Schreier multisets and the s-step Fibonacci sequences

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2022 Polymath Jr REU:
https://geometrynyc.wixsite.com/polymathreu

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

https://arxiv.org/pdf/2304.05409.pdf

Integers Conference, University of Georgia, May 18th, 2023

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We count sets in $\mathcal{A}_n$.

- $\mathcal{A}_1 = \{\{1\}\} \Rightarrow |\mathcal{A}_1| = 1$
- $\mathcal{A}_2 = \{\{2\}\} \Rightarrow |\mathcal{A}_2| = 1$
- $\mathcal{A}_3 = \{\{2,3\},\{3\}\} \Rightarrow |\mathcal{A}_3| = 2$
- $\mathcal{A}_4 = \{\{2,4\},\{3,4\},\{4\}\} \Rightarrow |\mathcal{A}_4| = 3$
- $\mathcal{A}_5 = \{\{5\},\{2,5\},\{3,5\},\{4,5\},\{3,4,5\}\} \Rightarrow |\mathcal{A}_5| = 5$
- $\ldots$

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- $\mathcal{A}_4 = \{\{2, 4\}, \{3, 4\}, \{4\}\} \Rightarrow |\mathcal{A}_4| = 3$
- $\mathcal{A}_5 = \{\{5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}\} \Rightarrow |\mathcal{A}_5| = 5$
- $\ldots$

$(|\mathcal{A}_n|)_{n=1}^{\infty}$: $1, 1, 2, 3, 5, 8, \ldots$ is Fibonacci!

Theorem

Define $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$. Then $|\mathcal{A}_n| = F_n$ for all $n \geqslant 1$.

Will give proof from A. Bird's post:

https://outofthenormmaths.wordpress.com/2012/05/13/jozef-schreier-schreier-sets-and-the-fibonacci-sequence/

Proof that $|\mathcal{A}_n| = F_n$

$$\mathcal{A}_n := \{A \subset \{1, \ldots, n\} : n \in A \text{ and } \min A \geqslant |A|\}.$$

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Verify that $|\mathcal{A}_1| = |\mathcal{A}_2| = 1$. For $n \geqslant 1$, we want $|\mathcal{A}_{n+2}| = |\mathcal{A}_{n+1}| + |\mathcal{A}_n|$.

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Both R_{n+2} and S_{n+2} are one-to-one. Furthermore,

$$R_{n+2}(\mathcal{A}_{n+1}) = \{A \in \mathcal{A}_{n+2} : n+1 \notin A\}$$
$$S_{n+2}(\mathcal{A}_n) = \{A \in \mathcal{A}_{n+2} : n+1 \in A\}.$$

$$\implies |\mathcal{A}_n| + |\mathcal{A}_{n+1}| = |S_{n+2}(\mathcal{A}_n)| + |R_{n+2}(\mathcal{A}_{n+1})| = |\mathcal{A}_{n+2}|.$$

□

s-step Fibonacci sequence

For $s \geqslant 2$, the s-step Fibonacci sequence: $F^{(s)}_{2-s} = \cdots = F^{(s)}_0 = 0$, $F^{(s)}_1 = 1$, and

$$F^{(s)}_n = F^{(s)}_{n-1} + \cdots + F^{(s)}_{n-s}, \text{ for } n \geqslant 2.$$

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$s = 2$ gives the Fibonacci sequence.
$s = 3$ gives the Tribonacci sequence

$$0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots.$$

Problem

Generate the s-step Fibonacci sequence using the Schreier condition.

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Use multisets!

Schreier multisets and the *s*-step Fibonacci sequences

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$\{4, 5, 5\}$ and $\{5, 6, 6, 10, 12\}$ are Schreier, but $\{2, 3, 3\}$ is not.

Schreier multisets and the s-step Fibonacci sequences

Define $\mathcal{A}_n^{(s-1)} :=$

$$\{A \subset \{\underbrace{1, \ldots, 1}_{s-1}, \ldots, \underbrace{n-1, \ldots, n-1}_{s-1}, n\} \;:\; n \in A \text{ and } A \text{ is Schreier}\}.$$

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Theorem (Chu, Irmak, Miller, Szalay, and Zhang, 2023)

For $n \in \mathbb{N}$ and $s \geqslant 2$, have $|\mathcal{A}_n^{(s-1)}| = F_n^{(s)}$.

Proof ingredient - generalized binomials vs s-step Fibonacci numbers

For $s \geqslant 1$, $\binom{n}{k}_s$ counts ways to distribute k identical objects among n labelled boxes, each has capacity s.

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Recall the well-known identity

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Belbachir, Bouroubi, and Khelladi (2008) generalized (1):

$$\forall n \geqslant 0, \forall s \geqslant 1 : \quad \sum_{k=0}^{\lfloor sn/(s+1) \rfloor} \binom{n-k}{k}_s = F_{n+1}^{(s+1)}.$$

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Trivially, $\{n\} \in \mathcal{A}_n^{(s-1)}$. An extension of $\{n\}$ by k elements is a choice of k elements from

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Therefore,

$$|\mathcal{A}_n^{(s-1)}| = \sum_{k=0}^{\lfloor (n-1)(s-1)/s \rfloor} \binom{n-1-k}{k}_{s-1} = F_n^{(s)}.$$

□

Yet another way to obtain the Fibonacci sequence

Multisets & Fibonacci

Fix $\mathbf{s} = (s_n)_{n=1}^{\infty} \subset \mathbb{Z}_{\geqslant 0}$ satisfying

$$s_n \geqslant k, \forall n \geqslant 2k+1, \forall k \geqslant 1.$$

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Turns out that $|\mathcal{B}_n^{\mathbf{s}}| = F_n$ for $n \in \mathbb{N}$. We prove a more general result.

Multisets & other recurrences

Fix $u \geqslant 2$. Define $(K_n^{(u)})_{n=1}^{\infty}$:

$$K_1^{(u)} = \cdots = K_u^{(u)} = 1 \text{ and } K_n^{(u)} = K_{n-1}^{(u)} + K_{n-u}^{(u)}, \quad n \geqslant u+1.$$

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Given $\mathbf{s} = (s_n)_{n=1}^{\infty} \subset \mathbb{Z}_{\geqslant 0}$, let

$$\mathcal{B}_n^{\mathbf{s},u} := \{B \subset \{\underbrace{1,\ldots,1}_{s_1},\underbrace{2,\ldots,2}_{s_2},\ldots,\underbrace{n,\ldots,n}_{s_n}\} : \min B \geqslant u|B|+1\}.$$

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Theorem (Chu, Irmak, Miller, Szalay, and Zhang, 2023)

Fix $u \geqslant 2$ and $\mathbf{s} = (s_n)_{n=1}^{\infty} \subset \mathbb{Z}_{\geqslant 0}$ such that $s_n \geqslant k$ for all $n \geqslant uk+1$ and $k \geqslant 1$. We have

$$|\mathcal{B}_n^{\mathbf{s},u}| = K_n^{(u)}, \quad n \in \mathbb{N}.$$

Nonlinear Schreier condition

Nonlinear Schreier condition & decompositions

Define, for $n, p \in \mathbb{N}$,

$$\mathcal{A}_n^p \; := \; \{S \subset \{1, \ldots, n\} \; : \; \min S \geqslant |S|^p \text{ and } n \in S\}.$$

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For $(n, p) \in (\mathbb{N}, \mathbb{Z}_{\geqslant 0})$, let $K_{n,p}$ count decompositions of n where the smallest part is greater than the number of parts raised to the p^{th} power:

$$K_{n,p} \; := \; \left| \left\{ (x_1, \ldots, x_k) \; : \; \sum_{i=1}^{k} x_i = n \text{ and } \min_{1 \leqslant i \leqslant k} x_i > k^p \right\} \right| .$$

$K_{n,0} = F_{n-1}$ for $n \geqslant 1$.

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$K_{13,1} = 12$ because

$$\begin{aligned} 13 &= 3+10 = 10+3 = 4+9 = 9+4 \\ &= 5+8 = 8+5 = 6+7 = 7+6 \\ &= 4+4+5 = 4+5+4 = 5+4+4. \end{aligned}$$

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When $p = 1$, we regain $|\mathcal{A}_n^1| = K_{n+1,0} = F_n$.

Further research

Problem

Counting sets satisfying linear Schreier-type condition $p \min S \geqslant q|S|$ $(p, q \in \mathbb{N})$ has been done. However, less is known about nonlinear conditions.

We counted sets F that satisfying $\min F \geqslant |F|^s$, where $s \in \mathbb{N}_{\geqslant 2}$. Further research can investigate other nonlinear conditions.

Thanks to 2022 Polymath Jr REU: https://geometrynyc.wixsite.com/polymathreu.

Paper: https://arxiv.org/pdf/2304.05409.pdf.