Advances in Number Theory and Random Matrix Theory

Investigations of Zeros Near the Central Point of Elliptic Curve L-Functions

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Measures of Spacings: 1-Level Density and Families

L-function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.

C_f = analytic conductor, scale factor for low zeros.

$\phi(x)$ a compactly supported even Schwartz function.

$$D_{1,f}(\phi) \ = \ \sum_j \phi\left(\frac{\log C_f}{2\pi}\gamma_{f,j}\right)$$

- individual zeros contribute in limit
- most of contribution is from low zeros
- average over similar curves (family)

$$D_{1,\mathcal{F}}(\phi) \ = \ \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} D_{1,f}(\phi).$$

Limiting Behavior

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} D_{1,f}(\phi) = \lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_{f,j} \log C_f}{2\pi}\right)$$

$$= \int \phi(x) W_{1,\mathcal{G}(\mathcal{F})}(x) dx$$

$$= \int \widehat{\phi}(y) \widehat{W}_{1,\mathcal{G}(\mathcal{F})}(y) dy.$$

Density Conjecture: Distribution of low zeros of L-functions agree with the distribution of eigenvalues near 1 of a classical compact group.

Elliptic Curves:

$$E_t : y^2 \;=\; x^3 + A(T)x + B(T), \;\; A(T), B(T) \in \mathbb{Z}(T).$$

$$a_{E_t}(p) \;=\; -\sum_{x \bmod p} \left(\frac{x^3 + A(t)x + B(t)}{p}\right) \;=\; a_{E_{t+mp}}(p)$$

$$L(E_t, s) \;=\; \sum_{n=1}^{\infty} \frac{a_{E_t}(n)}{n^s} \;=\; \prod_p L_p(E_t, s).$$

By GRH: All non-trivial zeros on the critical line.

Rational solutions: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$.

Birch and Swinnerton-Dyer Conjecture:
Geometric rank $r =$ analytic rank (order of vanishing at central point).

Tools to Study Low Zeros

- explicit formula relating zeros and Fourier coeffs;
- averaging formulas for the family;
- conductors easy to control (constant or monotone).

1-Level Expansion

$$
\begin{aligned}
D_{1,\mathcal{F}_N}(\phi) &= \frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_j \phi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) \\
&= \frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\widehat{\phi}(0)+\phi_i(0) \\
&\quad - \frac{2}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_p \frac{\log p}{\log C_{E_t}}\frac{1}{p}\,\widehat{\phi}\left(\frac{\log p}{\log C_{E_t}}\right)a_{E_t}(p) \\
&\quad - \frac{2}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_p \frac{\log p}{\log C_{E_t}}\frac{1}{p^2}\,\widehat{\phi}\left(2\frac{\log p}{\log C_{E_t}}\right)a_{E_t}^2(p) \\
&\quad + O\left(\frac{\log\log C_{E_t}}{\log C_{E_t}}\right)
\end{aligned}
$$

Want to move $\frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}$, Leads us to study

$$
A_{r,\mathcal{F}}(p) = \frac{1}{p}\sum_{t \bmod p} a_{E_t}^r(p), \quad r = 1 \text{ or } 2.
$$

Input

For many families

$$(1): A_{1,\mathcal{F}}(p) \ = \ -r \ + \ O(p^{-1})$$
$$(2): A_{2,\mathcal{F}}(p) \ = \ p \ + \ O(p^{1/2})$$

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(T)$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} -A_{1,\mathcal{F}}(p)\log p \ = \ r$$

Surfaces with $j(T)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p + O\left(p^{1/2}\right).$$

One-Level Result

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \phi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \phi(x) W_{\mathcal{G}}(x)dx + r\phi(0$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Family zeros seem independent.

Twist generic families of rank r_1 and r_2 then resulting family has symplectic symmetry and rank 0 (though potential lower order correction proportional to $r_1 r_2$).

Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$.

Natural sub-families

- Curves of rank r.
- Curves of rank $r + 2$.

Question: Does the sub-family of rank $r + 2$ curves in a rank r family behave like the sub-family of rank $r+2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r + 2$?

Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model:

Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$:

$$d\varepsilon_{2r}(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

with $1 \le j,k \le N-r$.

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

Small support, as conductors $\to$ infinity the 1-level densities for one-param families agree with $\rho_{r,\mathrm{Indep}}$ and not $\rho_{r,\mathrm{Inter}}$.

Curve E, conductor C_E, expect first zero $\frac{1}{2}+i\gamma_{E,1}$ with $\gamma_{E,1} \approx \frac{1}{\log C_E}$.

If r zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log C_E}$, can detect in 1-level density:

$$\frac{1}{|\mathcal{F}_N|} \sum_{E\in\mathcal{F}_N} \sum_j \phi\left(\gamma_{E,j}\frac{\log C_E}{2\pi}\right).$$

Testing Random Matrix Theory Predictions

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: what percent have rank $\geq r+2$?

2. **First (Normalized) Zero above Central Point:** Do extra zeros at the central point affect the distribution of zeros near the central point?

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(T)$.

Density Conjecture $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r	$\approx$	32%
Percent with rank r+1	$\approx$	48%
Percent with rank r+2	$\approx$	18%
Percent with rank r+3	$\approx$	2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$.

Data on Excess Rank

$$y^2 + y = x^3 + Tx$$

Each data set 2000 curves from start.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{17}, but on logarithmic scale still small.

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

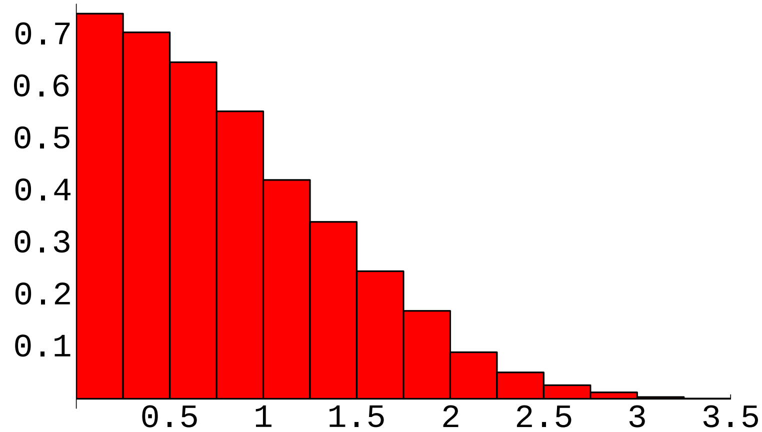

Figure 1a: 1st norm. evalue above 1: 23,040 SO(4) matrices
Mean $= .709$, Std Dev of the Mean $= .601$, Median $= .709$

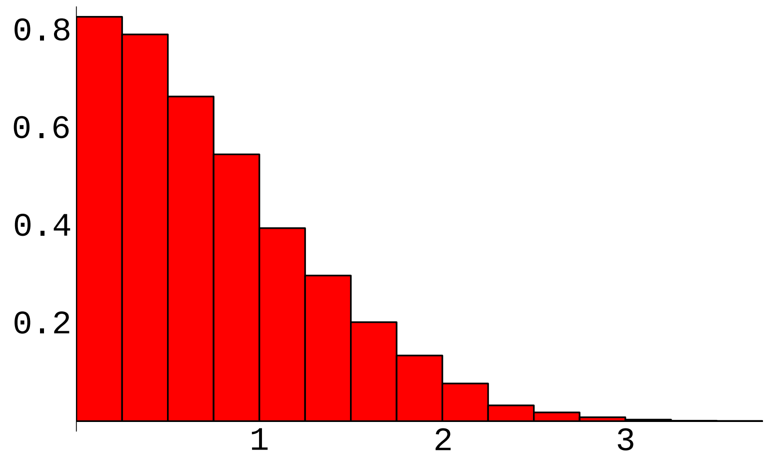

Figure 1b: 1st norm. evalue above 1: 23,040 SO(6) matrices
Mean $= .635$, Std Dev of the Mean $= .574$, Median $= .635$

RMT: Theoretical Results ($N \to \infty$)

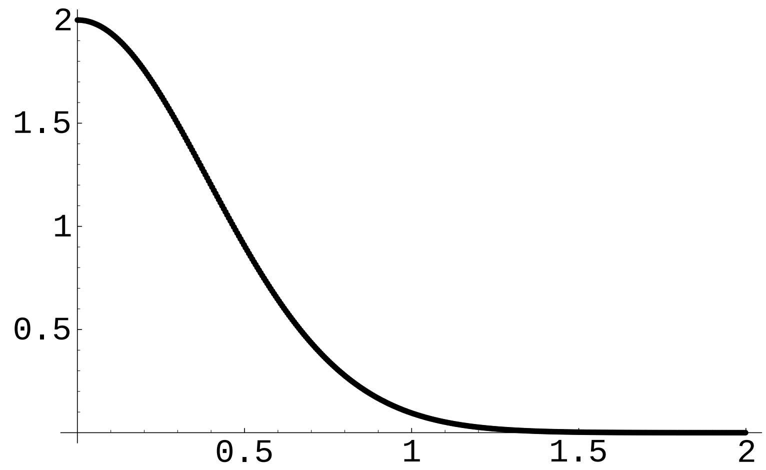

Figure 1c: 1st norm. evalue above 1: SO(even)

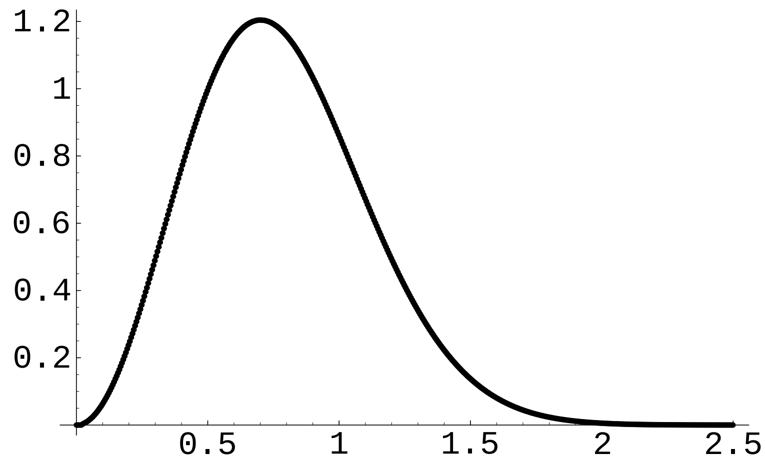

Figure 1d: 1st norm. evalue above 1: SO(odd)

Rank 0 Curves: 1st Normalized Zero above Central Point

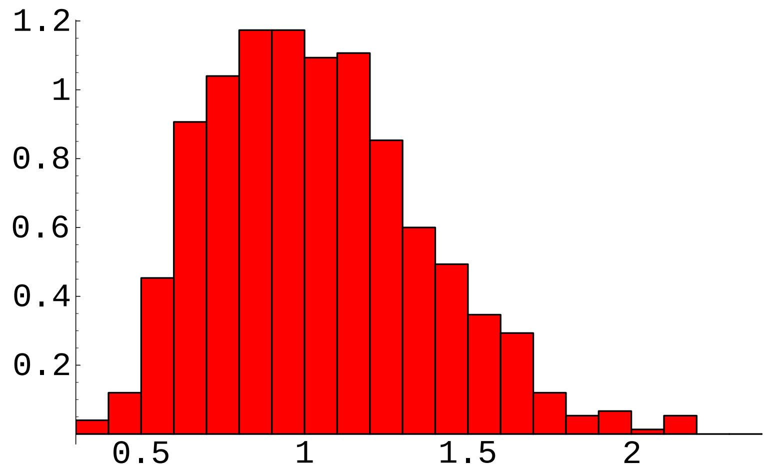

Figure 2a: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, $\sigma_\mu = .32$

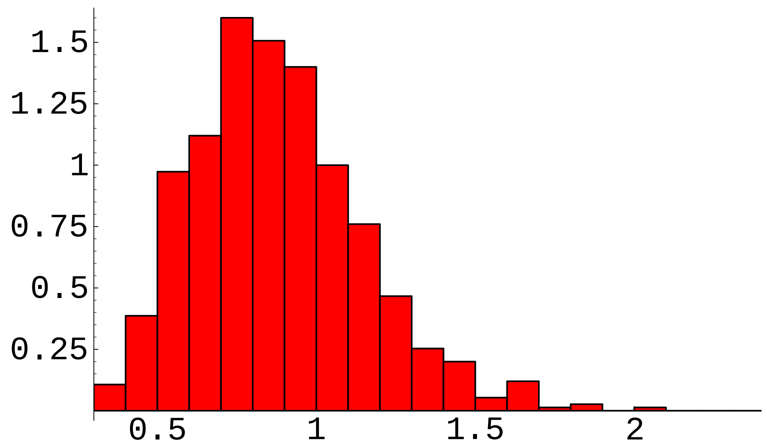

Figure 2b: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

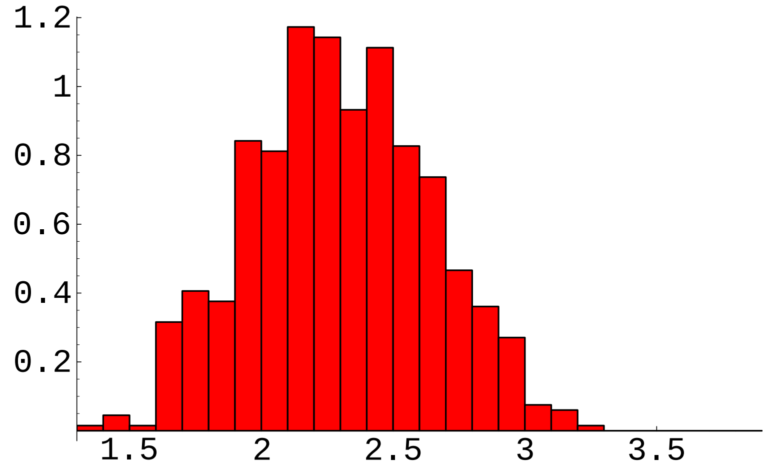

Figure 3a: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

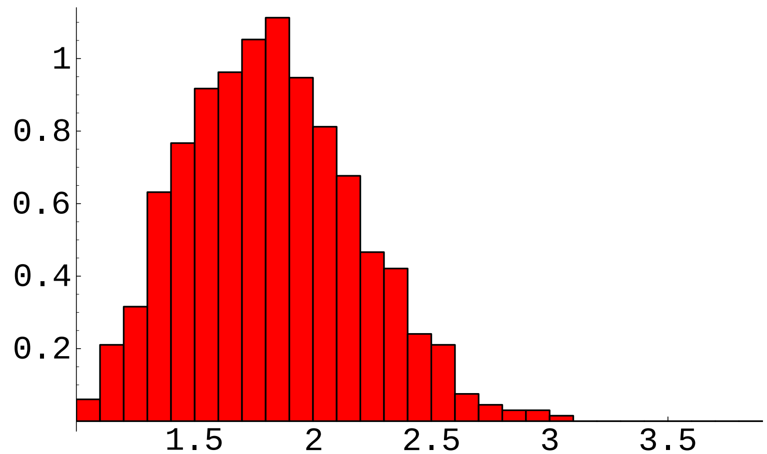

Figure 3b: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

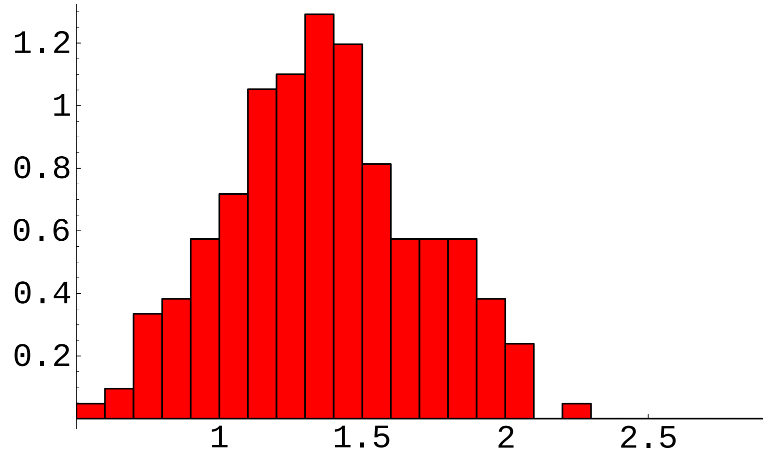

Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

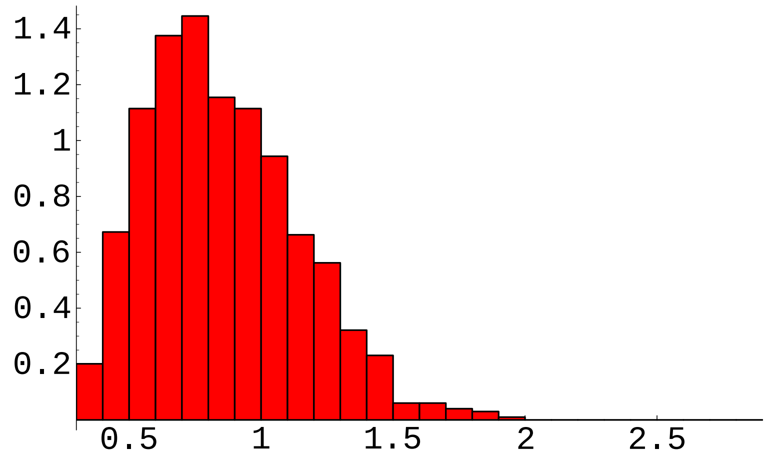

Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	**Median** $\widetilde{\mu}$	**Mean** μ	**StDev** σ_μ	**log(conductor)**	**Number**
1: [0,1,1,1,T]	1.28	1.33	0.26	[3.93, 9.66]	7
2: [1,0,0,1,T]	1.39	1.40	0.29	[4.66, 9.94]	11
3: [1,0,0,2,T]	1.40	1.41	0.33	[5.37, 9.97]	11
4: [1,0,0,-1,T]	1.50	1.42	0.37	[4.70, 9.98]	20
5: [1,0,0,-2,T]	1.40	1.48	0.32	[4.95, 9.85]	11
6: [1,0,0,T,0]	1.35	1.37	0.30	[4.74, 9.97]	44
7: [1,0,1,-2,T]	1.25	1.34	0.42	[4.04, 9.46]	10
8: [1,0,2,1,T]	1.40	1.41	0.33	[5.37, 9.97]	11
9: [1,0,-1,1,T]	1.39	1.32	0.25	[7.45, 9.96]	9
10: [1,0,-2,1,T]	1.34	1.34	0.42	[3.26, 9.56]	9
11: [1,1,-2,1,T]	1.21	1.19	0.41	[5.73, 9.92]	6
12: [1,1,-3,1,T]	1.32	1.32	0.32	[5.04, 9.98]	11
13: [1,-2,0,T,0]	1.31	1.29	0.37	[4.73, 9.91]	39
14: [-1,1,-3,1,T]	1.45	1.45	0.31	[5.76, 9.92]	10
All Curves	1.35	1.36	0.33	[3.26, 9.98]	209
Distinct Curves	1.35	1.36	0.33	[3.26, 9.98]	196

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(conductor)	Number
1: [0,1,1,1,T]	0.80	0.86	0.23	[15.02, 15.97]	49
2: [1,0,0,1,T]	0.91	0.93	0.29	[15.00, 15.99]	58
3: [1,0,0,2,T]	0.90	0.94	0.30	[15.00, 16.00]	55
4: [1,0,0,-1,T]	0.80	0.90	0.29	[15.02, 16.00]	59
5: [1,0,0,-2,T]	0.75	0.77	0.25	[15.04, 15.98]	53
6: [1,0,0,T,0]	0.75	0.82	0.27	[15.00, 16.00]	130
7: [1,0,1,-2,T]	0.84	0.84	0.25	[15.04, 15.99]	63
8: [1,0,2,1,T]	0.90	0.94	0.30	[15.00, 16.00]	55
9: [1,0,-1,1,T]	0.86	0.89	0.27	[15.02, 15.98]	57
10: [1,0,-2,1,T]	0.86	0.91	0.30	[15.03, 15.97]	59
11: [1,1,-2,1,T]	0.73	0.79	0.27	[15.00, 16.00]	124
12: [1,1,-3,1,T]	0.98	0.99	0.36	[15.01, 16.00]	66
13: [1,-2,0,T,0]	0.72	0.76	0.27	[15.00, 16.00]	120
14: [-1,1,-3,1,T]	0.90	0.91	0.24	[15.00, 15.99]	48
All Curves	0.81	0.86	0.29	[15.00,16.00]	996
Distinct Curves	0.81	0.86	0.28	[15.00,16.00]	863

Rank 2 Curves: 1st Norm Zero: one-param of rank 0 over $\mathbb{Q}(T)$

first set $\log(\text{cond}) \in [15, 15.5)$; second set $\log(\text{cond}) \in [15.5, 16]$. Median $\widetilde{\mu}$, Mean μ, Std Dev (of Mean) σ_μ.

Family	$\widetilde{\mu}$	μ	σ_μ	**Number**	$\widetilde{\mu}$	μ	σ_μ	**Number**
1: [0,1,3,1,T]	1.59	1.83	0.49	8	1.71	1.81	0.40	19
2: [1,0,0,1,T]	1.84	1.99	0.44	11	1.81	1.83	0.43	14
3: [1,0,0,2,T]	2.05	2.03	0.26	16	2.08	1.94	0.48	19
4: [1,0,0,-1,T]	2.02	1.98	0.47	13	1.87	1.94	0.32	10
5: [1,0,0,T,0]	2.05	2.02	0.31	23	1.85	1.99	0.46	23
6: [1,0,1,1,T]	1.74	1.85	0.37	15	1.69	1.77	0.38	23
7: [1,0,1,2,T]	1.92	1.95	0.37	16	1.82	1.81	0.33	14
8: [1,0,1,-1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
9: [1,0,1,-2,T]	1.74	1.74	0.43	14	1.82	1.90	0.40	14
10: [1,0,-1,1,T]	2.00	2.00	0.32	22	1.81	1.94	0.42	18
11: [1,0,-2,1,T]	1.97	1.99	0.39	14	2.17	2.14	0.40	18
12: [1,0,-3,1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
13: [1,1,0,T,0]	1.89	1.88	0.31	20	1.82	1.88	0.39	26
14: [1,1,1,1,T]	2.31	2.21	0.41	16	1.75	1.86	0.44	15
15: [1,1,-1,1,T]	2.02	2.01	0.30	11	1.87	1.91	0.32	19
16: [1,1,-2,1,T]	1.95	1.91	0.33	26	1.98	1.97	0.26	18
17: [1,1,-3,1,T]	1.79	1.78	0.25	13	2.00	2.06	0.44	16
18: [1,-2,0,T,0]	1.97	2.05	0.33	24	1.91	1.92	0.44	24
19: [-1,1,0,1,T]	2.11	2.12	0.40	21	1.71	1.88	0.43	17
20: [-1,1,-2,1,T]	1.86	1.92	0.28	23	1.95	1.90	0.36	18
21: [-1,1,-3,1,T]	2.07	2.12	0.57	14	1.81	1.81	0.41	19
All Curves	1.95	1.97	0.37	350	1.85	1.90	0.40	388
Distinct Curves	1.95	1.97	0.37	335	1.85	1.91	0.40	366

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

- Observe the medians and means of the small conductor set to be larger than those from the large conductor set.

- For all curves the Pooled and Unpooled Two-Sample t-Procedure give t-statistics of 2.5 with over 600 degrees of freedom.

- For distinct curves the t-statistics is 2.16 (respectively 2.17) with over 600 degrees of freedom (about a 3% chance).

- Provides evidence against the null hypothesis (that the means are equal) at the $.05$ confidence level (though not at the $.01$ confidence level).

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

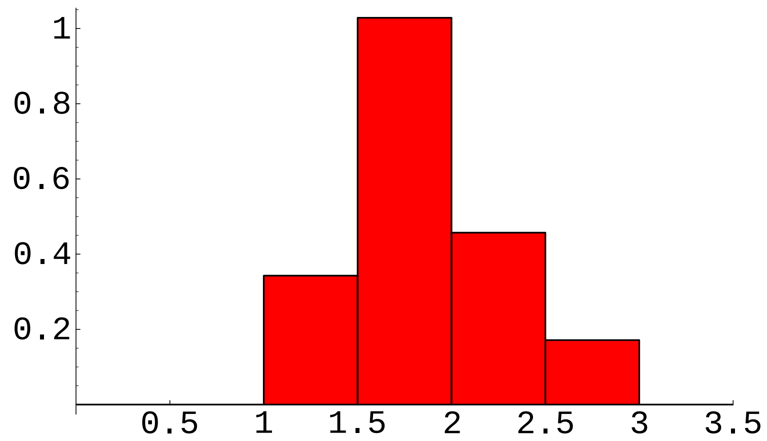

Figure $5a$:35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

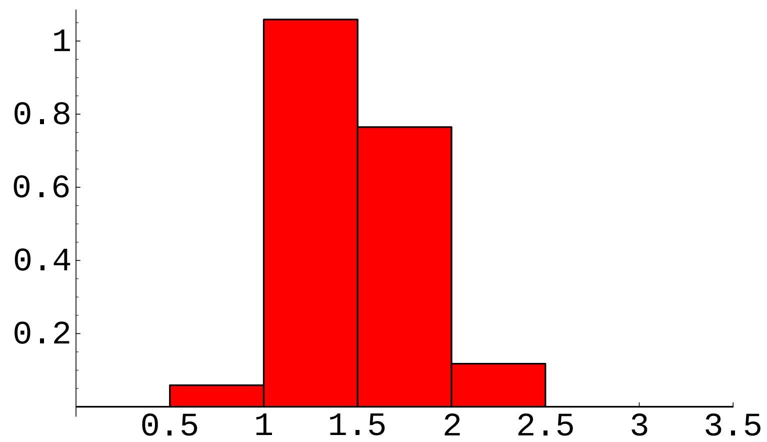

Figure $5b$:34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Rank 2 Curves: 1st Norm Zero: rank 2 one-param over $\mathbb{Q}(T)$

$\log(\mathrm{cond}) \in [15, 16]$, $t \in [0, 120]$, median is 1.64.

Family	Mean	Standard Deviation	log(conductor)	Number
1: [1,T,0,-3-2T,1]	1.91	0.25	[15.74,16.00]	2
2: [1,T,-19,-T-1,0]	1.57	0.36	[15.17,15.63]	4
3: [1,T,2,-T-1,0]	1.29		[15.47, 15.47]	1
4: [1,T,-16,-T-1,0]	1.75	0.19	[15.07,15.86]	4
5: [1,T,13,-T-1,0]	1.53	0.25	[15.08,15.91]	3
6: [1,T,-14,-T-1,0]	1.69	0.32	[15.06,15.22]	3
7: [1,T,10,-T-1,0]	1.62	0.28	[15.70,15.89]	3
8: [0,T,11,-T-1,0]	1.98		[15.87,15.87]	1
9: [1,T,-11,-T-1,0]				
10: [0,T,7,-T-1,0]	1.54	0.17	[15.08,15.90]	7
11: [1,T,-8,-T-1,0]	1.58	0.18	[15.23,25.95]	6
12: [1,T,19,-T-1,0]				
13: [0,T,3,-T-1,0]	1.96	0.25	[15.23, 15.66]	3
14: [0,T,19,-T-1,0]				
15: [1,T,17,-T-1,0]	1.64	0.23	[15.09, 15.98]	4
16: [0,T,9,-T-1,0]	1.59	0.29	[15.01, 15.85]	5
17: [0,T,1,-T-1,0]	1.51		[15.99, 15.99]	1
18: [1,T,-7,-T-1,0]	1.45	0.23	[15.14, 15.43]	4
19: [1,T,8,-T-1,0]	1.53	0.24	[15.02, 15.89]	10
20: [1,T,-2,-T-1,0]	1.60		[15.98, 15.98]	1
21: [0,T,13,-T-1,0]	1.67	0.01	[15.01, 15.92]	2
All Curves	1.61	0.25	[15.01, 16.00]	64

Repulsion or Attraction?

Conductors in $[15, 16]$; first set is rank 0 curves from 14 one-parameter families of rank 0 over $\mathbb{Q}$; second set rank 2 curves from 21 one-parameter families of rank 0 over $\mathbb{Q}$. The t-statistics exceed 6.

Family	**2nd vs 1st Zero**	**3rd vs 2nd Zero**	**Number**
Rank 0 Curves	2.16	3.41	863
Rank 2 Curves	1.93	3.27	701

The repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Can also interpret as attraction.

Comparison b/w One-Param Families of Different Rank

First normalized zero above the central point.

- The first family is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;
- the second family is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

Family	Median	Mean	Std. Dev.	Number
Rank 2 Curves (Rank 0 Families)	1.926	1.936	0.388	701
Rank 2 Curves (Rank 2 Families)	1.642	1.610	0.247	64

- t-statistic is 6.60, indicating the means differ.
- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on *how* we choose the curves.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	701 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- While the normalized zeros are repelled from the central point (and by different amounts for the two sets), the *differences* between the normalized zeros are statistically independent of this repulsion (t-statistics < 2).

- While for a given range of log-conductors the average second normalized zero of a rank 0 curve is close to the average first normalized zero of a rank 2 curve, they are not the same and the additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Spacings b/w Norm Zeros: rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	23 Rank 4 Curves	t-Statistic
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Conclusions and Future Work

- Theoretical supports the Independent Model and Birch and Swinnerton-Dyer Conjecture for one-parameter families over $\mathbb{Q}(T)$ as the conductors tend to infinity.

- Experimental suggests a different answer for finite conductors:

  $\diamond$ First normalized zero is repelled by zeros at the central point.

  $\diamond$ The more central point zeros the greater the repulsion.

  $\diamond$ Repulsion decreases as the conductor increases.

  $\diamond$ Difference b/w adjacent normalized zeros stat. indep. of the repulsion.

- What is the right model for rank $r+2$ curves from rank r one-parameter families over $\mathbb{Q}(T)$: Independent, Interaction or other?

- Unlike the excess rank investigations, noticeable convergence to the limiting theoretical results as we increase the conductors.

Appendices

The first appendix list various standard conjectures. The second appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor C_E, one needs about $\sqrt{C_E}\log C_E$ Fourier coefficients. The third is the statement (with assumptions) of the main theoretical result for the one-level density of one-parameter families of Elliptic curves over $\mathbb{Q}(T)$.

Appendix I: Standard Conjectures

Generalized Riemann Hypothesis (for Elliptic Curves) *Let $L(s, E)$ be the (normalized) L-function of the elliptic curve E. Then the non-trivial zeros of $L(s, E)$ satisfy Re$(s) = \frac{1}{2}$.*

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let E be an elliptic curve of geometric rank r over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$, T is the subset of torsion points). Then the analytic rank (the order of vanishing of the L-function at the central point) is also r.*

Tate's Conjecture for Elliptic Surfaces [Ta] *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to **C** and satisfies $-$ord$_{s=2} L_2(\mathcal{E}, s) =$ rank $NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line Re$(s) = 2$.*

Most of the 1-param families we investigate are rational surfaces, where Tate's conjecture is known. See [RSi].

Appendix II: Numerically Approximating Ranks: Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \quad \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

Algorithm for $L^r(s, E)$: I

$$\begin{aligned}
\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\
&= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\
&= \int_1^\infty f(iy/\sqrt{N})(y^{s-1} + \epsilon y^{1-s})dy.
\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1} + \epsilon(-1)^k y^{1-s})dy.$$

At $s = 1$,

$$\Lambda^{(k)}(E,1) = (1 + \epsilon(-1)^k) \int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s, E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy \\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-r}}{n^r\cdot n!}x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t;\\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12};\\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3};\\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160};\\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\,e^{-2\pi ny/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

Appendix III: 1-Level Density

Definitions:

$$D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log C_E}{2\pi} \gamma_E^{(j_i)} \right)$$

$D_{n,\mathcal{F}}^{(r)}(\phi)$: n-level density with contribution of r zeros at central point removed.

$\mathcal{F}_N$: Rational one-parameter family, $t \in [N, 2N]$, conductors monotone.

ASSUMPTIONS

1-parameter family of Ell Curves, rank r over $\mathbb{Q}(T)$, rational surface. Assume

- GRH;
- $j(t)$ non-constant;
- Sq-Free Sieve if $\Delta(t)$ has irr poly factor of deg ≥ 4.

Pass to positive percent sub-seq where conductors polynomial of degree m.

ϕ_i even Schwartz, support σ_i:

- $\sigma_1 < \min\left(\frac{1}{2}, \frac{2}{3m}\right)$ for 1-level
- $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for 2-level.

MAIN RESULT

Theorem (Miller 2004): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D^{(r)}_{n,\mathcal{F}_N}(\phi) \longrightarrow \int \phi(x) W_{\mathcal{G}}(x) dx,$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

1 and 2-level densities confirm Katz-Sarnak, B-SD predictions for small support.

Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$,
   $9t+1$ Square-Free: all even.

2. $y^2 = x^3 \pm 4(4t+2)x$,
   $4t+2$ Square-Free:
   $+$ all odd, $-$ all even.

3. $y^2 = x^3 + tx^2 - (t+3)x + 1$,
   $t^2 + 3t + 9$ Square-Free: all odd.

First two rank 0 over $\mathbb{Q}(T)$, third is rank 1.

Without 2-Level Density, couldn't say *which* orthogonal group.

Examples (cont)

Rational Surface of Rank 6 over $\mathbf{Q}(t)$:

$$\begin{aligned} y^2 \ = \ & x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ & +(2ct - D)(t^2 + 2t - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8,916,100,448,256,000,000 \\ B &= -811,365,140,824,616,222,208 \\ C &= 26,497,490,347,321,493,520,384 \\ D &= -343,107,594,345,448,813,363,200 \\ a &= 16,660,111,104 \\ b &= -1,603,174,809,600 \\ c &= 2,149,908,480,000 \end{aligned}$$

Need GRH, Sq-Free Sieve to handle sieving.

Appendix IV: t-Statistics

The Pooled Two-Sample t-Procedure is

$$t \;=\; (\overline{X_1} - \overline{X_2}) \Big/ s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where $\overline{X_i}$ is the sample mean of n_i observations of population i, s_i is the sample standard deviation and

$$s_p \;=\; \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

is the pooled variance; t has a t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

The Unpooled Two-Sample t-Procedure is

$$t \;=\; (\overline{X_1} - \overline{X_2}) \Big/ \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}};$$

this is approximately a t distribution with

$$\frac{(n_1 - 1)\,(n_2 - 1)\,(n_2 s_1^2 + n_1 s_2^2)^2}{(n_2 - 1)n_2^2 s_1^4 + (n_1 - 1)n_1^2 s_2^4}$$

degrees of freedom

Bibliography

[BMW] B. Bektemirov, B. Mazur and M. Watkins, *Average Ranks of Elliptic Curves*, preprint.

[BEW] B. Berndt, R. Evans and K. Williams, *Gauss and Jacobi Sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts, vol. **21**, Wiley-Interscience Publications, John Wiley & Sons, Inc., New York, 1998.

[Bi] B. Birch, *How the number of points of an elliptic curve over a fixed prime field varies*, J. London Math. Soc. **43**, 1968, $57-60$.

[BS] B. Birch and N. Stephens, *The parity of the rank of the Mordell-Weil group*, Topology **5**, 1966, $295-299$.

[BSD1] B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. I*, J. reine angew. Math. **212**, 1963, $7-25$.

[BSD2] B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. reine angew. Math. **218**, 1965, $79-108$.

[BCDT] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over* $\mathbf{Q}$*: wild* 3*-adic exercises*, J. Amer. Math. Soc. **14**, no. 4, 2001, $843-939$.

[Br] A. Brumer, *The average rank of elliptic curves I*, Invent. Math. **109**, 1992, $445-472$.

[BHB3] A. Brumer and R. Heath-Brown, *The average rank of elliptic curves III*, preprint.

[BHB5] A. Brumer and R. Heath-Brown, *The average rank of elliptic curves V*, preprint.

[BM] A. Brumer and O. McGuinness, *The behaviour of the Mordell-Weil group of elliptic curves*, Bull. AMS **23**, 1991, $375-382$.

[CW] J. Coates and A. Wiles, *On the conjecture of Birch and Swinnterton-Dyer*, Invent. Math. **39**, 1977, $43-67$.

[Cr] Cremona, *Algorithms for Modular Elliptic Curves*, Cambridge University Press, 1992.

[Di] F. Diamond, *On deformation rings and Hecke rings*, Ann. Math. **144**, 1996, $137-166$.

[Fe1] S. Fermigier, *Zéros des fonctions L de courbes elliptiques*, Exper. Math. **1**, 1992, $167 - 173$.

[Fe2] S. Fermigier, *Étude expérimentale du rang de familles de courbes elliptiques sur* $\mathbb{Q}$, Exper. Math. **5**, 1996, $119 - 130$.

[FP] E. Fouvrey and J. Pomykala, *Rang des courbes elliptiques et sommes d'exponentelles*, Monat. Math. **116**, 1993, $111 - 125$.

[GM] F. Gouvéa and B. Mazur, *The square-free sieve and the rank of elliptic curves*, J. Amer. Math. Soc. **4**, 1991, $45 - 65$.

[Go] D. Goldfeld, *Conjectures on elliptic curves over quadratic fields*, Number Theory (Proc. Conf. in Carbondale, 1979), Lecture Notes in Math. **751**, Springer-Verlag, 1979, $108 - 118$.

[Gr] Granville, *ABC Allows Us to Count Squarefrees*, International Mathematics Research Notices **19**, 1998, $991 - 1009$.

[He] H. Helfgott, *On the distribution of root numbers in families of elliptic curves*, preprint.

[Ho] C. Hooley, *Applications of Sieve Methods to the Theory of Numbers*, Cambridge University Press, Cambridge, 1976.

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91**, 2000, $55 - 131$.

[Kn] A. Knapp, *Elliptic Curves*, Princeton University Press, Princeton, 1992.

[KS1] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.

[KS2] N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36**, 1999, $1 - 26$.

[Ko] V. Kolyvagin, *On the Mordell-Weil group and the Shafarevich-Tate group of modular elliptic curves*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, $429 - 436$.

[Mai] L. Mai, *The analytic rank of a family of elliptic curves*, Canadian Journal of Mathematics **45**, 1993, $847 - 862$.

[Mes1] J. Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Mathematica **58**, 1986, $209 - 232$.

[Mes2] J. Mestre, *Courbes elliptiques de rang $\geq$ 11 sur* $\mathbf{Q}(t)$, C. R. Acad. Sci. Paris, ser. 1, **313**, 1991, $139 - 142$.

[Mes3] J. Mestre, *Courbes elliptiques de rang $\geq$ 12 sur* $\mathbf{Q}(t)$, C. R. Acad. Sci. Paris, ser. 1, **313**, 1991, $171 - 174$.

[Mi] P. Michel, *Rang moyen de familles de courbes elliptiques et lois de Sato-Tate*, Monat. Math. **120**, 1995, $127 - 136$.

[Mil1] S. J. Miller, 1- *and* 2-*Level Densities for Families of Elliptic Curves: Evidence for the Underlying Group Symmetries*, P.H.D. Thesis, Princeton University, 2002, http://www.math.princeton.edu/∼sjmiller/thesis/thesis.pdf.

[Mil2] S. J. Miller, 1- *and* 2-*level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **140** (2004), 952-992.

[Mil3] S. J. Miller, *Variation in the number of points on elliptic curves and applications to excess rank*, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120.

[Mil4] S. J. Miller, *Investigations of zeros near the central point of elliptic curve L-functions* (with an appendix by E. Dueñez), to appear in Experimental Mathematics.

[Mor] Mordell, *Diophantine Equations*, Academic Press, New York, 1969.

[Na1] K. Nagao, *On the rank of elliptic curve $y^2 = x^3 - kx$*, Kobe J. Math. **11**, 1994, $205 - 210$.

[Na2] K. Nagao, *Construction of high-rank elliptic curves*, Kobe J. Math. **11**, 1994, $211 - 219$.

[Na3] K. Nagao, *$\mathbb{Q}(t)$-rank of elliptic curves and certain limit coming from the local points*, Manuscr. Math. **92**, 1997, $13 - 32$.

[Ri] Rizzo, *Average root numbers for a non-constant family of elliptic curves*, preprint.

[Ro] D. Rohrlich, *Variation of the root number in families of elliptic curves*, Compos. Math. **87**, 1993, $119 - 151$.

[RSi] M. Rosen and J. Silverman, *On the rank of an elliptic surface*, Invent. Math. **133**, 1998, $43 - 67$.

[Ru] M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L-functions*, P.H.D. Thesis, Princeton University, 1998, http://www.ma.utexas.edu/users/miker/thesis/thesis.html.

[RS] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Journal of Math. **81**, 1996, $269 - 322$.

[Sh] T. Shioda, *Construction of elliptic curves with high-rank via the invariants of the Weyl groups*, J. Math. Soc. Japan **43**, 1991, $673 - 719$.

[Si1] J. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **106**, Springer-Verlag, Berlin - New York, 1986.

[Si2] J. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **151**, Springer-Verlag, Berlin - New York, 1994.

[Si3] J. Silverman, *The average rank of an algebraic family of elliptic curves*, J. reine angew. Math. **504**, 1998, $227-236$.

[Sn] N. Snaith, *Derivatives of random matrix characteristic polynomials with applications to elliptic curves*, preprint.

[St1] N. Stephens, *A corollary to a conjecture of Birch and Swinnerton-Dyer*, J. London Math. Soc. **43**, 1968, $146-148$.

[St2] N. Stephens, *The diophantine equation $X^3+Y^3=DZ^3$ and the conjectures of Birch and Swinnerton-Dyer*, J. reine angew. Math. **231**, 1968, $16-162$.

[ST] C. Stewart and J. Top, *On ranks of twists of elliptic curves and power-free values of binary forms*, Journal of the American Mathematical Society **40**, number 4, 1995.

[Ta] J. Tate, *Algebraic cycles and the pole of zeta functions*, Arithmetical Algebraic Geometry, Harper and Row, New York, 1965, $93-110$.

[TW] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. Math. **141**, 1995, $553-572$.

[Wa] L. Washington, *Class numbers of the simplest cubic fields*, Math. Comp. **48**, number 177, 1987, $371-384$.

[Wi] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. Math **141**, 1995, $443-551$.

[Yo1] M. Young, *Lower order terms of the 1-level density of families of elliptic curves*, IMRN **10** (2005), 587–633.

[Yo2] M. Young, *Low-Lying Zeros of Families of Elliptic Curves*, JAMS, to appear.

[ZK] D. Zagier and G. Kramarz, *Numerical investigations related to the L-series of certain elliptic curves*, J. Indian Math. Soc. **52** (1987), 51–69.