+2} - p^2 + 1}{p^2(p-1)p^{2n}}\right] \\ &\approx .77132. \end{aligned}$$

Instead of studying large density sets avoiding 3-term progressions, we can also try to maximize the upper density.

Definition (Upper Density)

The upper density of a set $A \subset \mathcal{H}$ is

$$\limsup_{N\to\infty} \frac{|A \cap \{q \in \mathcal{H} : \text{Norm}[q] \leq N\}|}{|\{q \in \mathcal{H} : \text{Norm}[q] \leq N\}|}$$

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Study lower bounds for the supremum of upper densities of 3-term progression-free sets.

Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density. Consider

$$S_N = \left(\frac{N}{4}, N\right]$$

Then the quaternions with norm in S_N have no 3-term progressions in their norms, and thus no 3-term progressions in the elements themselves.

By spacing out copies of $\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}$, we construct a set with upper density

$$\lim_{N\to\infty} \frac{\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}}{\{q \in \mathcal{H} : \text{Norm}[q] \leq N\}} \approx .946589.$$

Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density. Consider

$$S_N = \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

Then the quaternions with norm in S_N have no 3-term progressions in their norms, and thus no 3-term progressions in the elements themselves.

By spacing out copies of $\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}$, we construct a set with upper density

$$\lim_{N\to\infty} \frac{\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}}{\{q \in \mathcal{H} : \text{Norm}[q] \leq N\}} \approx .946589.$$

Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density. Consider

$$S_N = \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

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Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density. Consider

$$S_N = \left(\frac{N}{40}, \frac{N}{36}\right] \cup \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$$

Then the quaternions with norm in S_N have no 3-term progressions in their norms, and thus no 3-term progressions in the elements themselves.

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Lower Bound for the Supremum

For a lower bound, we construct a set with large upper density. Consider
$S_N = \left(\frac{N}{48}, \frac{N}{45}\right] \cup \left(\frac{N}{40}, \frac{N}{36}\right] \cup \left(\frac{N}{32}, \frac{N}{27}\right] \cup \left(\frac{N}{24}, \frac{N}{12}\right] \cup \left(\frac{N}{9}, \frac{N}{8}\right] \cup \left(\frac{N}{4}, N\right]$

Then the quaternions with norm in S_N have no 3-term progressions in their norms, and thus no 3-term progressions in the elements themselves.

By spacing out copies of $\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}$, we construct a set with upper density

$$\lim_{N\to\infty} \frac{\{q \in \mathcal{H} : \text{Norm}[q] \in S_N\}}{\{q \in \mathcal{H} : \text{Norm}[q] \leq N\}} \approx .946589.$$

The Greedy Set

Recall Rankin's greedy set, G_3^*:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21...$

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Norms of elements in our greedy set:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21...$

The Greedy Set

Recall Rankin's greedy set, G_3^*:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21 \ldots 48, 51 \ldots$

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Reasons for discrepancies:

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Norms of elements in our greedy set:
$1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 17, 19, 21 ... 48, 49, 51 ...$

Reasons for discrepancies: Try 31^2. Recall

$$S(\{N\}) \;=\; 24 \sum_{2 \nmid d \mid N} d.$$

Then $S(\{31^2\}) = 24 \sum_{2 \nmid d \mid 31^2} d$. However, the number of ways to write a quaternion of norm 31^2 as the square of a quaternion of norm 31 multiplied by a unit is

$$S(\{31^2\}) \;\geq\; 24 \sum_{2 \nmid 31 d \mid 31^2} d \;=\; 24 * 31 \sum_{2 \nmid d \mid 31} d \;>\; 24 S(\{31\}).$$