Variance of Gaussian Primes Across Sectors and the Hecke L-functions Ratios Conjecture

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History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} \;=\; \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ◇ n-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point.
- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ Predicts lower order terms to square-root level;
  - ◇ Fast computations.

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- Equivalently, an odd prime p is a sum of squares if and only if it splits in $\mathbb{Z}[i]$.

Angles of Gaussian Primes

- For an odd prime $p = a^2 + b^2 = (a+bi)(a-bi)$, define

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- By convention, we choose $0 \leq b \leq a$ so that $\frac{b}{a} \in [0, 1]$, i.e. $\theta_p \in [0, \frac{\pi}{4}]$ (well-defined!)

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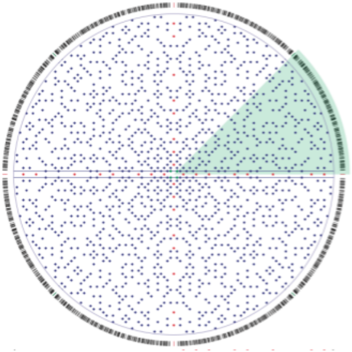
- 1919, Hecke: The Gaussian primes are uniformly distributed in the complex plane.



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- The expected value is easily calculated to be

$$\langle N_{K,X}(\theta) \rangle \;\sim\; \frac{X}{K}$$

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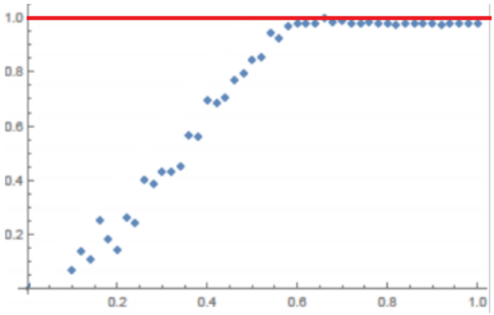
- For $K << X$, the question remains open.
- Rudnick and Waxman conjectured

$$Var(N_{K,X}(\theta)) \sim \frac{X}{K} \min\left\{1, 2\frac{\log K}{\log X}\right\}.$$

Fixing the Graph



Data vs. $Var(N_{K,X}(\theta)) \sim \frac{X}{K}$
(normalized by $\mathbb{E}[N_{K,X}(\theta)] = \frac{X}{K}$)

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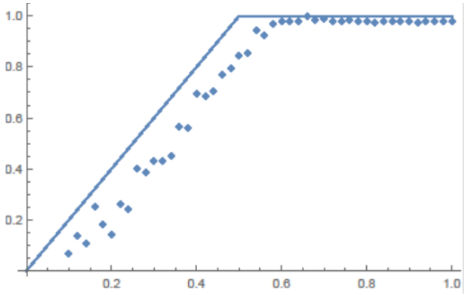

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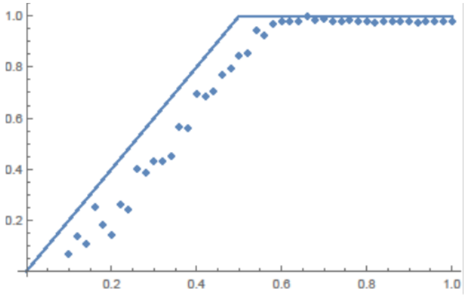

The conjecture does a **better** job of fitting the data than a total random distribution (marked in red), but to get a better fit, we'll need to incorporate lower-order terms

$\Longrightarrow$ ratios conjecture allows us to do this

Ingredients for the Ratios Recipe

- Approximate Functional Equation for our Hecke *L*-function:

$$L_k(s) \;=\; \sum_{n\geq 1} \frac{A_n}{n^s} + \pi^{2s-1}\frac{\Gamma\left(1-s+|2k|\right)}{\Gamma\left(s+|2k|\right)} + \text{Error}$$

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- Generalized Mobius function:

$$\frac{1}{L_k(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s},$$

where $\mu(n)$ is a multiplicative function explicitly defined at prime powers.

Procedure (Recipe)

- We wish to apply the ratios conjecture to

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- Breaking $F_K(\theta_{\mathfrak{a}} - \theta)$ into its Fourier components allows us to study the objects twisted by Hecke characters as we range over $k \in \mathbb{Z}$:

$$\sum_{\mathfrak{a}} \Lambda(\mathfrak{a}) \cdot \chi^{4k}(\mathfrak{a})\Phi\left(\frac{N(\mathfrak{a})}{X}\right).$$

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- Variance: subtract expected value and take magnitude:

$$\int_{(2)} \int_{(2)} \left\langle \frac{L_k'}{L_k}(s) \frac{L_k'}{L_k}(\overline{s'}) \right\rangle_k \tilde{\Phi}(s) \tilde{\Phi}(\overline{s'}) X^s X^{\overline{s'}} \, ds \, d\overline{s'}$$

where we average over k within $\langle \cdot \rangle$.

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$$\mathrm{Var}(\psi_{K,X}(\theta)) = \int_{(2)} \int_{(2)} \left\langle \frac{L_k'}{L_k}(s) \frac{L_k'}{L_k}(\overline{s'}) \right\rangle_k \tilde{\Phi}(s)\tilde{\Phi}(\overline{s'}) X^s X^{\overline{s'}} \, ds \, d\overline{s'}$$

- The ratios conjecture estimates $\left\langle \frac{L_k'}{L_k}(s) \frac{L_k'}{L_k}(\overline{s'}) \right\rangle_k$ as a product of zeta/L-function factors, and an arithmetic factor.

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- We expect the leading term to match the RMT prediction, and lower order terms captured by the ratios conjecture are smaller by factors of $\log X$.

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$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha}Y(\alpha,\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}\tilde{\Phi}(\frac{1}{2}+\alpha)\tilde{\Phi}(\frac{1}{2}+\beta)e^{\gamma\log K\alpha}e^{\gamma\log K\beta}\,da\overline{db}$$

$$+\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha}\pi^{2\beta}\frac{(2K)^{-\beta}}{1-\beta}Y(\alpha,-\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}\tilde{\Phi}(\frac{1}{2}+\alpha)\tilde{\Phi}(\frac{1}{2}+\beta)e^{\gamma\log K\alpha}e^{\gamma\log K\beta}\,da\overline{db}$$

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where

$$Y(\alpha,\beta,\gamma,\delta)$$
$$=\frac{\zeta(1+2\alpha)\zeta(1+2\beta)L(1+2\gamma)L(1+2\delta)\zeta(1+\alpha+\beta)L(1+\alpha+\beta)\zeta(1+\gamma+\delta)^2L(1+\gamma+\delta)^2}{\zeta(1+\alpha+\gamma)L(1+\alpha+\gamma)\zeta(1+\alpha+\delta)L(1+\alpha+\delta)\zeta(1+\beta+\gamma)L(1+\beta+\gamma)\zeta(1+\beta+\delta)L(1+\beta+\delta)}.$$

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- Goal is to match with RMT-model prediction

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