Closed-form moments in elliptic curve families and low-lying zeros

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Goals

FELLOWSHIP OF THE RING SEMINAR

(Sunda har i ened mente)

(Ash intya durbatuluk ash intya gimbatul. Ash intya thrakatuluk agh kala-ishi krimpatul)

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Conjectures and Theorems

Tate's Conjecture

Tate's Conjecture for Elliptic Surfaces

Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\text{ét}}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$-\text{ord}_{s=2} L_2(\mathcal{E}, s) \ = \ \text{rank}\, NS(\mathcal{E}/\mathbb{Q}),$$

where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line $\text{Re}(s) = 2$.

Theorem: Preliminaries

Consider a one-parameter family

$$\mathcal{E} : y^2 + a_1(T)xy + a_3(T)y \;=\; x^3 + a_2(T)x^2 + a_4(T)x + a_6(T).$$

Let $a_t(p) = p + 1 - N_p$, where N_p is the number of solutions mod p (including ∞). Define

$$A_{\mathcal{E}}(p) \;:=\; \frac{1}{p}\sum_{t(p)} a_t(p).$$

$A_{\mathcal{E}}(p)$ is bounded independent of p (Deligne).

Theorem: Preliminaries

Theorem

Rosen-Silverman (Conjecture of Nagao For an elliptic surface (a one-parameter family), assume Tate's conjecture. Then

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} -A_{\mathcal{E}}(p)\log p \ = \ \mathrm{rank}\ \mathcal{E}(\mathbb{Q}(t)).$$

Tate's conjecture is known for rational surfaces: An elliptic surface $y^2 = x^3 + A(t)x + B(t)$ is rational iff one of the following is true:

- $0 < \max\{3\deg A, 2\deg B\} < 12$;
- $3\deg A = 2\deg B = 12$ and $\mathrm{ord}_{t=0} t^{12}\Delta(t^{-1}) = 0$.

Conjectures: ABC, Square-Free

ABC Conjecture

Fix $\epsilon > 0$. For coprime positive integers a, b and c with $c = a + b$ and $N(a,b,c) = \prod_{p|abc} p$, $c \ll_\epsilon N(a,b,c)^{1+\epsilon}$.

Square-Free Sieve Conjecture

Fix an irreducible polynomial $f(t)$ of degree at least 4. As $N \to \infty$, the number of $t \in [N, 2N]$ with $D(t)$ divisible by p^2 for some $p > \log N$ is $o(N)$.

Conjectures: Restricted Sign

Restricted Sign Conjecture (for the Family $\mathcal{F}$)

Consider a 1-parameter family $\mathcal{F}$ of elliptic curves. As $N \to \infty$, the signs of the curves E_t are equidistributed for $t \in [N, 2N]$.

Fails: constant $j(t)$ where all curves same sign.
Rizzo:

$$E_t : y^2 = x^3 + tx^2 - (t+3)x + 1, \;\; j(t) = 256(t^2 + 3t + 9),$$

for every $t \in \mathbb{Z}$, E_t has odd functional equation,

$$E_t : y^2 = x^3 + \frac{t}{4}x^2 - \frac{36t^2}{t-1728}x - \frac{t^3}{t-1728}, \;\; j(t) = t,$$

as t ranges over $\mathbb{Z}$ in the limit 50.1859% have even and 49.8141% have odd functional equation.

Conjectures: Polynomial Mobius

Polynomial Moebius

Let $f(t)$ be an irreducible polynomial such that no fixed square divides $f(t)$ for all t. Then $\sum_{t=N}^{2N} \mu(f(t)) = o(N)$.

Conjectures: Polynomial Mobius

Helfgott shows the Square-Free Sieve and Polynomial Moebius imply the Restricted Sign conjecture for many families. More precisely, let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$.

Theorem

Equidistribution of Sign in a Family Let $\mathcal{F}$ be a one-parameter family with coefficients integer polynomials in $t \in [N, 2N]$. If $j(t)$ and $M(t)$ are non-constant, then the signs of E_t, $t \in [N, 2N]$, are equidistributed as $N \to \infty$. Further, if we restrict to good t, $t \in [N, 2N]$ such that $D(t)$ is good (usually square-free), the signs are still equidistributed in the limit.

Constructing Families with Moderate Rank

Constructing one-parameter families of elliptic curves over $\mathbb{Q}(T)$ with moderate rank (with Scott Arms and Álvaro Lozano-Robledo), Journal of Number Theory **123** (2007), no. 2, 388–402.

http://arxiv.org/pdf/math/0406579.pdf.

Mordell-Weil and Legendre Expansions

Mordell-Weil Theorem: Rational solutions:
$E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus$ Finite Group.
Question: how does r depend on E?

Attach an L-Function to E: As $\zeta(s)$ gives us information on primes, expect L-Function gives us information on E.

Review: Legendre Symbol: $\left(\frac{0}{p}\right) = 0$ and

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \bmod p \text{ has two solutions} \\ -1 & \text{if } x^2 \equiv a \bmod p \text{ has no solutions.} \end{cases}$$

Note $1 + \left(\frac{a}{p}\right)$ is the number of solutions to $x^2 \equiv a \bmod p$.

1-Level Expansion

$$D_{1,\mathcal{F}_N}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_j \phi\left(\gamma_{t,j}\frac{\log C_t}{2\pi}\right) + O\left(\frac{\log\log N}{\log N}\right)$$

$$= \frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\left[\widehat{\phi}(0)+\phi_i(0)\right]$$

$$-\frac{2}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_p \frac{1}{p}\frac{\log p}{\log C_E}\widehat{\phi}\left(\frac{\log p}{\log C_E}\right)a_t(p)$$

$$-\frac{2}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_p \frac{1}{p^2}\frac{\log p}{\log C_E}\widehat{\phi}\left(2\frac{\log p}{\log C_E}\right)a_t^2(p)$$

Want to move $\frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}$, leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t \bmod p} a_t^r(p), \quad r = 1 \text{ or } 2.$$

Input

For many families

$$\begin{aligned} A_{1,\mathcal{F}}(p) &= -rp + O(1) \\ A_{2,\mathcal{F}}(p) &= p^2 + O(p^{3/2}) \end{aligned}$$

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(T)$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} -\frac{A_{1,\mathcal{F}}(p)\log p}{p} = r$$

Surfaces with $j(T)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p^2 + O\left(p^{3/2}\right).$$

Rank 6 Family

Rational Surface of Rank 6 over $\mathbb{Q}(T)$:

$$\begin{aligned} y^2 &= x^3 + (2aT - B)x^2 + (2bT - C)(T^2 + 2T - A + 1)x \\ &\quad + (2cT - D)(T^2 + 2T - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8,916,100,448,256,000,000 \\ B &= -811,365,140,824,616,222,208 \\ C &= 26,497,490,347,321,493,520,384 \\ D &= -343,107,594,345,448,813,363,200 \\ a &= 16,660,111,104 \\ b &= -1,603,174,809,600 \\ c &= 2,149,908,480,000 \end{aligned}$$

Need GRH, Sq-Free Sieve to handle sieving.

Constructing Rank 6 Family

Idea: can explicitly evaluate linear and quadratic Legendre sums.

Use: a and b are not both zero mod p and $p > 2$, then for $t \in \mathbb{Z}$

$$\sum_{t=0}^{p-1} \left(\frac{at^2 + bt + c}{p}\right) = \begin{cases} (p-1)\left(\frac{a}{p}\right) & \text{if } p|(b^2 - 4ac) \\ -\left(\frac{a}{p}\right) & \text{otherwise.} \end{cases}$$

Thus if $p|(b^2 - 4ac)$, the summands are $\left(\frac{a(t-t')^2}{p}\right) = \left(\frac{a}{p}\right)$, and the t-sum is large.

Constructing Rank 6 Family

$$\begin{aligned} y^2 = f(x,T) &= x^3T^2 + 2g(x)T - h(x) \\ g(x) &= x^3 + ax^2 + bx + c,\ c \neq 0 \\ h(x) &= (A-1)x^3 + Bx^2 + Cx + D \\ D_T(x) &= g(x)^2 + x^3h(x). \end{aligned}$$

Note that $D_T(x)$ is one-fourth of the discriminant of the quadratic (in T) polynomial $f(x,T)$.
Our elliptic curve $\mathcal{E}$ is not written in standard form, as the coefficient of x^3 is T^2. This is harmless. As $y^2 = f(x,T)$, for the fiber at $T = t$ we have

$$a_t(p) = -\sum_{x(p)} \left(\frac{f(x,t)}{p}\right) = -\sum_{x(p)} \left(\frac{x^3t^2 + 2g(x)t - h(x)}{p}\right).$$

Constructing Rank 6 Family

We study $-pA_{\mathcal{E}}(p) = \sum_{x=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right)$.
When $x \equiv 0$ the t-sum vanishes if $c \not\equiv 0$, as it is just $\sum_{t=0}^{p-1}\left(\frac{2ct-D}{p}\right)$.

Assume now $x \not\equiv 0$. By the lemma on Quadratic Legendre Sums

$$\sum_{t=0}^{p-1}\left(\frac{x^3t^2+2g(x)t-h(x)}{p}\right) = \begin{cases}(p-1)\left(\frac{x^3}{p}\right) & \text{if } p \mid D_t(x)\\ -\left(\frac{x^3}{p}\right) & \text{otherwise.}\end{cases}$$

Goal:find coefficients a, b, c, A, B, C, D so that $D_t(x)$ has six distinct, non-zero roots that are squares.

Constructing Rank 6 Family

Assume we can find such coefficients. Then

$$\begin{aligned}
-pA_{\mathcal{E}}(p) &= \sum_{x=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right) = \sum_{x=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{x^3t^2+2g(x)t-h(x)}{p}\right)\\
&= \sum_{x=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right) + \sum_{x:D_t(x)\equiv 0}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right)\\
&\quad + \sum_{x:xD_t(x)\not\equiv 0}\sum_{t=0}^{p-1}\left(\frac{f(x,t)}{p}\right)\\
&= 0 + 6(p-1) - \sum_{x:xD_t(x)\not\equiv 0}\left(\frac{x^3}{p}\right) = 6p.
\end{aligned}$$

Constructing Rank 6 Family

We must find $a, \ldots, D$ such that $D_t(x)$ has six distinct, non-zero roots ρ_i^2:

$$\begin{aligned} D_t(x) &= g(x)^2 + x^3 h(x) \\ &= Ax^6 + (B + 2a)x^5 + (C + a^2 + 2b)x^4 \\ &\quad + (D + 2ab + 2c)x^3 \\ &\quad + (2ac + b^2)x^2 + (2bc)x + c^2 \\ &= A(x^6 + R_5x^5 + R_4x^4 + R_3x^3 + R_2x^2 + R_1x + R_0) \\ &= A(x - \rho_1^2)(x - \rho_2^2)(x - \rho_3^2)(x - \rho_4^2)(x - \rho_5^2)(x - \rho_6^2). \end{aligned}$$

Constructing Rank 6 Family

Because of the freedom to choose B, C, D there is no problem matching coefficients for the x^5, x^4, x^3 terms. We must simultaneously solve in integers

$$\begin{aligned} 2ac + b^2 &= R_2 A \\ 2bc &= R_1 A \\ c^2 &= R_0 A. \end{aligned}$$

For simplicity, take $A = 64R_0^3$. Then

$$\begin{array}{rclcrcl} c^2 &=& 64R_0^4 & \longrightarrow & c &=& 8R_0^2 \\ 2bc &=& 64R_0^3 R_1 & \longrightarrow & b &=& 4R_0 R_1 \\ 2ac + b^2 &=& 64R_0^3 R_2 & \longrightarrow & a &=& 4R_0 R_2 - R_1^2. \end{array}$$

Constructing Rank 6 Family

For an explicit example, take $r_i = \rho_i^2 = i^2$. For these choices of roots,

$$R_0 = 518400, \; R_1 = -773136, \; R_2 = 296296.$$

Solving for a through D yields

$$\begin{aligned}
A &= 64R_0^3 &&= 8916100448256000000 \\
c &= 8R_0^2 &&= 2149908480000 \\
b &= 4R_0R_1 &&= -1603174809600 \\
a &= 4R_0R_2 - R_1^2 &&= 16660111104 \\
B &= R_5A - 2a &&= -811365140824616222208 \\
C &= R_4A - a^2 - 2b &&= 26497490347321493520384 \\
D &= R_3A - 2ab - 2c &&= -343107594345448813363200
\end{aligned}$$

Constructing Rank 6 Family

We convert $y^2 = f(x,t)$ to $y^2 = F(x,T)$, which is in Weierstrass normal form. We send $y \to \frac{y}{T^2+2T-A+1}$, $x \to \frac{x}{T^2+2T-A+1}$, and then multiply both sides by $(T^2+2T-A+1)^2$. For future reference, we note that

$$\begin{aligned} T^2+2T-A+1 &= (T+1-\sqrt{A})(T+1+\sqrt{A}) \\ &= (T-t_1)(T-t_2) \\ &= (T-2985983999)(T+2985984001). \end{aligned}$$

We have

$$\begin{aligned} f(x,T) &= T^2x^3+(2x^3+2ax^2+2bx+2c)T-(A-1)x^3-Bx^2-Cx-D \\ &= (T^2+2T-A+1)x^3+(2aT-B)x^2+(2bT-C)x+(2cT-D) \\ F(x,T) &= x^3+(2aT-B)x^2+(2bT-C)(T^2+2T-A+1)x \\ &\quad +(2cT-D)(T^2+2T-A+1)^2. \end{aligned}$$

Constructing Rank 6 Family

We now study the $-pA_{\mathcal{E}}(p)$ arising from $y^2 = F(x,T)$. It is enough to show this is $6p + O(1)$ for all p greater than some p_0. Note that t_1, t_2 are the unique roots of $t^2 + 2t - A + 1 \equiv 0 \bmod p$. We find

$$-pA_{\mathcal{E}}(p) = \sum_{t=0}^{p-1}\sum_{x=0}^{p-1}\left(\frac{F(x,t)}{p}\right) = \sum_{t\neq t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{F(x,t)}{p}\right) + \sum_{t=t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{F(x,t)}{p}\right).$$

For $t \neq t_1, t_2$, send $x \longrightarrow (t^2+2t-A+1)x$. As $(t^2+2t-A+1) \not\equiv 0$, $\left(\frac{(t^2+2t-A+1)^2}{p}\right) = 1$. Simple algebra yields

$$\begin{aligned}-pA_{\mathcal{E}}(p) &= 6p + O(1) + \sum_{t=t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{f_t(x)}{p}\right) + O(1)\\ &= 6p + O(1) + \sum_{t=t_1,t_2}\sum_{x=0}^{p-1}\left(\frac{(2at-B)x^2+(2bt-C)x+(2ct-D)}{p}\right).\end{aligned}$$

Constructing Rank 6 Family

The last sum above is negligible (i.e., is $O(1)$) if

$$D(t) \ = \ (2bt - C)^2 - 4(2at - B)(2ct - D) \not\equiv 0(p).$$

Calculating yields

$$\begin{aligned} D(t_1) &= 4291243480243836561123092143580209905401856 \\ &= 2^{32} \cdot 3^{25} \cdot 7^5 \cdot 11^2 \cdot 13 \cdot 19 \cdot 29 \cdot 31 \cdot 47 \cdot 67 \cdot 83 \cdot 97 \cdot 103 \\ D(t_2) &= 4291243816662452751895093255391719515488256 \\ &= 2^{33} \cdot 3^{12} \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 173 \cdot 17389 \cdot 805873 \cdot 9447850813. \end{aligned}$$

Constructing Rank 6 Family

Hence, except for finitely many primes (coming from factors of $D(t_i)$, $a, \dots, D$, t_1 and t_2), $-A_{\mathcal{E}}(p) = 6p + O(1)$ as desired.

We have shown: There exist integers a, b, c, A, B, C, D so that the curve $\mathcal{E} : y^2 = x^3 T^2 + 2g(x)T - h(x)$ over $\mathbb{Q}(T)$, with $g(x) = x^3 + ax^2 + bx + c$ and $h(x) = (A-1)x^3 + Bx^2 + Cx + D$, has rank 6 over $\mathbb{Q}(T)$. In particular, with the choices of a through D above, $\mathcal{E}$ is a rational elliptic surface and has Weierstrass form

$$y^2 = x^3 + (2aT - B)x^2 + (2bT - C)(T^2 + 2T - A + 1)x + (2cT - D)(T^2 + 2T - A + 1)^2$$

Constructing Rank 6 Family

We show $\mathcal{E}$ is a rational elliptic surface by translating $x \mapsto x - (2aT - B)/3$, which yields $y^2 = x^3 + A(T)x + B(T)$ with $\deg(A) = 3, \deg(B) = 5$.

Therefore the Rosen-Silverman theorem is applicable, and because we can compute $A_{\mathcal{E}}(p)$, we know the rank is exactly 6 (and we never need to calculate height matrices). $\square$

Lower order terms

Variation in the number of points on elliptic curves and applications to excess rank, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120.

`http://arxiv.org/pdf/math/0506461v2.pdf`.

Explicit calculations

Let $n_{3,2,p}$ equal the number of cube roots of 2 modulo p, and set $c_0(p) = \left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right)$, $c_1(p) = \left[\sum_{x \bmod p} \left(\frac{x^3-x}{p}\right)\right]^2$ and $c_{3/2}(p) = p\sum_{x \bmod p}\left(\frac{4x^3+1}{p}\right)$.

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + Sx + T$	0	$p^3 - p^2$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p\equiv 2 \bmod 3 \\ 0 & p\equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p\equiv 1 \bmod 3 \\ 0 & p\equiv 3 \bmod 3 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - \left(\frac{-3}{p}\right)$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$

Explicit calculations

The first family is the family of all elliptic curves; it is a two parameter family and we expect the main term of its second moment to be p^3.

Note that except for our family $y^2 = x^3 + Tx^2 + 1$, all the families $\mathcal{E}$ have $A_{2,\mathcal{E}}(p) = p^2 - h(p)p + O(1)$, where $h(p)$ is non-negative. Further, many of the families have $h(p) = m_{\mathcal{E}} > 0$.

Note $c_1(p)$ is the square of the coefficients from an elliptic curve with complex multiplication. It is non-negative and of size p for $p \not\equiv 3 \bmod 4$, and zero for $p \equiv 1 \bmod 4$ (send $x \mapsto -x \bmod p$ and note $\left(\frac{-1}{p}\right) = -1$).

It is somewhat remarkable that all these families have a correction to the main term in Michel's theorem in the same direction, and we analyze the consequence this has on the average rank. For our family which has a $p^{3/2}$ term, note that on average this term is zero and the p term is negative.

Lower order terms and average rank

$$\frac{1}{N}\sum_{t=N}^{2N}\sum_{\gamma_t}\phi\left(\gamma_t\frac{\log R}{2\pi}\right) = \widehat{\phi}(0)+\phi(0)-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log R}\right)a_t(p)$$
$$-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)a_t(p)^2+O\left(\frac{\log\log R}{\log R}\right).$$

If ϕ is non-negative, we obtain a bound for the average rank in the family by restricting the sum to be only over zeros at the central point. The error $O\left(\frac{\log\log R}{\log R}\right)$ comes from trivial estimation and ignores probable cancellation, and we expect $O\left(\frac{1}{\log R}\right)$ or smaller to be the correct magnitude. For most families $\log R \sim \log N^a$ for some integer a.

Lower order terms and average rank (cont)

The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$.

Assume the second moment of $a_t(p)^2$ is $p^2 - m_{\mathcal{E}}p + O(1)$, $m_{\mathcal{E}} > 0$.

We have already handled the contribution from p^2, and $-m_{\mathcal{E}}p$ contributes

$$\begin{aligned} S_2 &\sim \frac{-2}{N}\sum_p \frac{\log p}{\log R}\widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{1}{p^2}\frac{N}{p}(-m_{\mathcal{E}}p) \\ &= \frac{2m_{\mathcal{E}}}{\log R}\sum_p \widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p^2}. \end{aligned}$$

Thus there is a contribution of size $\frac{1}{\log R}$.

Lower order terms and average rank (cont)

A good choice of test functions (see Appendix A of [ILS]) is the Fourier pair

$$\phi(x) \ = \ \frac{\sin^2(2\pi\frac{\sigma}{2}x)}{(2\pi x)^2}, \quad \widehat{\phi}(u) \ = \ \begin{cases} \frac{\sigma-|u|}{4} & \text{if } |u| \le \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Note $\phi(0) = \frac{\sigma^2}{4}$, $\widehat{\phi}(0) = \frac{\sigma}{4} = \frac{\phi(0)}{\sigma}$, and evaluating the prime sum gives

$$S_2 \ \sim \ \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2 \log R}\right) \frac{m_{\mathcal{E}}}{\log R}\ \phi(0).$$

Lower order terms and average rank (cont)

Let r_t denote the number of zeros of E_t at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$\begin{aligned}\frac{1}{N}\sum_{t=N}^{2N} r_t\phi(0) &\le \frac{\phi(0)}{\sigma} + \left(r+\frac{1}{2}\right)\phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}\phi(0) \\ \text{Ave Rank}_{[N,2N]}(\mathcal{E}) &\le \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}.\end{aligned}$$

$\sigma = 1$, $m_{\mathcal{E}} = 1$: for conductors of size 10^{12}, the average rank is bounded by $1 + r + \frac{1}{2} + .03 = r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier's observed $r + \frac{1}{2} + .40$.

$\sigma = 2$: lower order correction contributes .02 for conductors of size 10^{12}, the average rank bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$. Now in the ballpark of Fermigier's bound (already there without the potential correction term!).

Excess rank: Numerics

Excess rank

One-parameter family, rank $r(\mathcal{E})$ over $\mathbb{Q}(T)$.
For each $t \in \mathbb{Z}$ consider curves E_t.
RMT $\Longrightarrow$ 50% rank $r(\mathcal{E})$, 50% rank $r(\mathcal{E}) + 1$.
For many families, observe

$$\begin{array}{lclclcl} \text{Rank } r(\mathcal{E}) & = & 32\% & & \text{Rank } r(\mathcal{E})+1 & = & 48\% \\ \text{rank } r(\mathcal{E})+2 & = & 18\% & & \text{Rank } r(\mathcal{E})+3 & = & 2\% \end{array}$$

Problem: small data sets, sub-families, convergence rate log(conductor)?

Interval	Primes	Twin Primes Pairs
$[1, 10]$	$2, 3, 5, 7$ (40%)	$(3, 5), (5, 7)$ (20%)
$[11, 20]$	$11, 13, 17, 19$ (40%)	$(11, 13), (17, 19)$ (20%)

Small data can be misleading! Remember $\sum_{p \le x} \frac{1}{p} \sim \log\log x$.

Data on Excess Rank

$$y^2 + a_1xy + a_3y \ = \ x^3 + a_2x^2 + a_4x + a_6$$

Family: a_1 : 0 to 10, rest -10 to 10.
14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).

Rank r	$=$	28.60%	Rank $r+1$	$=$	47.56%
Rank $r+2$	$=$	20.97%	Rank $r+3$	$=$	2.79%
Rank $r+4$	$=$	.08%			

Data on excess rank (cont)

$$y^2 = x^3 + 16Tx + 32$$

Each data set runs over 2000 consecutive t-values.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{11}, but on logarithmic scale still small.

Numerically approximating analytic rank

Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^\infty f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \ \ \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

Algorithm for $L^r(s,E)$

$$\begin{aligned}\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_1^\infty f(iy/\sqrt{N})(y^{s-1}+\epsilon y^{1-s})dy.\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1}+\epsilon(-1)^k y^{1-s})dy.$$

At $s=1$,

$$\Lambda^{(k)}(E,1) = (1+\epsilon(-1)^k)\int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s,E)$: II

$$\Lambda^{(r)}(E,1) = 2\int_1^\infty f(iy/\sqrt{N})(\log y)^r dy = 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log y)^r dy.$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log \frac{1}{x}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-r}}{n^r \cdot n!} x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t; \\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12}; \\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3}; \\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160}; \\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

Expansion of $G_r(x)$ (cont)

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-2\pi n y/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.