Results in the Theory of Low-Lying Zeros

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Introduction

Measures of Spacings: n-Level Correlations

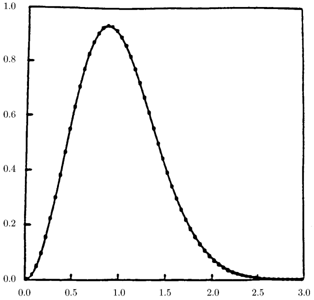
$\{\alpha_j\}$ increasing sequence of numbers, $B \subset \mathbb{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k\right\}}{N}$$

Instead of using a box, can use a smooth test function.

Measures of Spacings: n-Level Correlations

1. Normalized spacings of $\zeta(s)$ starting at 10^{20}. (Odlyzko)



70 million spacings between adjacent normalized zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Measures of Spacings: n-Level Correlations

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$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\dots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i\neq j_k\right\}}{N}$$

Instead of using a box, can use a smooth test function.

1. Spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

Let g_i be even Schwartz functions whose Fourier Transform is compactly supported, $L(s,f)$ an L-function with zeros $\frac{1}{2}+i\gamma_f$ and conductor Q_f:

$$D_{n,f}(g) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

- Properties of n-level density:
  - ◇ Individual zeros contribute in limit.
  - ◇ Most of contribution is from low zeros.
  - ◇ Average over similar L-functions (family).

n-Level Density

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, g_k an even Schwartz function: $D_{n,\mathcal{F}}(g) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, n-level density converges to

$$\int g(\vec{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\vec{x})d\vec{x} \quad = \quad \int \widehat{g}(\vec{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\vec{u})d\vec{u}.$$

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r + 2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Example:
Dirichlet L-functions

Dirichlet Characters (m prime)

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g. Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$. The principal character χ_0 is given by

$$\chi_0(k) = \begin{cases} 1 & (k,m) = 1 \\ 0 & (k,m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on g.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^l$. Hence for each $l, 1 \le l \le m-2$ we have

$$\chi_l(k) = \begin{cases} \zeta_{m-1}^{la} & k \equiv g^a(m) \\ 0 & (k,m) > 0 \end{cases}$$

Dirichlet L-Functions

Let χ be a primitive character mod m. Let

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}.$$

$c(m,\chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned}$$

where

$$\epsilon = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases}$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \quad = \quad -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \;=\; -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
\end{aligned}
$$

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Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.
$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

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$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).$$

Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula

Let ϕ be an even Schwartz function with compact support $(-\sigma, \sigma)$, let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\begin{aligned}
&\sum \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right) \\
= &\int_{-\infty}^{\infty} \phi(y) dy \\
&- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}} \\
&- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p) + \bar{\chi}^2(p)]p^{-1} \\
&+ O\left(\frac{1}{\log m}\right).
\end{aligned}$$

Expansion

$\{\chi_0\} \cup \{\chi_l\}_{1 \le l \le m-2}$ are all the characters mod m.
Consider the family of primitive characters mod a prime m ($m-2$ characters):

$$\begin{aligned} & \int_{-\infty}^{\infty} \phi(y) dy \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1} \\ + & O\left(\frac{1}{\log m}\right). \end{aligned}$$

Note can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise.} \end{cases}$$

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For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} -1 + m - 1 & p \equiv 1(m) \\ -1 & \text{otherwise.} \end{cases}$$

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$$\sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} -1+m-1 & p \equiv 1(m) \\ -1 & \text{otherwise.} \end{cases}$$

Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$\begin{aligned} & \frac{-2}{m-2}\sum_{p}^{m^\sigma} \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}} \\ + \quad & 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma} \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}} \\ \ll \quad & \frac{1}{m}\sum_{p}^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p\equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}} \end{aligned}$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}$$

$$\ll\quad \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}}+\sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}}\ll\frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}+\sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac{1}{2}}$$

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$$\ll\quad \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}}+\sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}}\ll\frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}+\sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac{1}{2}}$$

$$\ll\quad \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}+\frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}\ll\frac{1}{m}m^{\sigma/2}.$$

Second Sum

$$\frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_{p}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)+\bar{\chi}^2(p)}{p}.$$

$$\sum_{\chi\neq\chi_0}[\chi^2(p)+\bar{\chi}^2(p)] = \begin{cases} 2(m-2) & p\equiv\pm1(m) \\ -2 & p\not\equiv\pm1(m)\end{cases}$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\ll \frac{1}{m-2}\sum_{p}^{m^{\sigma/2}} p^{-1} + \frac{2m-2}{m-2}\sum_{p\equiv\pm1(m)}^{m^{\sigma/2}} p^{-1}$$

$$\ll \frac{1}{m-2}\sum_{k}^{m^{\sigma/2}} k^{-1} + \sum_{k\equiv1(m)}^{m^{\sigma/2}} k^{-1} + \sum_{k\equiv-1(m)}^{m^{\sigma/2}} k^{-1}$$

Increasing support

Notation:

$$\psi(x) \ := \ \sum_{n \le x} \Lambda(n), \qquad \psi(x,q,a) \ := \ \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n),$$

$$E(x,q,a) \ := \ \psi(x,q,a) - \frac{\psi(x)}{\phi(q)}.$$

If we assume GRH, we have that

$$\psi(x) \ = \ x + O(x^{\frac{1}{2}}(\log x)^2), \qquad E(x,q,a) = O(x^{\frac{1}{2}}(\log x)^2).$$

Increasing support

Our first result uses GRH and the following de-averaging hypothesis, which depends on a parameter $\eta \in [0, 1]$.

$$\sum_{Q/2<q\leq Q} \left|\psi(x;q,1) - \frac{\psi(x)}{\phi(q)}\right|^2 \ll$$

$$Q^{\eta-1} \sum_{Q/2<q\leq Q} \sum_{\substack{1\leq a\leq q:\\(a,q)=1}} \left|\psi(x;q,a) - \frac{\psi(x)}{\phi(q)}\right|^2.$$

Trivially true for $\eta = 1$.

Theorem (Fiorilli-M): GRH and above hypothesis give $(-4+2\eta, 4-2\eta)$.

Increasing support

Can get 1-level density for arbitrary finite support, under a hypothesis of Montgomery.
For any a, q such that $(a, q) = 1$ and $q \le x$, we have

$$\psi(x; q, a) - \frac{\psi(x)}{\phi(q)} \ll_\epsilon x^\epsilon \left(\frac{x}{q}\right)^{1/2}.$$

Following weaker version suffices: For any a, q such that $(a, q) = 1$ and $q \le x$, we have

$$\psi(x; q, 1) - \frac{\psi(x)}{\phi(q)} \ll_\epsilon \frac{x^{\frac{1}{2}+\epsilon}}{q^\theta}.$$

Theorem (Fiorilli-M): GRH and above hypothesis give arbitrary support.

Identifying Family Symmetry and Lower Order Terms

Some Number Theory Results

- Orthogonal: holomorphic cuspidal newforms: Iwaniec-Luo-Sarnak, Hughes-Miller, Ricotta-Royer, ... Elliptic curves: Miller, Young. Maass forms: Amersi, Alpoge, Iyer, Lazarev, Miller and Zhang.

- Symplectic: Quadratic Dirichlet characters: Rubinstein, Gao, Levinson-Miller, and Entin, Roddity-Gershon and Rudnick: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.

- Unitary: Dirichlet characters: Fiorilli-Miller, Hughes-Rudnick. Cuspidal GL(3) Maass forms: Goldfeld-Kontorovich.

Main Tools

1. Control of conductors: Usually monotone, gives scale to study low-lying zeros.
2. Explicit Formula: Relates sums over zeros to sums over primes.
3. Trac Formulas: Orthogonality of characters in Fiorilli-Miller, Gao, Hughes-Rudnick, Levinson-Miller, Rubinstein. Petersson formula in Iwaniec, Luo and Sarnak, ... Kuznetsov in Amersi et al, Goldfeld-Kontorovich.

Applications of n-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Applications of n-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \le \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.
- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation:
- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake params $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu} \widehat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right) \frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\lambda_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu$$

$$S_1(\mathcal{F}) = -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right]$$

$$S_2(\mathcal{F}) = -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]$$

The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2, k}(p)\}_{k=1}^{nm} = \{\alpha_{\pi_1, i}(p) \cdot \alpha_{\pi_2, j}(p)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Breaks analysis of compound families into simple ones.

Some Results: Rankin-Selberg Convolution of Families: Proofs

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Moments of Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\sum_{k=1}^{nm} \alpha_{\pi_1 \times \pi_2, k}(p)^\nu = \sum_{i=1}^{n} \alpha_{\pi_1, i}(p)^\nu \sum_{j=1}^{m} \alpha_{\pi_2, j}(p)^\nu.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F}\times\mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Breaks analysis of compound families into simple ones.

Takeaways

Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.
- First moment zero save for families of elliptic curves.
- Higher moments control convergence and can depend on arithmetic of family.

Open Problem:

Develop a theory of lower order terms to split the universality and see the arithmetic.

Lower Order Terms

Convolve families of elliptic curves with ranks r_1 and r_2: see lower order term of size $r_1 r_2$ (over logarithms).

Difficulty is isolating that from other errors (often of size $\log\log R/\log R$). Study weighted moments

$$\begin{aligned} A_{r,\mathcal{F}}(p) &:= \frac{1}{W_R(\mathcal{F})} \sum_{\substack{f\in\mathcal{F}\\ f\in S(p)}} w_R(f)\lambda_f(p)^r \\ A'_{r,\mathcal{F}}(p) &:= \frac{1}{W_R(\mathcal{F})} \sum_{\substack{f\in\mathcal{F}\\ f\notin S(p)}} w_R(f)\lambda_f(p)^r \\ S(p) &:= \{f\in\mathcal{F} : p \nmid N_f\}. \end{aligned}$$

Main difficulty in 1-level density is evaluating

$$S(\mathcal{F}) = -2\sum_p \sum_{m=1}^{\infty} \frac{1}{W_R(\mathcal{F})} \sum_{f\in\mathcal{F}} w_R(f) \frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \frac{\log p}{\log R}\, \widehat{\phi}\left(m\frac{\log p}{\log R}\right).$$

Fourier Coefficient Expansion

$$\begin{aligned}
S(\mathcal{F}) &= -2\sum_p \sum_{m=1}^{\infty} \frac{A'_{m,\mathcal{F}}(p)}{p^{m/2}} \frac{\log p}{\log R}\, \widehat{\phi}\left(m\frac{\log p}{\log R}\right) \\
&\quad -2\widehat{\phi}(0)\sum_p \frac{2A_{0,\mathcal{F}}(p)\log p}{p(p+1)\log R} + 2\sum_p \frac{2A_{0,\mathcal{F}}(p)\log p}{p\log R}\,\widehat{\phi}\left(2\frac{\log p}{\log R}\right) \\
&\quad -2\sum_p \frac{A_{1,\mathcal{F}}(p)}{p^{1/2}}\frac{\log p}{\log R}\,\widehat{\phi}\left(\frac{\log p}{\log R}\right) + 2\widehat{\phi}(0)\frac{A_{1,\mathcal{F}}(p)(3p+1)}{p^{1/2}(p+1)^2}\frac{\log p}{\log R} \\
&\quad -2\sum_p \frac{A_{2,\mathcal{F}}(p)\log p}{p\log R}\,\widehat{\phi}\left(2\frac{\log p}{\log R}\right) + 2\widehat{\phi}(0)\sum_p \frac{A_{2,\mathcal{F}}(p)(4p^2+3p+1)\log p}{p(p+1)^3\log R} \\
&\quad -2\widehat{\phi}(0)\sum_p\sum_{r=3}^{\infty} \frac{A_{r,\mathcal{F}}(p)p^{r/2}(p-1)\log p}{(p+1)^{r+1}\log R} + O\left(\frac{1}{\log^3 R}\right) \\
&= S_{A'}(\mathcal{F}) + S_0(\mathcal{F}) + S_1(\mathcal{F}) + S_2(\mathcal{F}) + S_A(\mathcal{F}) + O\left(\frac{1}{\log^3 R}\right).
\end{aligned}$$

Letting $\widetilde{A}_{\mathcal{F}}(p) := \frac{1}{W_R(\mathcal{F})}\sum_{f\in S(p)} w_R(f)\frac{\lambda_f(p)^3}{p+1-\lambda_f(p)\sqrt{p}}$, by the geometric series formula we may replace $S_A(\mathcal{F})$ with $S_{\tilde{A}}(\mathcal{F})$, where

$$S_{\tilde{A}}(\mathcal{F}) = -2\widehat{\phi}(0)\sum_p \frac{\widetilde{A}_{\mathcal{F}}(p)p^{3/2}(p-1)\log p}{(p+1)^3\log R}.$$

Family Dependent Lower Order Terms: Miller '09

$\mathcal{F}_{k,N}$ the family of even weight k and prime level N cuspidal newforms, or just the forms with even (or odd) functional equation.

Up to $O(\log^{-3} R)$, as $N \to \infty$ for test functions ϕ with $\text{supp}(\widehat{\phi}) \subset (-4/3, 4/3)$ the (non-conductor) lower order term is

$$-1.33258 \cdot 2\widehat{\phi}(0)/\log R.$$

Note the lower order corrections are independent of the distribution of the signs of the functional equations.

Family Dependent Lower Order Terms: Miller '09

CM example, with or without forced torsion: $y^2 = x^3 + B(6T+1)^\kappa$ over $\mathbb{Q}(T)$, with $B \in \{1,2,3,6\}$ and $\kappa \in \{1,2\}$.

CM, sieve to $(6T+1)$ is $(6/\kappa)$-power free. If $\kappa = 1$ then all values of B the same, if $\kappa = 2$ the four values of B have different lower order corrections; in particular, if $B = 1$ then there is a forced torsion point of order three, $(0, 6T+1)$.

Up to errors of size $O(\log^{-3} R)$, the (non-conductor) lower order terms are approximately

$$\begin{aligned}
B = 1, \kappa = 1: &\quad -2.124 \cdot 2\widehat{\phi}(0)/\log R,\\
B = 1, \kappa = 2: &\quad -2.201 \cdot 2\widehat{\phi}(0)/\log R,\\
B = 2, \kappa = 2: &\quad -2.347 \cdot 2\widehat{\phi}(0)/\log R\\
B = 3, \kappa = 2: &\quad -1.921 \cdot 2\widehat{\phi}(0)/\log R\\
B = 6, \kappa = 2: &\quad -2.042 \cdot 2\widehat{\phi}(0)/\log R.
\end{aligned}$$

Family Dependent Lower Order Terms: Miller ’09

CM example, with or without rank:
$y^2 = x^3 - B(36T+6)(36T+5)x$ over $\mathbb{Q}(T)$, with $B \in \{1,2\}$. If $B=1$ the family has rank 1, while if $B=2$ the family has rank 0.

Sieve to $(36T+6)(36T+5)$ is cube-free. Most important difference between these two families is the contribution from the $S_{\widetilde{\mathcal{A}}}(\mathcal{F})$ terms, where the $B=1$ family is approximately $-.11 \cdot 2\widehat{\phi}(0)/\log R$, while the $B=2$ family is approximately $.63 \cdot 2\widehat{\phi}(0)/\log R$.

This large difference is due to biases of size $-r$ in the Fourier coefficients $a_t(p)$ in a one-parameter family of rank r over $\mathbb{Q}(T)$.

Main term of the average moments of the p^{th} Fourier coefficients are given by the complex multiplication analogue of Sato-Tate in the limit, for each p there are lower order correction terms which depend on the rank.

Family Dependent Lower Order Terms: Miller '09

Non-CM Example: $y^2 = x^3 - 3x + 12T$ over $\mathbb{Q}(T)$. Up to $O(\log^{-3} R)$, the (non-conductor) lower order correction is approximately

$$-2.703 \cdot 2\widehat{\phi}(0)/\log R,$$

which is very different than the family of weight 2 cuspidal newforms of prime level N.

Data for Elliptic Curve Famillies
Dueñez, Huynh, Keating, Miller and Snaith

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

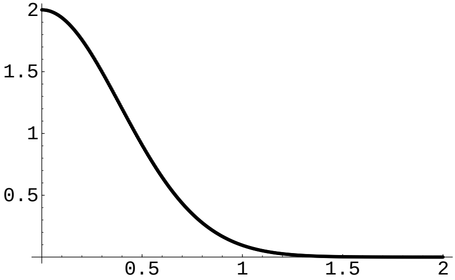
$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd.} \end{cases}$$

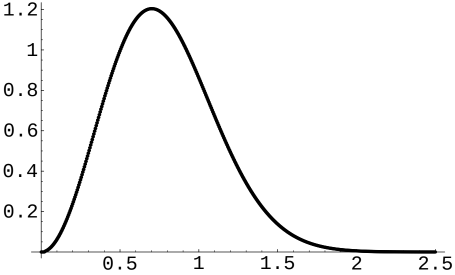
Supports Katz-Sarnak, B-SD, and Independent model in limit.

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

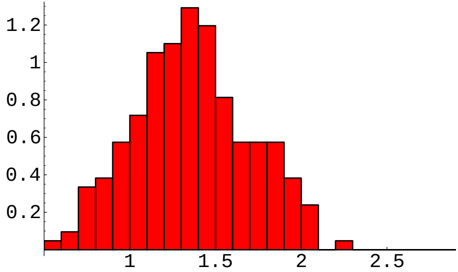


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

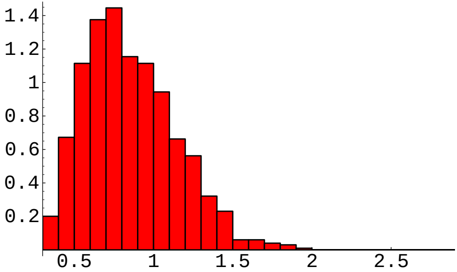


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $=$ .81, mean $=$ .86.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i. e., shifted by the same amount).

New Model for Finite Conductors

- Replace conductor N with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with L-function Ratios Conj.
  - ◇ Do the number theory computation.

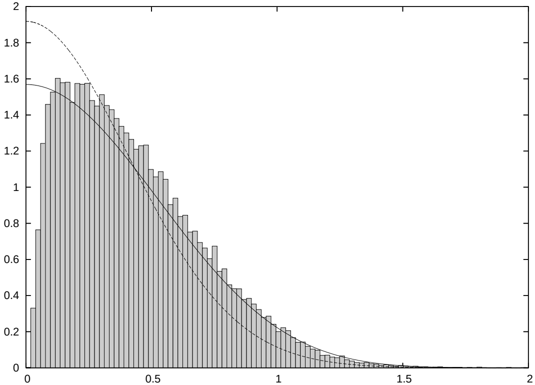
- Excised Orthogonal Ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.

- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

Open Problem:

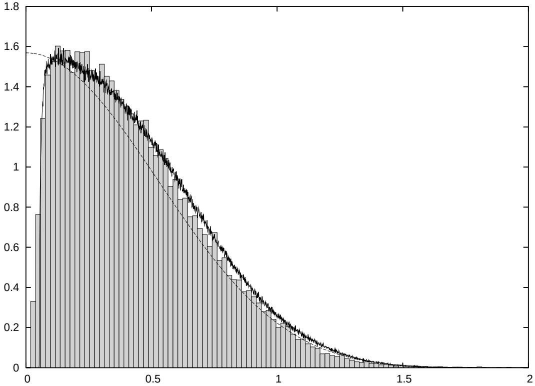
Generalize to other families (ongoing with Nathan Ryan).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Ratio's Conjecture

History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} = \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  ◇ n-level correlations and densities;
  ◇ mollifiers;
  ◇ moments;
  ◇ vanishing at the central point;

- Advantages:
  ◇ RMT models often add arithmetic ad hoc;
  ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) = \sum_{m\le x} \frac{a_m}{m^s} + \epsilon \mathbb{X}_L(s) \sum_{n\le y} \frac{a_n}{n^{1-s}};$$

  - ⋄ ϵ sign of the functional equation,
  - ⋄ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

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  ◇ ϵ sign of the functional equation,
  ◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}}\omega_f\sum_{\gamma} g\left(\gamma\frac{\log N_f}{2\pi}\right) \;=\; \frac{1}{2\pi i}\int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\cdots)g\left(\cdots\right)$$

  ◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha}R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

Procedure (Recipe)

- Use approximate functional equation to expand numerator.

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- Expand denominator by generalized Mobius function: cusp form

$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Extend the m and n sums to infinity (complete the products).

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- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Procedure ('Illegal Steps')

- Use approximate functional equation to expand numerator.
- Expand denominator by generalized Mobius function: cusp form

$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.
- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

1-Level Prediction from Ratio's Conjecture

$$\begin{aligned} & A_E(\alpha,\gamma) \\ = \quad & Y_E^{-1}(\alpha,\gamma) \times \prod_{p|M}\left(\sum_{m=0}^{\infty}\left(\frac{\lambda(p^m)\omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)}{p^{1/2+\gamma}}\frac{\lambda(p^m)\omega_E^{m+1}}{p^{m(1/2+\alpha)}}\right)\right) \times \\ & \prod_{p\nmid M}\left(1+\frac{p}{p+1}\left(\sum_{m=1}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}}\right.\right. \\ & \left.\left. + \frac{1}{p^{1+2\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}}\right)\right) \end{aligned}$$

where

$$Y_E(\alpha,\gamma) = \frac{\zeta(1+2\gamma)L_E(\text{sym}^2,1+2\alpha)}{\zeta(1+\alpha+\gamma)L_E(\text{sym}^2,1+\alpha+\gamma)}.$$

Huynh, Morrison and Miller confirmed Ratios' prediction, which is

1-Level Prediction from Ratio's Conjecture

$$\frac{1}{X^*}\sum_{d\in\mathcal{F}(X)}\sum_{\gamma_d} g\left(\frac{\gamma_d L}{\pi}\right)$$

$$= \frac{1}{2LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right)+\frac{\Gamma'}{\Gamma}\left(1+\frac{i\pi\tau}{L}\right)+\frac{\Gamma'}{\Gamma}\left(1-\frac{i\pi\tau}{L}\right)\right]d\tau$$

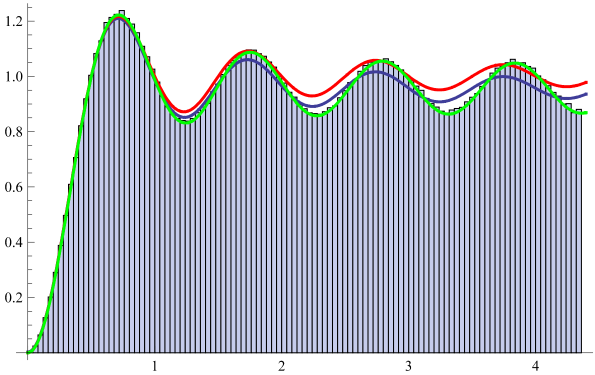
$$+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\left(-\frac{\zeta'}{\zeta}\left(1+\frac{2\pi i\tau}{L}\right)+\frac{L_E'}{L_E}\left(\mathrm{sym}^2,1+\frac{2\pi i\tau}{L}\right)-\sum_{\ell=1}^{\infty}\frac{(M^\ell-1)\log M}{M^{\left(2+\frac{2i\pi\tau}{L}\right)\ell}}\right)d\tau$$

$$-\frac{1}{L}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g(\tau)\frac{\log M}{M^{(k+1)(1+\frac{\pi i\tau}{L})}}d\tau+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\sum_{p\nmid M}\frac{\log p}{(p+1)}\sum_{k=0}^{\infty}\frac{\lambda(p^{2k+2})-\lambda(p^{2k})}{p^{(k+1)(1+\frac{2\pi i\tau}{L})}}d\tau$$

$$-\frac{1}{LX^*}\int_{-\infty}^{\infty} g(\tau)\sum_{d\in\mathcal{F}(X)}\left[\left(\frac{\sqrt{M}|d|}{2\pi}\right)^{-2i\pi\tau/L}\frac{\Gamma(1-\frac{i\pi\tau}{L})}{\Gamma(1+\frac{i\pi\tau}{L})}\frac{\zeta(1+\frac{2i\pi\tau}{L})L_E(\mathrm{sym}^2,1-\frac{2i\pi\tau}{L})}{L_E(\mathrm{sym}^2,1)}\right.$$

$$\left.\times A_E\left(-\frac{i\pi\tau}{L},\frac{i\pi\tau}{L}\right)\right]d\tau+O(X^{-1/2+\varepsilon});$$

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.

$\diamond$ Green: all lower order terms.

Excised Orthogonal Ensembles

Excised Orthogonal Ensemble: Preliminaries

Characteristic polynomial of $A \in \mathrm{SO}(2N)$ is

$$\Lambda_A(e^{i\theta}, N) := \det(I - Ae^{-i\theta}) = \prod_{k=1}^{N}(1 - e^{i(\theta_k - \theta)})(1 - e^{i(-\theta_k - \theta)}),$$

w ith $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N}$ the eigenvalues of A.

Motivated by the arithmetical size constraint on the central values of the L-functions, consider Excised Orthogonal Ensemble $T_{\mathcal{X}}$: $A \in \mathrm{SO}(2N)$ with $|\Lambda_A(1, N)| \geq \exp(\mathcal{X})$.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.

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where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.
The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) := C_{\mathcal{X}} \cdot N \int_0^{\pi} \cdots \int_0^{\pi} H(\log|\Lambda_A(1,N)| - \mathcal{X}) \times \\ \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N,$$

Here $H(x)$ denotes the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0, \end{cases}$$

and $C_{\mathcal{X}}$ is a normalization constant

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$$R_1^{T_{\mathcal{X}}}(\theta_1) = \frac{C_{\mathcal{X}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{Nr} \frac{\exp(-r\mathcal{X})}{r} R_1^{J_N}(\theta_1; r-1/2, -1/2) dr$$

where $C_{\mathcal{X}}$ is a normalization constant and

$$R_1^{J_N}(\theta_1; r-1/2, -1/2) = N \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} w^{(r-1/2,-1/2)}(\cos\theta_j) \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N$$

is the one-level density for the Jacobi ensemble J_N with weight function

$$w^{(\alpha,\beta)}(\cos\theta) = (1-\cos\theta)^{\alpha+1/2}(1+\cos\theta)^{\beta+1/2}, \qquad \alpha = r - 1/2 \text{ and } \beta = -1/2.$$

Results

- With $C_{\mathcal{X}}$ normalization constant and $P(N, r, \theta)$ defined in terms of Jacobi polynomials,

$$R_1^{T_{\mathcal{X}}}(\theta) = \frac{C_{\mathcal{X}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\exp(-r\mathcal{X})}{r} 2^{N^2+2Nr-N} \times$$
$$\times \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)} \times$$
$$\times (1-\cos\theta)^r \frac{2^{1-r}}{2N+r-1} \frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)} P(N, r, \theta)\, dr.$$

- Residue calculus implies $R_1^{T_{\mathcal{X}}}(\theta) = 0$ for $d(\theta, \mathcal{X}) < 0$ and

$$R_1^{T_{\mathcal{X}}}(\theta) = R_1^{\mathrm{SO}(2N)}(\theta) + C_{\mathcal{X}} \sum_{k=0}^{\infty} b_k \exp((k+1/2)\mathcal{X}) \quad \text{for } d(\theta, \mathcal{X}) \geq 0,$$

where $d(\theta, \mathcal{X}) := (2N-1)\log 2 + \log(1-\cos\theta) - \mathcal{X}$ and b_k are coefficients arising from the residues. As $\mathcal{X} \to -\infty$, θ fixed, $R_1^{T_{\mathcal{X}}}(\theta) \to R_1^{\mathrm{SO}(2N)}(\theta)$.

Numerical check

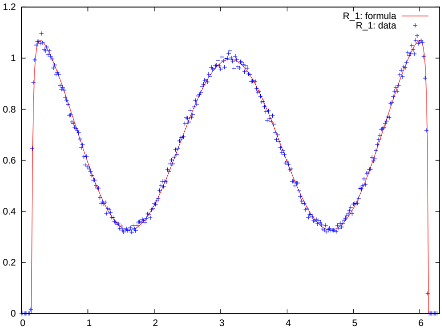

Figure: One-level density of excized $SO(2N), N = 2$ with cut-off $|\Lambda_A(1, N)| \geq 0.1$. The red curve uses our formula. The blue crosses give the empirical one-level density of 200,000 numerically generated matrices.

Theory vs Experiment

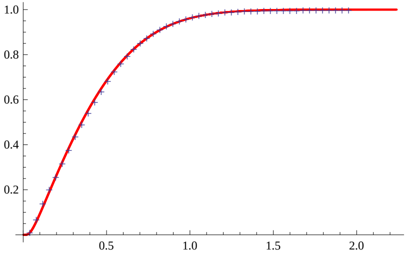

Figure: Cumulative probability density of the first eigenvalue from 3×10^6 numerically generated matrices $A \in SO(2N_{\rm std})$ with $|\Lambda_A(1, N_{\rm std})| \geq 2.188 \times \exp(-N_{\rm std}/2)$ and $N_{\rm std} = 12$ red dots compared with the first zero of even quadratic twists $L_{E_{11}}(s, \chi_d)$ with prime fundamental discriminants $0 < d \leq 400{,}000$ blue crosses. The random matrix data is scaled so that the means of the two distributions agree.

Cuspidal Newforms
Hughes-Miller

Modular Form Preliminaries: Explicit Formula

Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned}\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi)\\ &\quad + O\left(\frac{\log\log R}{\log R}\right)\\ P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.\end{aligned}$$

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

$$R(n,q) = \sum_{a \bmod q}^{*} e(an/q) = \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the summation to be over all a relatively prime to q.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\operatorname{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m \le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{dy}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$ and φ is Euler's totient function.

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{rclcrcl} u_2 & = & x_1x_2 & \qquad & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1}\end{array}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}.$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) \;=\; \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

3-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\int_{x_3=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)\widehat{\phi}\left(\frac{\log x_3}{\log R}\right)$$

$$* \quad J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2x_3N}}{c}\right)\frac{dx_1dx_2dx_3}{\sqrt{x_1x_2x_3}}$$

Change variables as below and get Jacobian:

$$\begin{aligned} u_3 &= x_1x_2x_3 \qquad & x_3 &= \frac{u_3}{u_2} \\ u_2 &= x_1x_2 \qquad & x_2 &= \frac{u_2}{u_1} \\ u_1 &= x_1 \qquad & x_1 &= u_1 \end{aligned}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} & 0 \\ 0 & -\frac{u_3}{u_2^2} & \frac{1}{u_2} \end{vmatrix} = \frac{1}{u_1u_2}.$$

n-Level Density: Determinant Expansions from RMT

- U(N), $\mathrm{U}_k(N)$: $\det\Big(K_0(x_j,x_k)\Big)_{1\le j,k\le n}$
- USp(N): $\det\Big(K_{-1}(x_j,x_k)\Big)_{1\le j,k\le n}$
- SO(even): $\det\Big(K_1(x_j,x_k)\Big)_{1\le j,k\le n}$
- SO(odd): $\det\big(K_{-1}(x_j,x_k)\big)_{1\le j,k\le n} + \sum_{\nu=1}^n \delta(x_\nu)\det\Big(K_{-1}(x_j,x_k)\Big)_{1\le j,k\ne\nu\le n}$

where

$$K_\epsilon(x,y) = \frac{\sin\big(\pi(x-y)\big)}{\pi(x-y)} + \epsilon\frac{\sin\big(\pi(x+y)\big)}{\pi(x+y)}.$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. Leads to harder combinatorics but allows us to appeal to the result from ILS.

Support for n-Level Density

Careful book-keeping gives (originally just had $\frac{1}{n-1/2}$)

$$\sigma_n \; < \; \frac{1}{n-1}.$$

n-Level Density is trivial for $\sigma_n < \frac{1}{n}$, non-trivial up to $\frac{1}{n-1}$.

Expected $\frac{2}{n}$. Obstruction from partial summation on primes.

Support Problems: 2-Level Density

Partial Summation on p_1 first, looks like

$$\sum_{\substack{p_1 \\ p_1 \neq p_2}} S(m^2, p_1 p_2 N, c) \frac{2 \log p_1}{\sqrt{p_1} \log R} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) J_{k-1}\left(4\pi \frac{\sqrt{m^2 p_1 p_2 N}}{c}\right)$$

Similar to ILS, obtain ($c = bN$):

$$\sum_{\substack{p_1 \leq x_1 \\ p_1 \nmid b}} S(m^2, p_1 p_2 N, c) \frac{\log p}{\sqrt{p}} = \frac{2\mu(N)}{\phi(b)} \widetilde{R}(m^2, b, p_2) x_1^{\frac{1}{2}} + O\left(b(bx_1 N)^{\epsilon}\right)$$

$\sum_{p_1}$ to $\int_{x_1}$, error $\ll b(bN)^{\epsilon} m \sqrt{p_2 N} N^{\sigma_2/2}/bN$, yields

$$\sqrt{N} \sum_{m \leq N^{\epsilon}} \frac{1}{m} \sum_{b \leq N^5} \frac{1}{bN} \sum_{p_2 \leq N_2^{\sigma}} \frac{1}{\sqrt{p_2}} \frac{b(bN)^{\epsilon} m \sqrt{p_2 N} N^{\frac{\sigma_2}{2}}}{bN}$$

$$\ll N^{\frac{1}{2}+\epsilon'+\sigma_2+\frac{1}{2}+\frac{\sigma_2}{2}-2} \ll N^{\frac{3}{2}\sigma_2 - 1 + \epsilon'}$$

Support Problems: n-Level Density: Why is $\sigma_2 < 1$?

- If no $\sum_{p_2}$, have above *without* the N^σ which arose from $\sum_{p_2}$, giving

$$\ll \; N^{\frac{1}{2}+\epsilon'+\frac{1}{2}+\frac{\sigma_1}{2}-2} \; = \; N^{\frac{1}{2}\sigma_1-1+\epsilon'}.$$

- Fine for $\sigma_1 < 2$. For 3-Level, have two sums over primes giving N^{σ_3}, giving

$$\ll \; N^{\frac{1}{2}+\epsilon'+2\sigma_3+\frac{1}{2}+\frac{\sigma_3}{2}-2} \; = \; N^{\frac{5}{2}\sigma_3-1+\epsilon'}$$

- n-Level, have an additional $(n-1)$ prime sums, each giving N^{σ_n}, yields

$$\ll \; N^{\frac{1}{2}+\epsilon'+(n-1)\sigma_n+\frac{1}{2}+\frac{\sigma_n}{2}-2} \; = \; N^{\frac{(2n-1)}{2}\sigma_n-1+\epsilon'}$$

Conclusion and References

Conclusion and Future Work

- In the limit: Birch and Swinnerton-Dyer, Katz-Sarnak appear true.
- Finite conductors: model with Excised Ensembles (cut-off on characteristic polynomials due to discretization at central point).
- Future Work: Joint with Nathan Ryan and his students, looking at other GL2 families (and hopefully higher) to study the relationship between repulsion at finite conductors and central values.
- Future Work: Further explore lower order terms and combinatorics.

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Appendix I:
New Cuspidal Results
Iyer, Miller and Triantafillou

Goal:

Prove n-level densities agree for $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$.

Philosophy:

Number theory harder - adapt tools to get an answer.
Random matrix theory easier - manipulate known answer.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\text{supp}(\widehat{\Psi}) \subset (-2, 2)$. Then

$$\sum_{m \le N^\epsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y) \widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right) \frac{dy}{\log R}$$

$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x) \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k \log\log kN}{\log kN}\right),$$

where $R = k^2 N$, φ is Euler's totient function, and $R(n,q)$ is a Ramanujan sum.

Number Theory Side: Iyer-Miller-Triantafillou: $\mathrm{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Sequence of Lemmas - New Contributions Arise

1. Apply Petersson Formula
2. Restrict Certain Sums
3. Convert Kloosterman Sums to Gauss Sums
4. Remove Non-Principal Characters
5. Convert Sums to Integrals

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Typical Argument

If any prime is 'special', bound error terms

$$\ll \frac{1}{\sqrt{N}} \sum_{p_1,\dots,p_n \le N^\sigma} \sum_{m \le N^\epsilon} \sum_{r=1}^{\infty} \frac{m^2 r}{r p_1} \frac{\sqrt{p_1 \cdots p_n}}{r p_1 \sqrt{N}} \frac{1}{\sqrt{p_1 \cdots p_n}}$$

$$\ll N^{-1+\epsilon'} \left(\sum_{p \le N^\sigma} 1 \right)^{n-1} \ll N^{-1+(n-1)\sigma+\epsilon'}$$

Bounds fail for large support - new terms arise.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

We want to evaluate

$$\sum_{n_1,\dots,n_n}\left[\prod_{i=1}^{n}\hat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]J_{k-1}\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)$$

$$=\frac{1}{2\pi i}\int_{\mathfrak{Re}(s)=1}\left[\prod_{i=1}^{n}\sum_{n_i}\hat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]$$

$$\times\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)^{-s}G_{k-1}(s)ds$$

Converting Sums to Integrals - Hughes-Miller: $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

Important Observation

$$\sum_{r=1}^{\infty} \hat{\phi}\left(\frac{\log r}{\log R}\right) \frac{\chi_0(r)\Lambda(r)}{r^{(1+s)/2}\log R} = \phi\left(\frac{1-s}{4\pi i}\log R\right) + \mathcal{E}(s),$$

where

$$\mathcal{E}(s) = -\frac{1}{2\pi i}\int_{\mathfrak{Re}(z)=c} \phi\left(\frac{(2z-1-s)\log R}{4\pi i}\right)\frac{L'}{L}(z,\chi_0)dz.$$

For convenience, rename expressions, $X = Y + Z$.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

Important Observation

By the binomial theorem,

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Re(s)=1} X^n \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds \\
=&\frac{1}{2\pi i}\int_{\Re(s)=1} (Y+Z)^n \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds \\
=&\sum_{j=0}^{n}\binom{n}{j}\frac{1}{2\pi i}\int_{\Re(s)=1} Y^{n-j}Z^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds.
\end{aligned}$$

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

$Z = \mathcal{E}(s)$ is easy to bound (shift coutours), get

$$\frac{1}{2\pi i}\int_{\mathfrak{Re}(s)=1} Y^{n-j} Z^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s) ds$$
$$\ll N^{(n-j)\sigma/2+\epsilon''}$$

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

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For $\sigma < \frac{1}{n-1}$, only $j = 0$ term is non-negligible.

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For $\sigma < \frac{1}{n-1}$, only $j = 0$ term is non-negligible.

For $\sigma < \frac{1}{n-2}$, $j = 1$ term also non-negligible.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

$Z = \mathcal{E}(s)$ is easy to bound (shift coutours), get

$$\frac{1}{2\pi i}\int_{\mathfrak{Re}(s)=1} Y^{n-j}Z^j\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$
$$\ll N^{(n-j)\sigma/2+\epsilon''}$$

For $\sigma < \frac{1}{n-1}$, only $j = 0$ term is non-negligible.

For $\sigma < \frac{1}{n-2}$, $j = 1$ term also non-negligible.

Unfortunately, $Z = \mathcal{E}(s)$ is hard to compute with.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Key idea: Recall $X = Y + Z$, write $Z = X - Y$.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Key idea: Recall $X = Y + Z$, write $Z = X - Y$.

$$\begin{aligned} & \frac{n}{2\pi i}\int_{\mathfrak{Re}(s)=1} Y^{n-1}Z\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds \\ = & \frac{n}{2\pi i}\int_{\mathfrak{Re}(s)=1} XY^{n-1}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds \\ - & \frac{n}{2\pi i}\int_{\mathfrak{Re}(s)=1} Y^{n}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds. \end{aligned}$$

Hughes-Miller handle Y^n term, XY^{n-1} term is similar.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

When the dust clears, we see

$$\sum_{n_1,\dots,n_n} \left[\prod_{i=1}^{n} \hat{\phi}\left(\frac{\log n_i}{\log R}\right) \frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right] J_{k-1}\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)$$
$$=(1-n)\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\phi(0)^n\right)$$
$$+n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1-|x_2|))}{2\pi x_1}dx_1dx_2 - \frac{1}{2}\phi^n(0)\right)+o(1)$$

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Theorem

Fix $n \geq 4$ and let ϕ be an even Schwartz function with $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$. Then, the nth centered moment of the 1-level density for holomorphic cusp forms is

$$\frac{1+(-1)^n}{2}(-1)^n(n-1)!!\left(2\int_{-\infty}^{\infty}\hat{\phi}(y)^2|y|dy\right)^{n/2}$$
$$\pm(-2)^{n-1}\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\phi(0)^n\right)$$
$$\mp(-2)^{n-1}n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1}dx_1dx_2 - \frac{1}{2}\phi^n(0)\right).$$

Compare to RMT (see Appendix I).

Appendix II:
Random Matrix Theory: New Combinatorial Vantage
Iyer-Miller-Triantafillou

n-Level Density: Katz-Sarnak Determinant Expansions

Example: $\mathrm{SO(even)}$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_1) \cdots \widehat{\phi}(x_n) \det \Big(K_1(x_j, x_k) \Big)_{1 \le j,k \le n} dx_1 \cdots dx_n,$$

where

$$K_1(x,y) = \frac{\sin\big(\pi(x-y)\big)}{\pi(x-y)} + \frac{\sin\big(\pi(x+y)\big)}{\pi(x+y)}.$$

n-Level Density: Katz-Sarnak Determinant Expansions

Example: $\mathrm{SO(even)}$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\widehat{\phi}(x_1)\cdots\widehat{\phi}(x_n)\det\left(K_1(x_j,x_k)\right)_{1\le j,k\le n}dx_1\cdots dx_n,$$

where

$$K_1(x,y)=\frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)}+\frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

Problem: n-dimensional integral - looks very different.

Preliminaries

Easier to work with cumulants.

$$\sum_{n=1}^{\infty} C_n \frac{(it)^n}{n!} = \log \widehat{P}(t),$$

where P is the probability density function.

$$\mu_n' = C_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m \mu_{n-m}',$$

where μ_n' is uncentered moment.

Preliminaries

Manipulating determinant expansions leads to analysis of

$$K(y_1, \ldots, y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!} \sum_{\epsilon_1,\ldots,\epsilon_n=\pm 1} \prod_{\ell=1}^{m} \chi_{\{|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j| \leq 1\}},$$

where

$$\eta(\ell, j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^{\ell} \lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell} \lambda_k. \end{cases}$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\text{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Hughes-Miller solved for $\text{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-1}, \frac{1}{n-1})$.

New Complications: If $\text{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-2}, \frac{1}{n-2})$,

1. $\eta(\ell, j)\epsilon_j y_j$ need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed m, λ_j, ϵ_j).

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1. $\eta(\ell, j)\epsilon_j y_j$ need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed m, λ_j, ϵ_j).

Solution: Double count terms and subtract a correcting term ρ_j.

Generating Function Identity - $\lambda_i \geq 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x} = \frac{1}{1-(1-e^x)} = \sum_{m=0}^{\infty} (1-e^x)^m$$

Generating Function Identity - $\lambda_i \geq 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x} = \frac{1}{1-(1-e^x)} = \sum_{m=0}^{\infty} (1-e^x)^m$$

$$= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{\lambda=1}^{\infty} \frac{1}{\lambda!} x^{\lambda} \right)^m = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m}{\lambda_1! \cdots \lambda_m!} \right) x^n$$

Generating Function Identity - $\lambda_i \geq 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x} = \frac{1}{1-(1-e^x)} = \sum_{m=0}^{\infty} (1-e^x)^m$$

$$= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{\lambda=1}^{\infty} \frac{1}{\lambda!} x^{\lambda} \right)^m = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m}{\lambda_1! \cdots \lambda_m!} \right) x^n$$

$$\Rightarrow (-1)^n = \sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m n!}{\lambda_1! \cdots \lambda_m!}.$$

New Result: Dealing With 'ρ_j's

All $\lambda_i, \lambda_i' \geq 1$.

$$\rho_j = \sum_{m=1}^{n} \sum_{l=1}^{m} \sum_{\substack{\lambda_1+\cdots+\lambda_{\ell-1}=j-1 \\ \lambda_\ell=1 \\ \lambda_{\ell+1}+\cdots+\lambda_m=n-j}} \frac{(-1)^m}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

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$$= n \sum_{m=1}^{n} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} (-1) \frac{(-1)^{m-1}(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!}$$

New Result: Dealing With 'ρ_j's

All $\lambda_i, \lambda_i' \geq 1$.

$$\begin{aligned}
\sum_{j=1}^{n} \rho_j &= \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\substack{\lambda_1+\cdots+\lambda_{\ell-1}=j-1 \\ \lambda_\ell=1 \\ \lambda_{\ell+1}+\cdots+\lambda_m=n-j}} \frac{(-1)^m}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \\
&= n \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} \frac{(-1)^m}{m} \frac{(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!} \\
&= n \sum_{m=1}^{n} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} (-1) \frac{(-1)^{m-1}(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!} \\
&= n(-1)^n.
\end{aligned}$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\text{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

After Fourier transform identities:

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After Fourier transform identities:

$$\begin{aligned} C_n^{SO}(\phi) \; &= \frac{(-1)^{n-1}}{2}\left(\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\phi(0)^n\right) \\ &+ \frac{n(-1)^n}{2}\left(\int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-1}(x_1) \right. \\ &\left. \frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2}\phi^n(0)\right). \end{aligned}$$

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Agrees with number theory!

Recap

1. Difficult to compare n-dimensional integral from RMT with NT in general. Harder combinatorics worthwhile to appeal to result from ILS.

2. Solve combinatorics by using cumulants; support restrictions translate to which terms can contribute.

3. Extend number theory results by bounding Bessel functions, Kloosterman sums, etc. New terms arise and match random matrix theory prediction.

4. Better bounds on percent of forms vanishing to large order at the center point.

Appendix III:
Dirichlet Characters: Square-Free conductors

Dirichlet Characters: m Square-free

Fix an r and let $m_1, \dots, m_r$ be distinct odd primes.

$$\begin{aligned} m &= m_1 m_2 \cdots m_r \\ M_1 &= (m_1 - 1)(m_2 - 1) \cdots (m_r - 1) = \phi(m) \\ M_2 &= (m_1 - 2)(m_2 - 2) \cdots (m_r - 2). \end{aligned}$$

M_2 is the number of primitive characters mod m, each of conductor m. A general primitive character mod m is given by $\chi(u) = \chi_{l_1}(u)\chi_{l_2}(u) \cdots \chi_{l_r}(u)$. Let $\mathcal{F} = \{\chi : \chi = \chi_{l_1}\chi_{l_2} \cdots \chi_{l_r}\}$.

$$\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) p^{-\frac{1}{2}} \sum_{\chi \in \mathcal{F}} [\chi(p) + \bar{\chi}(p)]$$

$$\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big) p^{-1} \sum_{\chi \in \mathcal{F}} [\chi^2(p) + \bar{\chi}^2(p)]$$

Characters Sums

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} m_i - 1 - 1 & p \equiv 1(m_i) \\ -1 & \text{otherwise.} \end{cases}$$

Define

$$\delta_{m_i}(p,1) = \begin{cases} 1 & p \equiv 1(m_i) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{\chi \in \mathcal{F}} \chi(p) &= \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p) \\ &= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \prod_{i=1}^{r} \left(-1 + (m_i - 1)\delta_{m_i}(p,1) \right). \end{aligned}$$

Expansion Preliminaries

$k(s)$ is an s-tuple $(k_1, k_2, \dots, k_s)$ with $k_1 < k_2 < \cdots < k_s$.
This is just a subset of $(1, 2, \dots, r)$, 2^r possible choices for $k(s)$.

$$\delta_{k(s)}(p,1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p,1).$$

If $s = 0$ we define $\delta_{k(0)}(p,1) = 1 \; \forall p$.
Then

$$\begin{aligned} & \prod_{i=1}^{r} \Big(-1 + (m_i - 1)\delta_{m_i}(p,1) \Big) \\ = \quad & \sum_{s=0}^{r} \sum_{k(s)} (-1)^{r-s} \delta_{k(s)}(p,1) \prod_{i=1}^{s} (m_{k_i} - 1). \end{aligned}$$

First Sum

$$\ll \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \frac{1}{M_2}\left(1+\sum_{s=1}^{r}\sum_{k(s)}\delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1)\right).$$

As $m/M_2 \leq 3^r$, $s=0$ sum contributes

$$S_{1,0}=\frac{1}{M_2}\sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}\sigma-1},$$

hence negligible for $\sigma < 2$.

First Sum

$$\ll \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \frac{1}{M_2}\left(1 + \sum_{s=1}^{r}\sum_{k(s)} \delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1)\right).$$

Now we study

$$\begin{aligned} S_{1,k(s)} &= \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\sum_{p}^{m^\sigma} p^{-\frac{1}{2}}\delta_{k(s)}(p,1) \\ &\ll \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\sum_{n\equiv 1(m_{k(s)})}^{m^\sigma} n^{-\frac{1}{2}} \\ &\ll \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\frac{1}{\prod_{i=1}^{s}(m_{k_i})}\sum_{n}^{m^\sigma} n^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}\sigma-1}. \end{aligned}$$

First Sum

There are 2^r choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2}\sigma - 1},$$

which is negligible as m goes to infinity for fixed r if $\sigma < 2$.
Cannot let r go to infinity.
If m is the product of the first r primes,

$$\begin{aligned} \log m &= \sum_{k=1}^{r} \log p_k \\ &= \sum_{p \leq r} \log p \ \approx \ r \end{aligned}$$

Therefore

$$6^r \ \approx \ m^{\log 6} \ \approx \ m^{1.79}.$$

Second Sum Expansions and Bounds

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) = \begin{cases} m_i - 1 - 1 & p \equiv \pm 1 (m_i) \\ -1 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \sum_{\chi \in \mathcal{F}} \chi^2(p) &= \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}^2(p) \cdots \chi_{l_r}^2(p) \\ &= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \\ &= \prod_{i=1}^{r} \Big(-1 + (m_i - 1)\delta_{m_i}(p,1) + (m_i - 1)\delta_{m_i}(p,-1) \Big). \end{aligned}$$

Second Sum Expansions and Bounds

Handle similarly as before. Say

$$\begin{aligned} p &\equiv 1 \bmod m_{k_1}, \dots, m_{k_a} \\ p &\equiv -1 \bmod m_{k_{a+1}}, \dots, m_{k_b} \end{aligned}$$

How small can p be?
$+1$ congruences imply $p \geq m_{k_1} \cdots m_{k_a} + 1$.
-1 congruences imply $p \geq m_{k_{a+1}} \cdots m_{k_b} - 1$.
Since the product of these two lower bounds is greater than $\prod_{i=1}^{b}(m_{k_i} - 1)$, at least one must be greater than $\left(\prod_{i=1}^{b}(m_{k_i} - 1)\right)^{\frac{1}{2}}$.
There are 3^r pairs, yielding

$$\text{Second Sum} = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}.$$

Summary

Agrees with Unitary for $\sigma < 2$ for square-free $m \in [N, 2N]$ from:

Theorem

- *m square-free odd integer with $r = r(m)$ factors;*
- $m = \prod_{i=1}^{r} m_i$;
- $M_2 = \prod_{i=1}^{r}(m_i - 2)$.

Then family $\mathcal{F}_m$ of primitive characters mod m has

$$\begin{aligned} \textit{First Sum} &\ll \frac{1}{M_2} 2^r m^{\frac{1}{2}\sigma} \\ \textit{Second Sum} &\ll \frac{1}{M_2} 3^r m^{\frac{1}{2}}. \end{aligned}$$