NOTES ON TALKS AT THE SIMONS SYMPOSIUM ON FAMILIES OF AUTOMORPHIC FORMS AND THE TRACE FORMULA

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ABSTRACT. The following are notes on the talks at the Simons Symposium on Families of Automorphic Forms and the Trace Formula, held from January 26 to February 1 in Puerto Rico. **All** errors should be attributed solely to the typist, Steven J. Miller; these notes were TeX-ed in real time (with no effort made to go back and correct mistakes!).. See

https://web.math.princeton.edu/~templier/families/

for more information about the conference.

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Key words and phrases. Families, Automorphic Forms, Trace Formula.

1. FAMILIES OF AUTOMORPHIC FORMS OF COHOMOLOGICAL TYPE (CALEGARI)

Fix a reductive group G, fix an infinity type $\pi_\infty \in \widehat{G}$.

Basic problem: count the multiplicity $m(\pi_\infty, \Gamma)$ in $\mathrm{L}^2_{\mathrm{cusp}}(\Gamma \setminus G)$ as Γ varies. Usually $G = \mathbb{G}(\mathbb{R})$. For example, $\mathbb{G} = \mathrm{SL}(N)/\mathbb{Q}$ and $\Gamma \subset \mathbb{G}(\mathbb{Z})$.

deGeorge-Wallach:

$$\lim_{N\to\infty} \frac{m(\pi_\infty, \Gamma(N))}{\mathrm{vol}(\Gamma(N) \setminus G/K)} \text{ is } \begin{cases} \neq 0 & \text{if } \pi_\infty \text{ D.S.} \\ = 0 & \text{if } \pi_\infty \text{ D.S.} \end{cases} \tag{1.1}$$

(the limit exists in the first case, and is non-zero).

If we take $G = \mathrm{GL}_2(\mathbb{R})$ then π_∞ D.S. iff π is a classical modular form of weight $k \geq 2$, and π_∞ is a HLDS iff π is a classical modular form of weight 1.

General problem: if G does not have D.S., do there exist cuspidal automorphic forms? The answer is yes. Two sources: via functoriality and via trace formula.

Consider $\mathrm{GL}_M(\mathbb{R})$.

- Symplectic / orthogonal groups. Arises via functoriality.

Deficit: the resulting π are not genuinely from G. Even though it is on $\mathrm{GL}_M(\mathbb{R})$ it knows it is not really from there and came from something smaller.

Want a technique to produce forms *not* coming from smaller groups. Can use Weyl law. This allows us to count certain classes of automorphic forms and "almost all" such form have to genuinely come from G. Just asymptotically number is proportional to volume, those coming from smaller are not enough. Deficit is allowing π_∞ to vary.

What π_∞ are of interest to me?

- Cohomological type: These representations contribute to the Betti cohomology of the arithmetic quotients $X(\Gamma) = \Gamma \setminus G/K$. When talking about π_∞ automorphic form satisfying conditions. Very specific way have some functional equations, say assume eigenvalue zero and harmonic form, can compute using deRham cohomology, arise in cohomology of a manifold.
- Holomorphic limits of D.S.: Loosely speaking, these contribute to the coherence cohomology of the Shimura variety (in H^0). From a formula point of view, no distinction between different weights, satisfy a formula for modular forms, $\overline{\mathrm{H}^0(w^{\otimes k})}$.

Want to count motives or something on the automorphic side, so want to restrict.

Can vary Γ in several ways. Two types of families. Horizontal: $\Gamma(N)$ with $N \in \mathbb{N}$. Vertical: $\Gamma(p^k)$ with p a fixed prime and $k \in \mathbb{N}$.

Let's look at classical weight 1 modular forms of level $N = Mp^k$ for M fixed. What do we know about these? Weight 1 modular forms are in bijection to Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ odd, irreducible conductor N. Possibilities are A_4, A_5, S_4, D_*. Up to twists, the first three occur only finitely often. As go up the tower, all but finitely many (up to twists) are dihedral down here and are induced from GL_1 (and some quadratic extension).

Conjecture A: Fix π_∞. $\mathrm{L}_{\mathrm{cusp}}(\Gamma(Mp^k) \setminus G)$. Up to twist, there will only be $O(1)$ forms which don't come from a smaller group (if π_∞ is not DS).

Conjecture B: Let A be the set of all algebraic π_∞. Make the same conjecture.

Example: Take GL_2/K, π_∞ equal weight 0. $\mathrm{H}^1(\Gamma(N) \setminus \mathbb{H}^3) \leftarrow E/K$.

Consider $\mathbb{G} = \mathrm{GL}(n)/\mathbb{Q}$. We should have representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)/\sim$, tame level M, all $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ p-adic HT cond. By compactness argument land in $\mathrm{GL}_n(\overline{Z}_p) \overset{\overline{\rho}}{\to} \mathrm{GL}_{\mathrm{N}}(\mathbb{F}_p)$. $\mathcal{X}_{\mathrm{glob}}$ is the rep. variety of $G_{\mathbb{Q},Sp}$.

Fix $\overline{\rho}$, $\coprod_{\overline{\rho}}$ nice space $B(1)^2$, $\mathcal{X}_{\mathrm{loc}}$ is a nice space $B(1)^3$ and $\mathcal{X}_{\mathrm{loc}}^{\mathrm{dR}}$ is a countable union of nice subspaces $B(1)$. Have $\mathcal{X} \to \mathcal{X}_{\mathrm{loc}}$ (finite) and $\mathcal{X}_{\mathrm{glob}}^{\mathrm{dR}} \to \mathcal{X}_{\mathrm{loc}}^{dR}$.

In general can do a computation that

$$\dim \mathcal{X}_{\mathrm{loc}} - \mathrm{codim} \mathcal{X}_{\mathrm{glob}} - \mathrm{codim} X_{\mathrm{loc}}^{dR} = \begin{cases} 0 & \text{if } \pi_\infty \text{ D.S.} \\ -\ell_0 > 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Examples: Conjecture (Ash-Pollock): $\mathrm{H}^2_{\mathrm{cusp}}(\mathrm{SL}_3\mathbb{Z}, \bigvee_K)$ (with $\bigvee_K$ finite dim alg rep of $\mathrm{SL}_3\mathbb{R}$) equals the space of sym^2 from $\mathrm{GL}(2)/\mathbb{Q}$.

Let $F = \mathbb{Q}(\sqrt{-2})$, $3 = \pi\overline{\pi}$, D/F division alg ram at $\pi, \overline{\pi}$ with inv $1/n, -1/n$, Γ equals $\mathcal{O}_p^\times \to \mathrm{GL}_n(\mathbb{C})$,

$$\dim H^*(X(\Gamma(\pi^k)), \mathbb{C}) = H^*_{\mathrm{inv}}(X(\Gamma(\pi^k), \mathbb{C}) =^? \tag{1.3}$$

Regard as weak evidence towards problem, but suggest that disproving will be hard as no trivial source.

One last example. Let F be a real quadratic field. Look at Hilbert Modular Forms of weight (k_1, k_2) with $k_1 \equiv k_2 \bmod 2$. If $k_1, k_2 \geq 2$ D.S., if $k_1 = k_2 = 1$ (???), or $k_1 = 1$ and $k_2 \neq 1$. If weight is $(2k+1m1)$ then $\mathrm{H}^0(X, \mathcal{E}_k)$ (essentially same dimension as H^1. What are such forms? Can induce grossencharacters. My students Specter and May found a non-CM form. On the other hand, have the following theorem: For certain N and totally odd χ satisfying some conditions (such exist), the dimension of the space $S_{(1,2k+1)}(\Gamma(N), \chi)$ equals CM forms for all k.

2. Limit multiplicities for principal congruence subgroups of GL_n (Muller)

Much of the following is joint work with T. Finis and E. Lapid.

2.1. **Part 1.** Let G be a semi-simple real Lie group of non-compact type. Let $\widehat{G}$ be the unitary dual equipped with its natural measure / topology. On $\widehat{G}$ we have the Plancherel measure μ_{PL}, whose support is the tempered spectrum / dual $\widehat{H}_{\mathrm{temp}} \subset \widehat{G}$. Consider a lattice $\Gamma \subset G$. Have a representation $G \to \mathrm{Aut}(L^2(\Gamma \setminus G))$. Consider R_Γ^{disc}, $L^2_{\mathrm{disc}} = \widehat{\oplus_{\pi \in \widehat{G}}} m_\Gamma(\pi)\mathcal{H}_\pi$.

Problem: Information about $m_\Gamma(\pi)$.

We saw in the previous talk that you can fix a representation and can consider a sequence of lattices:

$$\frac{m_{\Gamma(N)}(\pi)}{\mathrm{vol}(\Gamma(N) \setminus G)} \to \begin{cases} d(\pi) & \text{if } \pi \in \widehat{G}_d \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Consider

$$\mu_\Gamma \ = \ \frac{1}{\mathrm{vol}(\Gamma \setminus G)} \sum_{\pi \in \widehat{G}} m_\Gamma(\pi)\delta_\pi, \tag{2.2}$$

where (Γ_n) is a sequence of lattices with $\mathrm{vol}(\Gamma_n \setminus G) \to \infty$.

Definition 2.1. *We say (Γ_n) satisfies the limit multiplicity property (LMP) if*

(1) *For all $A \subset \widehat{G}_{\mathrm{temp}}$ (Jordan measurable subset, which means A is bounded and $\mu_{\mathrm{PL}}(\partial A) = 0$) we have $\mu_{\Gamma_n} \to \mu_{\mathrm{PL}}$.*
(2) *For all $A \subset \widehat{G} \setminus \widehat{G}_{\mathrm{temp}}$ we have $\mu_{\Gamma_n} \to 0$.*

An equivalent statement: $f \in C_c(\widehat{G})$ then $\mu_{\Gamma_n} \to_{n \to \infty} \mu_{\mathrm{PL}}(f)$.

Results: $\Gamma \supset \Gamma_1 \supset \Gamma_2 \cdots$ normal subgroups of finite index with $\cap \Gamma_i = \{1\}$. $\Gamma \setminus G$ cocompact: DeGeorge, Wallach, Delorme.

Generalizations: (Albest, Bergeron, et al) Consider $K \subset G$ maximal compact, $\widetilde{X} = K/G$, (Γ_n), $X_n = \Gamma_n \setminus \widetilde{X}$, X_n BS-convergent to $\widetilde{X}$ if for all $R > 0$ we have $\mathrm{vol}((X_n)_{<R})/\mathrm{vol}(X_n) \to_{n\to\infty} 0$. We have the following theorem: (Γ_n) unif. discrete sequence, $X_n \to_{\mathrm{BS}} \widetilde{X}$ implies LMP holds for (Γ_n).

2.2. **Part 2: Non-uniform case.** Back in the case of towers of finite index that intersect to $\{1\}$: $\Gamma \supset \Gamma_1 \supset \Gamma_2 \cdots$. Take $\pi \in \widehat{G}_d$, $\lim_{n\to\infty} m_{\Gamma_n}(\pi)/\mathrm{vol}\,(\Gamma_n \setminus G) = d(\pi)$. Results by Rohlfs-Speh, Savin, Clozel $\geq C > 0$. $\mathbb{Q}$-rank 1: Deitmar, Hoffmann.

2.3. **Part 3: General case.** $\mathbb{G}/F$ real, F finite field, $\mathbb{P}_0 \subset \mathbb{G}$ minimal parabolic, M_0. S finite set of places, $S_\infty \subset S$. $F_S = \prod_{\nu \in S} F_\nu$, $\mathbb{A}^S = \prod_{\nu \notin S} F_\nu$. $K \subset \mathbb{G}(\mathbb{A}^S)$ spec compact, $\mu_K = \mathrm{vol}\,(K)\,/\mathrm{vol}\,\big(G(F) \setminus G(\mathbb{A})^1\big)$. Consider

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A})^1)} m(\pi)\mathrm{dim}(\pi^S)^k \delta_{\pi_S}. \tag{2.3}$$

Let $\mathcal{K} = \{K_i, i \in I\}$. Measures $\mu_K \to \mu_{\mathrm{PL}}$ if

(1) For all $A \subset \Pi(G(F_S)^1_{\mathrm{temp}})$ meas J, $\mu_K(A) \to \mu_{\mathrm{PL}}(A)$.
(2) Bounded $A \subset \Pi(G(F_S)^1) \setminus \Pi_{\mathrm{temp}}(G(F_S)^1)$, $\mu_K(A) \to 0$ if for all $\epsilon > 0$ have $|\mu_K(A) - \mu_{\mathrm{PL}}(A)| \geq \epsilon$ for only finitely many $K \in mathcal K$.

Theorem 2.2. *LMS holom for $\mathbb{G} = \mathrm{GL}_n$ and $\mathcal{K}(\mathfrak{n})$, $\mathfrak{n} \subset \mathcal{O}_F$ prime to S.*

How to approach? Want to handle it by using the trace formula. First step is the density theorem of Sauveigeot: $\mathcal{H}(G(F_S)) = C_+^\infty$, compactly supported bi-K_S finite.

$$h \in \mathcal{H}(G(F_S)) \ \mapsto \ \mathrm{tr}\pi(h) \ = \ \widehat{h}. \tag{2.4}$$

Function of $\Pi(G(F_S)^1)$, var $|\chi|$. Get $\mu_k(\widehat{h})$ is $\mathrm{tr}R_{\mathrm{disc}}(h \otimes 1_{\mathbb{K}^S}$.

Theorem 2.3 (Sauvergeot)**.** *Let $\mathcal{K} = \{k_i : i \in I\}$ be such that for all $h \in \mathcal{H}(G(F_S)^1)$, $\mu_K(\widehat{h}) \to h(1)$. Then LMP holds.*

Apply trace formula (non-invariant trace formula of Arthur). $h \in C_c^\infty(G(\mathbb{A})^1)$. $J_{\mathrm{spec}}(h) = J_{\mathrm{geom}}(h)$. Here $J_{\mathrm{spec}}, J_{\mathrm{geom}}$ are sums of distributions on $G(\mathbb{A})^1$.

Take $L^2(G(F) \setminus G((A)^1)$.

$$J_{\mathrm{spec}}(h) \ = \ \mathrm{tr}R_{\mathrm{disc}}(h) + \sum_{\substack{[M] \\ M \neq G}} J_{M,\mathrm{spec}}(h), \tag{2.5}$$

with M a Levi group, $M \supset M_0$. $J_{M,\mathrm{spec}}$ defined by Eisenstein series. $J_{\mathrm{geom}}(h) = \mathrm{vol}\left(G(F) \setminus G(\mathbb{A}^1\right) h(1) +$ orbital integrals.

(a) We have

$$J_{\mathrm{spec}}(h \otimes 1_K) - \mathrm{tr}R_{\mathrm{disc}}(h \otimes 1_K) \ \to \ 0. \tag{2.6}$$

(b) Take $h \in \mathcal{H}(G(F_S)^1)$.

$$J_{\mathrm{geom}}(h \otimes 1_K) - \mathrm{vol}\left(\ \right)(h(1)) \ \to \ 0. \tag{2.7}$$

Q, P parabolic with Levi M, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}^1))$.

$M_{Q|P}(\pi, \lambda) : \mathcal{A}_\pi^2 \to \mathcal{A}_\pi^2(P)$, $\lambda \in G_{M,\mathbb{C}}^\times$ intertwining operators.

$$M_{Q,P}(\pi, \lambda) \ \leftrightarrow \ n_\alpha(\pi, \lambda), R_{Q|P}(\pi, \lambda). \tag{2.8}$$

Theorem 2.4 (Finis, Lapid, M–)**.** *$J_{M,\mathrm{spec}}$ can be expressed in terms of rank one Int. op and log der.*

$\mathbb{G} = \mathrm{GL}_n$, $M = \mathrm{GL}(m) \times \mathrm{GL}(n_2)$. $\pi = \pi_1 \otimes \pi_2$, $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(n; \mathbb{A})$:

$$n(\pi, S) \ = \ \frac{L(s, \pi_1 \times \widetilde{\pi}_2)}{\epsilon() N(\pi_1 \times \pi_2)^{h-S} L(s+1, \pi_1 \times \widetilde{\pi}_2)}. \tag{2.9}$$

3. Families of L-functions and their symmetry (Templier)

The following is joint work with P. Sarnak and S.-W. Shin, available on the arXiv:

http://arxiv.org/abs/1401.5507.

We'll work on $\mathrm{GL}(n)/\mathbb{Q}$.

Let π be an automorphic cusp form, a_π the coefficients. The Sato-Tate problem asks for the equidistribution for the a_π as we vary p. To be precise, let's take $\{a_\pi(p) : p < x\}$ with $x \to \infty$. Is this equidistributed with respect to $\mu_{\mathrm{ST}}(\pi)$?

Taylor and his colleagues proved it holds if π is holomorphic on $\mathrm{GL}(2)$.

Example: Consider $E/\mathbb{Q}$. Can write $p^{1/2}a(p) = p + 1 - \#E(\mathbb{F}_p)$. Hasse proved $|a(p)| \leq 2$. It is equidistributed for $\mu_{\mathrm{ST}} = \sqrt{4-a^2}da/2\pi$. Not known in other cases. Even giving a recipe for $\mu_{\mathrm{ST}}(\pi)$ is hard. If π is algebraic there is a recipe to give the conjectured Sato-Tate measure.

Let $\mathcal{F}$ be a family of forms, and let $\mathcal{F}(y)$ denote all elements of the family whose conductor $C(\pi)$ is less than y. This always picks out a finite set.

Limit multiplicity, Weyl's law, cohomological growth: asks for the distribution of $a_\pi(p)$ for a fixed prime as $y \to \infty$. Refined version is to let the prime p vary as well.

Sato-Tate for the family: Consider $\{a_\pi(p) : p < x, \pi \in \mathcal{F}(y)\}$ where now both $x, y \to \infty$.

Riemann:

$$\#\{0 \leq \gamma \leq T, \zeta(\sigma + i\gamma) = 0\} \sim \frac{T}{2\pi}\log T \tag{3.1}$$

as $T \to \infty$.

Montgomery pair correlation conjecture: Instead of counting zeros we count pairs of zeros, and consider the gap between them. We believe

$$\#\{0 \leq \gamma \neq \gamma' \leq T, \frac{2\pi a}{\log T} \leq \gamma - \gamma' \leq \frac{2\pi b}{\log T}\} \sim N(T)\int_a^b \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right)dx, \tag{3.2}$$

where the integrand is a Dyson kernel related to random matrix theory.

Katz-Sarnak heuristics: zeros of $L(s,\pi)$ as π ranges through a family should agree with either $\mathrm{U}(\infty)$, $\mathrm{Sp}(\infty)$ or $\mathrm{SO}(\infty)$ (or one of the subgroups. The question is how to determine the symmetry type? This is table 2 in their BAMS '99 paper.

Siegel modular forms: $f(z)$ with $z \in \mathcal{H}_{2g}$, have $f((Az+B)(Cz+D)^{-1}) = \det(Cz+D)^k f(z)$, where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ and weight $k < y$.

Let $t_\pi(p)$ be the Satake parameters, $\widehat{G(\mathcal{O}_p)} \supset \widehat{G(\mathcal{O}_p)}^{\mathrm{unr}} \subset \widehat{T}/W$, $\widehat{G(\mathcal{O}_p)}^{\mathrm{temp}} \supset \widehat{G(\mathcal{O}_p)}^{\mathrm{temp,\ unr}} \simeq \widehat{T}_c/W$ with $\widehat{T} = \mathbb{C}^g$.

Get the Sato-Tate measure μ_{ST} by a push forward of the Haar measure on $\widehat{G}_c$.

Main result with Shin is:

Theorem 3.1 (Shin-T). *Sato-Tate for family $\mathcal{F}$ under some assumptions: π_∞ is a discrete series, $\Phi \in C_c^\infty(\widehat{T})$, then*

$$\frac{1}{\pi(x)}\frac{1}{|\mathcal{F}(y)|}\sum_{\substack{\pi\in\mathcal{F}(y)\\ p<x}} \Phi(t_\pi(p)) \to \int \Phi\mu_{\mathrm{ST}} \tag{3.3}$$

as soon as $x < y^c$ for some c which depends on G and is positive.

Consider

$$\left| \frac{1}{|\mathcal{F}(y)|} \sum_{\pi \in \mathcal{F}(y)} \Phi(t_\pi(p)) - \int \Phi \mu^p_{P'} \right| \ll p^A y^{-\delta} \tag{3.4}$$

for some $\delta > 0$ and $A < \infty$. Ingredients of the proof include the Arthur trace formula and some extra steps. The extra steps are difficult (integrals, convergence of trace formula, Fourier transform on the geometric side).

Conjecture 1 (Montgomery, Katz-Sarnak, Sarnak-Shin-T)**.** *The low-lying zeros of $\pi \in \mathcal{F}$ follow the following heuristics (moments of $\mu_{\mathrm{ST}}(\mathcal{F})$) for a good family:*

Homogeneity type	Symmetry
unitary	U(∞)
orthogonal	Sp(∞)
symplectic $\epsilon = 1$	SO(even)
symplectic $\epsilon = -1$	SO(odd)

Cannot calculate large enough support to distinguish the different densities if only do the 1-level, but can distinguish if calculate the 2-level density.
Harmonic family: Pick a bunch of local conditions. Positive Plancherel measure.
Geometric family: $X \to W \subset \mathbb{A}^m$, assume map is smooth. Have $w \in W(\mathbb{Z})$, X_w Hasse-Weil L-function, $D(w) \neq 0$, $L(s, X_w)$. Similar result holds due to Deligne-Katz equidistribution theorem.

$$\begin{aligned} i_1(\mathcal{F}) &= \int |\mathrm{tr}(t)|^2 \mu_{\mathrm{ST}}(t)\ (-1) \\ i_2(\mathcal{F}) &= \int_T \mathrm{tr}(t)^2 \mu_{\mathrm{ST}}(t) \\ i_3(\mathcal{F}) &= \int_T \mathrm{tr}(t^2) \mu_{\mathrm{ST}}(t); \end{aligned} \tag{3.5}$$

all of these are integers. Fluctuation of ϵ. Alluded to a few times that there is some type of Sato-Tate group $H(\mathcal{F})$ which gives $\mu_{\mathrm{ST}}(\mathcal{F})$; unfortunately the Sato-Tate group can be difficult to define. The value of $i_3(\mathcal{F})$ is in $\{0, -1, 1\}$ and is the invariant of the family.

4. Using model theory to obtain uniform bounds for orbital integrals (Gordon)

Joint with Raf Cluckers and I. Halupczok.

G is a connected reductive group (over a local field K **or** over a global field F). Sometimes the local field can vary. Think of a number field, but a lot of the talk is about transferring between function fields and fields of characteristic zero. I want to talk about things on harmonic analysis on G that don't depend on the prime. Can do base change to completion, but also a completely different method which I'll explain today.

For the moment we'll be over a local field (until further notice). First let's review the classical results of Harish-Chandra. Consider orbital integrals. We'll always take f to be a test function (locally constant over K of compact support): $f \in C_c^\infty(G(K))$. For $\gamma \in G(K)$ semi-simple, we set

$$O_\gamma(f) \ = \ \int_{C_a(\gamma)\backslash G(K)} f(g^{-1}\gamma g) d^\times g. \tag{4.1}$$

Set

$$D(\gamma) \ = \ \prod_{\alpha\in\Phi} |1-\alpha(\gamma)| \tag{4.2}$$

(where Φ is a root system). Example: in GL_n: if γ over $\overline{F}$ is diagonal (say with entries $\lambda_1, \dots, \lambda_n$) then $\alpha(\gamma) = \lambda_i\lambda_{i+1}^{-1}$ for α-simple. Note $D(\gamma)$ vanishes if element is not regular. Consider

$$D^a(\gamma) \ = \ \prod_{\substack{\alpha\in\Phi \\ \alpha(\gamma)\neq 1}} |1-\alpha(\gamma)| \tag{4.3}$$

(which is non-zero for non-regular elements). Harish-Chandra proved that for $\gamma \in G^{\text{reg s.s.}}$ for fixed test function f there is a constant C (which can depend on f and the root datum and K) such that we have

$$|O(\gamma)|\,|D(\gamma)|^{1/2} \ \le \ C. \tag{4.4}$$

Kottwitz: for all γ we have

$$|O(\gamma)|\,|D^G(\gamma)|^{1/2} \ \le \ C. \tag{4.5}$$

Normalization of measures: γ-regular then $C_G(\gamma) = T$-torus. Have finitely many conjugacy classes of tori. Fix the normalization of invariant measures on torus (pick "canonical" compact subgroup in each T). Then d^*gdg/dt (fix some H.M. on G). Non-regular: $C_G(\gamma) = M$ an be one of finitely many reductive groups. Pick "canonical" measures.

Question 1: C: how does it depend on ν as K varies through completions F_ν? On f?

Question 2: π-irreducible admissible representation of $G(K)$. Let $f \in C_c^\infty(G)$, define

$$\pi(f) \ = \ \int_G \pi(g) f(g) dg. \tag{4.6}$$

Let

$$\widehat{f} \ = \ \mathrm{Tr}\pi(f) \ = \ \int_{G(K)} \Theta_\pi(g) f(g) dg, \tag{4.7}$$

where Θ_π is in L^1_{loc} (locally constant on $G^{\text{reg s.s.}}$. How to transfer this statement to $\mathbb{F}_q((t))$.

Will discuss a method based on model theory. Plan:

(1) Construct a family of test functions for Question 1.

(2) Language of logic (first order language).
(3) Define a class of functions "definable" in this language (will be called motivic functions).
(4) "Prove" that orbital integrals belong to this class.
(5) Model theory (and transfer) implies that if a motivic is bounded then it has to be bounded by q^a with a a constant. Have to assume $p \gg 0$ (so results will be for all but finitely many primes, but we won't have control over the bad primes).

Families of L-functions: Generators of the spherical Hecke algebra. F global (number field), G/F, F_ν - completion of F; $\overline{\mathcal{O}_\nu}$ - ring of integers. $G_\nu = G \times_F F_\nu$ - unramified for almost all ν. $G_\nu(\mathbb{Q})$ - hyperspecial max comp (for unramified), smooth scheme over F_ν.

Have the Cartan decomposition. G^{spl} - split form of G, fix $\rho : G^{\mathrm{spl}} \hookrightarrow \mathrm{GL}_n$ (over $\mathbb{Z}[1/R]$), in G^{spl} have T (diagonal), which is contained in B (upper triangular). Let A_ν - max split torus in $T_\nu \subset G_\nu$, $\lambda \in X_*(A_\nu) \leftarrow X_*(T) \subset \mathbb{Z}^n$. Test functions $\tau_\lambda^G = 1_{\underline{G}_\nu(Q)\lambda(\omega)\underline{G}_\nu(\mathcal{O}_\nu)}$ (with ω a normalizer). For example, on $\mathrm{GL}_2(\mathbb{Q}_p)$ have

$$G(\mathbb{Z}_p)\begin{pmatrix} p^n & 0 \\ 0 & p^m \end{pmatrix} G(\mathbb{Z}_p). \tag{4.8}$$

Let

$$\lambda \in \mathbb{Z}^n \quad \tau_\lambda^G \quad |O_\gamma(\tau_\lambda^G)|\,|D^G(\gamma)|^{1/2} \;\leq\; C(\nu,\lambda). \tag{4.9}$$

Answer: $|C(\lambda,\nu)| \leq q_\nu^{a+b||\lambda||}$ for some constants a, b which depend only on the root data of G and ρ.

Definition of language: 3 sorts of variables: $x_1,\ldots,x_m$ are valued field, $y_1,\ldots,y_n$ residue field, and $z_1,\ldots,z_n \in \mathbb{Z}$. Only allow addition on the z's, but allow addition and multiplication in other two; also have $\leq$ and $\equiv_d$ on last. Also need constants for each (0, 1). Have map from X's to Y's by taking first non-zero coefficient of ω-adic expansion $\overline{\mathrm{ac}}$, and map from X's to z's by valuation (ord). For example, $\mathrm{ord}(x) \equiv 0 \bmod 2$ and there exists a y such that $\overline{\mathrm{ac}}(x) = y^2$.

5. (GEE)

Joint with Matthew Emerton. Won't state any theorems, but will instead concentrate on how we think about things. The motivation is that on the Thursday discussion section we'll continue with the Taylor-Wiles method. Today we'll discuss the structures that underlie this. The idea is that some part of the structures we talk about will remind someone about something in a related area (trace formula, geometric Langlands), and start a dialogue on Thursday. Not too much motivation for this talk. For introductory / motivation, see my talks at the Arizona winter school (video on-line and 50-60 pages of notes on my homepage). Will manly work on $\mathrm{GL}_2/\mathbb{Q}$. Can generalize, but there are some things known here that are not known in higher cases.

Will assume $p > 2$. Won't discuss Maass forms, will talk about holomorphic forms $f \in S_R(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ where the weight k is at least 2. We have

$$f = \sum a_n q^n, \quad a_1 = 1, \ a_n \in \overline{\mathbb{Q}}. \tag{5.1}$$

Motivating question (which we'll ignore till Thursday): How does a_p behave as N, k vary? To be slightly more precise, let's assume p does not divide N.

Let $\pi = \otimes_{\ell \leq \infty} \pi_\ell$ cuspidal automorphic representation of $\mathrm{GL}_2/\mathbb{Q}$.

$$\pi \leftrightarrow \rho_\pi : G_{\mathbb{Q}} \to \mathrm{GL}_2/\mathbb{Q}_p, \tag{5.2}$$

where $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Dictionary is defined in local terms: $\ell \neq p$, $\pi_\ell \leftrightarrow \rho_\pi|_{G_\ell}$ (correspondence by the local Langlands correspondence, and $G_\ell = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

$G_\ell \rhd I_\ell$ (inertia), $G_\ell/I_\ell = \langle \mathrm{Frob}_\ell \rangle$, ℓ doesn't divide N_p, $\rho_\pi|_{I_\ell}$ trivial, $\mathrm{tr}\rho_\pi(\mathrm{Frob}_\ell) - a_\ell$.

$\rho_p i|_{a_p} \quad \pi_p$ (loses information in general). Idea is the space of Galois representations is a refinement / replacement for the Bernstein center. Has the nice property of being equidimensional.

Consider moduli spaces of Galois representations. G_p compact implies $\rho_\pi|_{a_p} \to \mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$. Also have a map $\overline{\rho}$ from G_p to $\mathrm{GL}_2(\mathbb{F}_p)$, with $\overline{\rho} : G_p \to \mathrm{GL}_2(\mathbb{F}_p)$ "rigid".

Moduli space of all $\rho : G_p \to \mathrm{GL}_2(\mathbb{Z}_p)$ which lift a fixed $\overline{\rho}$. $\mathcal{X}^{\mathrm{loc}}$. $\rho_\pi|_{a_p}$ is deRham, with Hodge-Tate weights $0, k-1$. Fix conductor, 1-dimensional laws, infinite union and Zariski dense filling space. $\overline{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_q)$, with $\overline{r}|_{G_p} = \overline{\rho}$ and $r : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p)$. Imagine is now a 2-dimensional space (image of the $\mathcal{X}^{\mathrm{glob}}$). $\mathcal{X}^{\mathrm{glob}} \hookrightarrow \mathcal{X}^{\mathrm{loc}}$. Global representations which should come from modular forms (Fontaine-Mazur).

Idea: vary tame level N, x a modular form. Taylor-Wiles method is a way of carefully choosing N so that we have some control on the x. Find that each strand either has no X or a Zariski-dense set of X; see Figure 1.

Let M_∞ be the glued module of modular forms. Highly non-canonical.

Conjecture: M_∞ **is** canonical. $\mathrm{GL}_2/\mathbb{Q}_p$: true: M_∞ "is" p-adic local Langlands.

Consider $\mathrm{Spec}R_p$, $R_p \quad M_\infty$, $R_p/p \quad M_\infty/p$. By the work of Kisin, need to understand the Hilbert-Samuel multiply (??). FM conj: $e(R_p/p) = e(M_\infty/p)$; the left is purely local and the right is the multiplicity of mod p modular forms. Can think / hope of left as geometric side and right as spectral side as a trace formula.

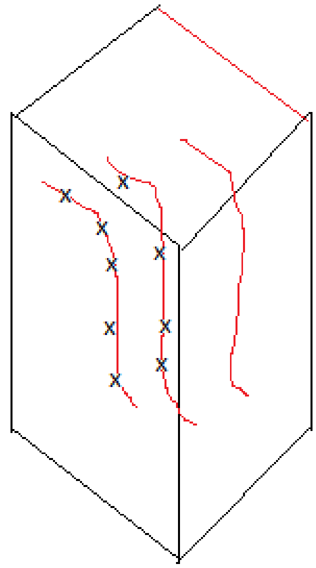

FIGURE 1. Image from varying tame level N.

6. Results on L-functions and Low-Lying Zeros (Miller)

Slides available online here:

http://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/SimonsTalk_LowlyingZer

7. Upper bounds for moments of L-functions (Soundararajan)

Will give a survey of progress in the past ten or fifteen years. Study objects such as

$$\int_0^1 |\zeta(1/2+it)|^{2k} dt, \tag{7.1}$$

where k is either a natural number or maybe a positive number. One would like to understand the growth (or even better find an asymptotic formula). A folklore conjecture says it should be asymptotic to $c_k T(\log T)^{k^2}$, where $c_k = a_k g_k$ with a_k an Euler product and g_k more mysterious. In this case

$$a_k \;=\; \prod_p \left(1-\frac{1}{p}\right)^{k^2} \left(\sum_{a=0}^{\infty} \frac{d_*\binom{a}{p}^2}{p^a}\right). \tag{7.2}$$

Hardy and Littlewood showed $g_1 = 1$ and $g_2 = 2$. There was a lot of progress in the '90s. Conrey and Ghosh conjectured $g_3 = 42$ and $g_4 = 24024$. Shortly afterwards Keating and Snaith came up with a general conjecture for g_k:

$$g_k \;=\; (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}. \tag{7.3}$$

There are analogues for families of L-functions (unitary, symplectic and orthogonal). The asymptotics for the moments reflect the family. For unitary consider Dirichlet characters modulo q, and conjecture

$$\sum_{\chi \bmod q} |L(1/2,\chi)|^{2k} \;\sim\; c_k q (\log q)^{k^2}. \tag{7.4}$$

For the symplectic case, consider

$$\sum_{|d|\le x} L(1/2,\chi_d)^k \;\sim\; c_k x (\log x)^{k(k+1)/2}. \tag{7.5}$$

For the orthogonal, let's look at quadratic twists of a fixed elliptic curve:

$$\sum_{|d|\le x} L(1/2, E\times\chi_d)^k \;\sim\; c_k x (\log x)^{k(k-1)/2}. \tag{7.6}$$

There are several ways to arrive at these constants and the powers of logarithms.

(1) The first is a beautiful idea of Keating and Snaith: use Random Matrix Theory. They proposed that the correspondence is more than just the zeros of L-functions correspond to eigenvalues of matrices, but the values of L-functions can be modeled by the values of the characteristic polynomials. Unfortunately this misses the arithmetic factor a_k, which must be added in.

(2) Conrey-Farmer-Keating-Rubinstein-Snaith: $\Lambda(s,\chi) = \epsilon_\chi \Lambda(1-s,\overline{\chi})$. Consider

$$\sum_{\chi \bmod q} \Lambda(1/2+\alpha_1,\chi)\cdots\Lambda(1/2+\alpha_k,\chi)\Lambda(1/2-\beta_1,\chi)\cdots\Lambda(1/2-\beta_k,\overline{\chi}). \tag{7.7}$$

Can trivially permute just the α's or just the β's, but can also switch an α with a β as that uses the functional equation twice. So symmetric in all α and β:

$$\sum_{m_1,\dots,m_k} \frac{\chi(m_1)\cdots\chi(m_k)}{m_1^{1/2+\alpha_1}\cdots m_k^{1/2+\alpha_k}} \sum_{n_1,\dots,n_k} \frac{\chi(n_1)\cdots\chi(n_k)}{n_1^{1/2-\beta_1}\cdots n_k^{1/2-\beta_k}}. \tag{7.8}$$

Consider

$$\sum_{m_1\cdots m_k = n_1\cdots n_k} \frac{1}{\prod m_j^{1/2+\alpha_j} n_j^{1/2-\beta_j}} \;\to\; \prod_{j,\ell} \zeta(1+\alpha_j-\beta_\ell). \tag{7.9}$$

Conjecture: Symmetrize:

$$\sum_{\pi \in S_{2k}/S_k \times S_k} \mathcal{Z}(\pi(\alpha, \beta)) \tag{7.10}$$

gives the $2k^{\text{th}}$ moment.

Small moments (Conrey, Iwaniec and S– for the sixth; Khandee and Li for the eight):

$$\sum_{q \leq Q} \sum_{\chi \bmod q} \int_{-\infty}^{\infty} |\Lambda(1/2 + it)|^{2k} dt, \tag{7.11}$$

and get the right answer for $k = 3$ or 4.

S– and Young: Under GRH, get

$$\sum_{|d| \leq x} L(1/2, E \times \chi_d)^2 \ \sim \ cx \log x. \tag{7.12}$$

Lower and Upper Bounds in Families: Lower bounds (Rudnick and S–, Radziwill and S–): Know first moment plus "ϵ" implies correct lower bounds for all $k \geq 1$. If have a lower bound then know some quantities are non-zero, gives a sharper version of non-vanishing. For example: $k \geq 1$ real, $\sum L(1/2, E \times \chi_d)^k \gg x(\log x)^{k(k-1)/2}$.

Upper bounds: Recent work of Radziwill and S–: If we know some moment plus ϵ implies correct upper bounds for all smaller moments. As a corollary: for $0 \leq k \leq 1$:

$$\sum_{|d| \leq x} L(1/2, E \times \chi_d)^k \ \ll \ x(\log x)^{k(k-1)/2} \ \ll \ \pi(x)(\log x)^{k(k-1)/2}. \tag{7.13}$$

If $0 < k < 1$ the exponents are negative. Tells us that except on a set of density $o(1)$ we have $L(1/2, E \times \chi_d) = O((\log x)^{-1/2+\epsilon})$. In the case of rank 0, $Ш(d) \ll \sqrt{d}/(\log |d|)^{1/2-\epsilon}$ almost surely.

If we assume GRH we can do much better. A few years ago S– proved an upper bound up to $(\log x)^{\epsilon}$. Recently Harper (2013) removed the power of logarithm and obtained a sharp result.

On GRH:

$$\log L(1/2, f) \ \leq \ \sum_{p \leq x} \frac{\lambda_f(p)}{\sqrt{p}} + \frac{1}{2}\left(\sum_{\text{prime squares}} \cdots\right) + \frac{\log \text{cond}}{\log x}. \tag{7.14}$$

(3) Selberg's Theorem from the 1940s: maybe t of size T, look at either the real or imaginary part of the logarithm of the zeta function on the critical line (or look at them simultaneously), $\log \zeta(1/2 + it)$. These behavior like a normal random variable with mean 0 and variance about $\frac{1}{2} \log \log T$.

$$\frac{1}{T} \text{meas}\left(t \leq T : \log |\zeta(1/2 + it)| \geq \lambda \sqrt{\frac{1}{2} \log \log T}\right) \ \sim \ \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} e^{-x^2/2} dx. \tag{7.15}$$

We are looking at $|\zeta(1/2 + it)| \geq e^v$. Frequency $e^{-v^2/\log\log T}$. We find

$$\int |\zeta|^{2k} dt \ = \ \int e^{2kv} (\text{freq of values of size } e^v), \tag{7.16}$$

with $v \sim k \log \log T$.

Conjectural analogues of Selberg's theorem: $|\ell| \leq x$: $\log L(1/2, \chi_d)$ is a Gaussian with mean $\sim \frac{1}{2} \log \log x$ and variance $\sim \log \log x$.

Consider positive functional equations and $\log L(1/2, E \times \chi_d)$ is Gaussian with mean $-\frac{1}{2} \log \log x$ and variance $\log \log x$.

Understanding the small moments (letting $k \to 0$) is trying to understand the distribution of $\log L(1/2, E\times \chi_d)$. Gives the distribution is bounded from above by a Gaussian.

8. EIGENVARIETIES (URBAN)

Start with $G/\mathbb{Q}$, Z, Q-split. $G/\mathbb{Q}_p)$ split (B,T) with $B = TN$. Let I be the Iwasawa subgroup which is a subset of $G(\mathbb{Q}_p)$, and let T^+ equal $\{t \in T(\mathbb{Q}_p) : tN(\mathbb{Z}_p)t^{-1} \subset N(\mathbb{Z}_q)\}$, which contains $T^{++} = \{t : \cap_M t^M N(\mathbb{Z}_p)t^{-M} = \{1\}\} \begin{pmatrix} p^\alpha & 0 \\ 0 & p^\beta \end{pmatrix}$.

$K^p \subset G(\mathbb{A}_f^p)$ with $K = K^p I$, $S_G(K) = G(\mathbb{Q}) \setminus G(\mathbb{A})/KK_\infty$.

V a $G(\mathbb{Q})$-rep, $\widetilde{V}$, $G(\mathbb{Q}) \setminus (G(\mathbb{A})/KK_\infty Z^\times V)$.

M K-rep, $\widetilde{M}$, $(G(\mathbb{Q}) \setminus G(\mathbb{A})/K_\infty Z_\infty \times M)/K$.

$G(\mathbb{A}_f)$ W

$\mathcal{X}$ weight space, $\mathcal{X}(\overline{O}_p) = \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times)$. Contains $\mathcal{X}^{\mathrm{alg}}$ = dominant alg weights.

$\lambda \in \mathcal{X}^{\mathrm{alg}}$ $\quad W_\lambda = (\mathrm{Ind}_{B^-}^G \lambda)^{\mathrm{alg}}$.

$\lambda \in \mathcal{X}_L$ $\quad \mathcal{A}_\lambda = \Big(\mathrm{Ind})B^- \cap I^I \lambda\Big)^{\ell a}$.

$W_\lambda \subset \mathcal{A}_\lambda$ if λ is algebraic.

$\Delta^{++} \subset \Delta^+ = IT^+I \quad (\Delta^+)^{-1}$ (which contains $G(\mathbb{Q}_p)$) acts on $\mathcal{A}_\lambda$ (which contains W_λ).

$W_\lambda^v \longleftarrow \mathcal{D}_\lambda$ which is continuous L-dual of $\mathcal{A}_\lambda$.

$H^0(\widetilde{S}_G, \mathcal{D}_\lambda) = \lim_{\overrightarrow{K^p}} H^0(S_G(K), \mathcal{D}_\lambda)$, $H^0(\widetilde{S}_G, W_\lambda^v)$.

$\mathcal{H}_p^+ = C_c^\infty(G(\mathbb{A}_f^p) \times \Delta^+, \mathbb{Q}_p)$, the left contains $\mathcal{H}_p^{++} = C_c^\infty(G(\mathbb{A}_f^p) \times \Delta^{++}, \mathbb{Q}_p)$.

Admissible rep of finite slope: (σ, V_σ) rep of $\mathcal{H}_p^+$ such that

- For all K^p, $V^K = 1_K(V)$ is finite dim.
- For all $t \in \Delta^-$, $u_t = 1_{ItI}$ is invertible. σ is real.

Let σ be real. We say σ is automorphic of weight λ if σ shows up in $H^q(\widetilde{S}_G, \mathcal{D}_\lambda)$ for some q. If π is classical automorphic and cohomological rep of $G(\mathbb{A})$ then $\pi = \pi^p \otimes \pi_f$.

$\pi|_{\mathcal{X}_p^+} \supset \sigma$ irreducible and λ algebraic.

σ is non ??? with respect to λ iff the ??? of the eigenvalues of u_t are small wrt λ.

K open compact which is spherical away from $\mathrm{SU}(p)$. $R_{s,p} = C(K^s \setminus G(\mathbb{A}_f^{\mathrm{sup}})/K^s \times \Delta^+, \mathbb{Q}_p)$.

Eigenvariety: $K \to S$.

$\mathcal{X} \leftarrow \xi_K$ $x = (\theta_\sigma, \lambda)$ such that (finite slope) $\theta_\sigma : R_{s,p} \to \overline{Q}_p$, θ_p shows up in $H(S_G(K), \mathcal{D}_\lambda)$. Rigid, analytic space (Agh-Staves, Xiong).

q $\; m_q^+(\sigma, \lambda)$ the multiplicity of σ in $H^q(\widetilde{S}_G, \mathcal{D}_\lambda)$.

$m^+(\sigma, \lambda) = \sum_q (-1)^q m_q^+(\sigma, \lambda)$.

Theorem: $\mathcal{H} \leftarrow \xi_K \supset \xi_K^{\mathrm{EP}} = \{x = (\theta_\sigma, \lambda)$ if $m^+(\sigma, \lambda) \neq 0$ and $\sigma^K \neq 0\}$.

If $G(\mathbb{R})$ does not have discrete series then $m^d(\sigma, \lambda) = 0$ if λ is regular.

$f^p \otimes u_t$. $I_G^\dagger(f, \lambda) = \mathrm{trace}(f : H^0(\widetilde{S}_G, \mathcal{D}_\lambda))$.

Theorem: $\lambda \mapsto I_G^\dagger(f, \lambda)$ is an analytic function on $\mathcal{X}$ multiplied by 1. $U \subset \mathcal{X}$. $\widetilde{D}_\lambda = \mathcal{D}_{U\lambda} \otimes L\mathbb{A}(U)$.

Remark: λ algebraic. One can show that

$$\oplus_{w:\ell(w)=j}\,\mathcal{D}_{w\times\lambda} \;\to\; W^v_\lambda \tag{8.1}$$

$I^{??}_G(f,\lambda)\xi(t)^\lambda \equiv I^\dagger_G(f,\lambda)$ mod $N(\lambda,t)$, with the left hand side equal to $\mathrm{tr}(f,H^0(--,W^v_?)$. $N(\lambda,t) = \mathrm{Inf}_{w\neq\mathrm{id}}|t^{\lambda-w\times\lambda}|$.

Corollary: $\lambda_n\to\lambda$ p-adically. $(\alpha^v,\lambda_n)\to\infty$ for all α^v simple root.

$I^\dagger_G(F,\lambda)=\lim_{k\to\infty}\xi(k)^{\lambda_n}I^\ell_G(f,\lambda_n)$. Arthur and Fronlu (?) computed $I^\ell_G(f,\lambda)$ for λ algebraic.

$$\xi(k)^\lambda I^\ell_G(f,\lambda) \;=\; \sum_w I_G(f^w, w\times\lambda). \tag{8.2}$$

p**-adic trace formula:** γ semi-simple element in $G(\mathbb{Q})$. $f_p\in C^\infty_c(\Delta^{++},\mathbb{Q}_p)$. $I_{G,\gamma}(f_p,\lambda)$.

$g\in\Delta^{++}$, $\lambda\in\mathcal{X}$, $\Phi^\dagger_G(g,\lambda)$ is the trace of g acting on $\mathcal{D}_\lambda$.

$\lambda\mapsto\Phi^\dagger_G(g,\lambda)$ is analytic.

$$I_{G,\gamma}(f,\lambda) \;=\; \int_{G(\mathbb{Q}_p)/G_\gamma(\mathbb{Q}_p)} f_p(x^{-1}\gamma x)\Phi^\dagger_G(x^{-1}\gamma x,\lambda)d\overline{x}. \tag{8.3}$$

Trace formula:

$$\begin{aligned} I^\dagger_G(f,\lambda) \quad &= \quad (-1)^{??}\sum_{\substack{\gamma\in G(\mathbb{Q})^{\mathrm{ss}}\\ \text{elliptic}}}(-1)^{\epsilon(\gamma)}\chi(G_\gamma,\mathrm{alg?})/L(\gamma)\\ &\qquad\times O_\gamma(f^p)I^\dagger_G(f_p,\lambda) \end{aligned} \tag{8.4}$$

with $f=f^p\otimes f_p$.

9. THE TRACE FORMULA AND PREHOMOGENEOUS VECTOR SPACES (HOFFMAN)

We start with induction of conjugacy classes. Here G is a linear algebraic group over a field F. We consider geometric conjugacy classes. What is the definition when G is reductive, $P = MN$ parabolic? Take a conjugacy class C in M, first get a conjugacy class in P: C' is the unique dense P-conjugacy class in CN. Take $\widetilde{C}$ to be the unique G-conjugacy class such that $\widetilde{C} \cap P = C'$. Usually one denotes $\widetilde{C}$ as the induced class from C, written $\mathrm{Ind}^G C$, with $CC' = \mathrm{Infl}^P C$ (the inflation). The union of all the inflations is P_{infl}: $P_{\mathrm{infl}} = \cup_C \mathrm{Infl}^P C$.

Theorem (L-L): If P, Q with levi component M then the induced classes $\mathrm{Ind}_P^G C = \mathrm{Ind}_Q^G C$ are the same (and hence will often omit the subscript).

Given any $\gamma \in G$ there are only finitely many P such that $\gamma \in P_{\mathrm{infl}}$. Tells us that a certain sum will be a finite sum.

Prehomogeneous varieties: V a G-variety, say it is prehomogeneous if there exists a dense orbit O: $p(gx) = \chi(g)p(x)$. Call V special if every p is constant.

p.v.: V vector space, action linear. It is regular if V^* is also prehomogeneous, and there exists p rel. inv. with dp/p a map from the regular orbit O to V^* (with the map $dp/p : O \to V^*$ a dominant homomorphism). There is a good theory if F is a number field:

$$Z(\phi, s_1, \dots, s_r) \;=\; \int_{G(\mathbb{A}/G(F)} |\chi_1(g)|^{s_1} \cdots |\chi_r(g)|^{s_r} \sum_{x \in O(F)} \phi(gx)dg. \tag{9.1}$$

Here ϕ is a Schwartz function: $\phi \in C_c^\infty(V)$. In general such integrals don't converge, classification where they do. If regular can apply Poisson summation and obtain several functional equations.

Canonical parabolic: $X \in \mathfrak{g}$ nilpotent. There exists H, Y such that $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$. Let $\mathfrak{q}_n = \{Z \in \mathfrak{q} : [H, Z] = n\mathbb{Z}\}$. $?? = \oplus_{n \geq 0} \mathfrak{g}_n$, $Q = \mathrm{Stab}(\mathfrak{q})$ will be a parabolic subgroup if $\mathfrak{g}$ is .... Called the maximal parabolic; since in characteristic zero can also consider the image under the exponential map (since nilpotent only finitely many terms). Get an element $\gamma = \exp X \in G(F)$.

Theorem: $\check{n} = \oplus_{n \geq 2} \mathfrak{g}_n$, $\check{n}' = \oplus_{n > 2} \mathfrak{q}_n$, $\check{n}/\check{n}'$ is regular p.v. under $L = \mathrm{Cont}_G \mathfrak{g}_0$.

Mean Value Formula: Theorem (Siegel/Weil/Ono): Let O be a homogeneous G variety over a number field F. Let $\pi_1(O(\mathbb{C})) = O$, $\pi_2(O(\mathbb{C})) = O$, $X^*(G) = O$, $[G(\mathbb{A}\xi \cap O(F) : G(F)]$ constant on $O(F)$. Then

$$\int_{G(\mathbb{A}/G(F)} \sum_{\xi \in O(F)} \phi(g\xi) \;=\; \int_{G(\mathbb{A})O(F)} \phi(x)dx. \tag{9.2}$$

Conjecture: There exists a normal subgroup N^C of P contained in N such that for $\gamma \in C'$ all elements of $\gamma N^C \cap C'$ have the same canonical parabolic and the prehomogeneous variety $\gamma N/N^C$ is special under $\mathrm{Stab}_P(\gamma N)$.

We now come to the geometric side of the trace formula. Representation of $\mathbb{G}^1$ on $L^2(\mathbb{N}P \setminus \mathbb{G}^1)$. $R_p(F)$, $f \in C_c^\infty(\mathbb{G}^1)$,

$$(R_p(f)\phi)(x) \;=\; \int_{G\mathbb{G}^1} K_P(x,y)\phi(y)dy, \tag{9.3}$$

with

$$K_P(x,y) \;=\; \sum_{\gamma\in P/N} \int_{\mathbb{N}} f(X^{-1}\gamma ny)dn. \tag{9.4}$$

When P is maximal parabolic:

$$-\,\widehat{\tau}_P \;=\; \mathrm{char}\{p \in P : \Delta_P(p) > \Delta(\exp T_p)\}\mathbb{K}. \tag{9.5}$$

Here Δ_P is the modular character.

For general parabolic P:

$$\tau_P(x) \;=\; \prod_{\substack{P'\supset P\\ \max.}} \tau_{P'}(x), \quad \tau_{\gamma P\gamma^{-1}}(\gamma x)^T \;=\; \tau_P^T(x), \quad T = (T_P)_P. \tag{9.6}$$

Arthur:

$$J^T(f) \;=\; \int \sum_{G\backslash\mathbb{G}^1} \sum_P K_P(x,x)\widehat{\tau}_P^{(}x)dx. \tag{9.7}$$

C geometric conjugacy class in G,

$$K_{P,C}(x,y) \;=\; \sum_{\substack{\gamma\in P/N\\ \mathrm{Ind}^G\gamma=C}} \int_{\mathbb{N}} f(x^{-1}\gamma ny)dn. \tag{9.8}$$

$J^T(f) = \sum_C J_C^T(f)$.

Selberg trace formula: $\mathrm{rk}_{\mathbb{R}} G = 1$.

Arthur trace formula: $\mathrm{rk}_{\mathbb{R}} G \geq 1$.

$\mathcal{F}$, $\mathcal{F}(y)$, $\frac{1}{|\mathcal{F}(y)|} \sum_{\pi \in \mathcal{F}(y)} \phi(\pi)$.

Lot of analytic problems not yet solved, want to discuss some. For GL(n) a number can be solved, want to indicate what is the difference between GL(n) and other groups. For the moment consider a reductive group over $\mathbb{Q}$, $G/\mathbb{Q}$ red. $\mathbb{A}_f$ will be the finite ring of adeles. Let $\mathbb{G}(\mathbb{A}^1) = \{g \in \mathbb{G}(\mathbb{A}) : |\chi(g)| = 1, \chi \in X(\mathbb{G})\}$.

Limit multiplicities, Weyl law.

$k \subset \mathbb{G}(\mathbb{A}_f)$ open compact.

$$\mu_K \;=\; \frac{1}{\mathrm{vol}\,(G(\mathbb{Q}) \setminus G(\mathbb{A})^1/K)} \sum_{\pi \in \Pi_{\mathrm{disc}}(G(\mathbb{A})^1} m(\pi) \dim(\pi_f)^{k_f} \delta_{\pi_\infty}. \tag{10.1}$$

Consider $A \subset \prod(G(\mathbb{R}))_{\mathrm{temp}}$, $\mu_k(A) \to \mu_{\mathrm{PL}}(A)$.

Reduction: $h \in C_c^\infty(G(\mathbb{R}))$, bi-$K_\infty$-finite.

$$\frac{1}{\mathrm{vol}\,(G(\mathbb{Q}) \setminus G(\mathbb{A})^1)} \mathrm{tr} R_{\mathrm{disc}}(h \otimes 1_K) \;\to_{N \to \infty}\; h(1). \tag{10.2}$$

$K \subset \mathbb{G}(\mathbb{A}_f)$, $k(N) \subset K$ cong subgroup.

(1) Control of residual spectrum: $L^2_{\mathrm{disc}} - L^2_{\mathrm{cusp}} \oplus L^2_{\mathrm{res}}$.

$\frac{1}{\mathrm{vol}} R_{\mathrm{cusp}}(h \otimes 1_k) \to h(1)$.

Moeglin, Waldsburger: Description of residual spectrum.

Trace formula: $f \in C_c^\infty(\mathbb{G}(\mathbb{A})^1)$, $J_{\mathrm{spec}}(f) = J_{\mathrm{geom}}(f)$, M Levi contains M_0.

The spectral side term is $\sum_{[M]} J_{\mathrm{spec,M}}(f)$, where the summands are $\mathrm{tr} R_{\mathrm{disc}}(f)$.

$P = MN$, $R_{\mathrm{disc,M}}$, $L^2_{\mathrm{disc}}(A_M M(\mathbb{Q}) \setminus M(\mathbb{A}))$.

$I_{P(\mathbb{A})}^{G(\mathbb{A})}(R_{\mathrm{disc,M}} \otimes e^{\langle x, H_M(\cdot)\rangle})$.

$\overline{A}_2 \supset A_2(P)$, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)$.

$A^2_\pi(P)$, $P, Q \in \mathcal{P}(M)$.

$M_{Q|P} : A^2_\pi(P) \to A^2_\pi(Q)$ intertwining operator.

P max, $\overline{P}$.

$$\begin{aligned} J_{\mathrm{spec,M}} \quad &= \quad \sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1)} \int_{\mathbb{R}} \mathrm{tr}\Big(M_{\overline{P}|P}(\pi, it)^{-1} \frac{d}{dt} M_{\overline{P}|P}(\pi, it) \\ &\qquad M_{P|P}(w, o) \rho_\pi(p, t, f) dt \Big). \end{aligned} \tag{10.3}$$

Theorem (Finis, Lapid, M–: $J_{\mathrm{spec}}(f)$ is absolutely convergent.

$M_{\overline{P}|P}(\pi, s) = n(\pi, s) R_{\overline{P}|P}(\pi, s)$, $n(\pi, s) = \prod_\nu (\pi_\nu, s)$.

$R_{\overline{P}|P}(\pi, s) = \otimes_{\nu \in S} R_{\overline{P}|P}(\pi_\nu, s)$ (local intertwining).

G satisfies temp winding numbers (TWN) if for all $M \in \mathcal{L}$, $M \neq G$, $\mathcal{F} \subset \prod(k_{M,\infty})$ finite, there exists $k > 1$ such that $\alpha \in \Sigma_M$ and $\epsilon > 0$

$$\int_{\mathbb{R}} \left| \frac{n'_\alpha(\pi, s)}{n_\alpha(\pi, s)} \right| (1 + |s|)^{-k} ds \;\ll_{\mathcal{F}, \epsilon}\; (1 + \Lambda_{\pi_\infty})^k. \tag{10.4}$$

Here $\mathbb{G} - \mathrm{GL}(n)$, rank P is 1, M is $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$, $\pi = \pi_1 \otimes \pi_2$, $\pi_i = \prod_{\text{cusp}}(\mathrm{GL}(n; \mathbb{A}))$,

$$n(\pi, s) \;=\; \frac{L(s, \pi_1 \times \widetilde{\pi_2})}{L(s+1, \pi_1 \times \widetilde{\pi_2})\epsilon(1/2, \pi_1 \times \widetilde{\pi_2})N(\pi_1 \times \widetilde{\pi_2}^{1/2-s})}. \tag{10.5}$$

Battery ran out.

11. SATO-TATE CONJECTURE FOR FAMILIES (SHIN)

Joint with Templier and Kottwitz and Cluckers-Gordon-Halupczok.

11.1. **Introduction.** Let $\mathcal{F}$ be a family of automorphic representations. Call this a "harmonic family". What is the distribution of the Satake parameters at p of $\pi \in \mathcal{F}$ as p, π vary? Eventually want to push p to infinity; π is ordered by analytic conductor and want that to go to infinity.

Goal is to explain the precise statement of the conjecture / theorem, and discuss some of the ingredients of the proof. For example, why we need a uniform bound on orbital integrals as p varies; very useful in this context.

11.2. **Families in level / weight aspect.** Hypothesis: G connected, reductive split group over $\mathbb{Q}$ and assume the center $Z(G) = 1$. Some of these assumptions are for simplicity. The essential hypothesis is that $G(\mathbb{R})$ has discrete series.

$N \geq 1$, ξ irreducible algebraic representation of G over $\mathbb{C}$. $\Gamma(N) := \mathrm{Ker}(G(\widehat{Z}) \to G(\mathbb{Z}/N))$, $\prod_\infty(\xi)$ equals discrete L-packet associated to ξ.

We now define a finite set whose union will be the family we consider. N is the level, ξ the weight:

$$\mathcal{F}(N, \xi) = \{\pi \subset L^2_{\mathrm{disc}}(G(\mathbb{R}) \setminus G(\mathbb{A})) : (\pi^\infty)(\xi)\}. \tag{11.1}$$

Level aspect: $N \to \infty$, ξ fixed; weight aspect: N fixed, $\xi \to \infty$.

11.3. **Satake theory.** Have $\widehat{T} \supset \widehat{T}_C$ and $\widehat{G} \supset \widehat{G}_C$ with $\widehat{T} \subset \widehat{G}$. Weyl group Ω. $\widehat{T}_C/\Omega \supset \widehat{G(\mathbb{Q}_p)} \subset \widehat{T}/\Omega$ with $\pi \mapsto x$ and $\mathrm{Ind}_B^{G(\mathbb{Q}_p)}\ \pi$, $x \in \mathrm{Hom}(T(\mathbb{Q}_p), \mathbb{C}^\times) - X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$.

$\mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p)) \overset{\sim}{\to} C[\widehat{T}/\Omega]$ by $\phi \to \widehat{\phi} : \pi \to \mathrm{tr}\pi(\phi)$.

Let $V := \mathrm{Lie}\widehat{G}/\mathrm{Lie}\widehat{T}$.

Have $\widehat{\mu}_p^{\mathrm{pl,ur}}(t) = \det(1 - \mathrm{ad}(t)|V)/\det(1 - p^{-1}\mathrm{ad}(t)|V)$ supported on $t \in \widehat{T}_C/\Omega$.

$$\lim_{p \to \infty} \widehat{\mu}_p^{\mathrm{pl,ur}} = \widehat{\mu}^{\mathrm{ST}} \tag{11.2}$$

by definition, induced by Haar on $\widehat{G}_C$.

11.4. **Theorem.** $(N, p) = 1$.

$$\widehat{\mu}_{N,\xi} := \frac{1}{\mathcal{F}(N, \xi)} \sum_{\pi \in \mathcal{F}(N,\xi)} \delta_{\pi_p} \tag{11.3}$$

(counting with multiplicity $m(\pi)\mathrm{dim}(\pi^\infty)^{P(N)}$.

Introduce the "metric" $|| \cdot || : X^*(T), X_*(T) \to R_{\geq 0}$.

In the case of $\mathrm{GL}(n)$: want $(m_1, \ldots, m_n) \mapsto \sup |m_i|$.

Write $|| \cdot || : \tau_\mu := 1_{G(\mathbb{Z}_p)\mu(p)G(\mathbb{Z}_p)} \mapsto ||\mu||$, with $\tau_\mu = \tau_{\mu,p}$ and $1_{\ldots} \in \mathcal{H}^{\mathrm{ur}}(G(\mathbb{Q}_p))$, with $||\xi|| := ||\text{highest weight of } \xi||$.

Theorem 11.1 (S– and Templier)**.** *Let $k \geq 1$, assume $||\mu|| \leq k$ and $\phi_p = \tau_\mu$. (Level)*

$$\widehat{\mu}_{N,\xi}(\widehat{\phi}_p) = \widehat{\mu}_p^{\mathrm{pl,ur}}(\widehat{\phi}_p) + O(p^{A+Bk}N^{-C}). \tag{11.4}$$

In the weight aspect have same main term but now the error is $O(p^{A+Bk}||\xi||^{-C})$.

Remark 11.2. *If p is fixed then $\lim \widehat{\mu}_{N,\xi} = \widehat{\mu}^{\mathrm{pl,ur}}$.*

Corollary 11.3. *If a plot of N or ξ versus p grows faster than polynomial then $\lim \widehat{\mu}_{N,\xi} = \widehat{\mu}^{\mathrm{ST}}$.*

Remark 11.4 (Application to low-lying zero statistics)**.** *Compute or estimate first or second moments of the Satake parameters of π at p. Look at $\widehat{\phi}(\pi_p)$ for suitable ϕ_p, get good estimates.*

11.5. **Idea of the proof.** $\widehat{\mu}_{N,\xi}(\widehat{\phi}_p) = J_{\mathrm{spec}}(\phi_p 1_{P^p(N)} \phi_\xi)$ at ∞ ("EP function"). Apply the trace formula, and get it equals $I_{\mathrm{geom}}(\cdots)$, which equals

$$\sum_{\substack{\gamma \in \overline{G(\mathbb{R})}/\sim \\ \text{elliptic}}} \mathrm{vol}\left(G(\mathbb{R}) \setminus G_\gamma(\mathbb{A})\right) O_\gamma(\phi_p) O_\gamma(1_{P^p(N)}) \mathrm{tr}\xi(\gamma) + \mathrm{Levi}. \tag{11.5}$$

Bound everything in the summand, bound the number of conjugacy classes where things are non-zero.

When $\gamma = 1$: main term: get

$$\widehat{\mu}^{\mathrm{pl,ur}}(\widehat{\phi}_p) \tag{11.6}$$

(which equals $\phi_p(1) = O_1(\phi_p)$).

All the effort is to control the error terms, arising when $\gamma \neq 1$. The difficulty comes from the following fact: p is moving around, ϕ_p can have arbitrarily large support. If increase level to infinity support gets smaller, eventually meets rational points and error goes away. Have to deal with / live with the fact that we have a varying set of conjugacy classes. Bound the number of conjugacy classes. Control the number of γ where the summand is non-zero. Looks quite harmless, but need some control on the volume. Break global problem into local problems. Have $D^G(\gamma)^{1/2} O(\phi_p)$ (work of Cluckers et al comes in here as things are varying and need uniform bounds, get something like $O(p^{A+Bk})$) and $D^G(\gamma)^{1/2} O(1_{P^p(N)})$ and $\mathrm{tr}\xi(\gamma)/\mathrm{dim}\xi$.

12. Symmetry breaking and the Gross Prasad Conjecture for orthogonal groups (Speh)

Slides online.

13. TRACES, CHARACTER SHEAVES, GEOMETRIC LANGLANDS (NADLER)

Plan:

(1) Dictionary between arithmetic/ geometric language. Primer in D-modules.
(2) What is a trace? Character sheaves.
(3) Discussion of traces in Geometric Langlands. Perspective if field theory.

13.1. **Dictionary.** We assume $k = \mathbb{C}$.

Arithmetic	Geometric
F number field	C smooth projective curve over k
K local field	$\mathbb{C}((t))$ power law series
k residue field	$\mathbb{C}$ residue field / base field
$\mathcal{O}$ ring of integers	$\mathbb{C}[[t]]$ power series
G_F Galois	$\pi_1(C)$ fund
$G(F) \setminus G(\mathbb{A})/G(\mathcal{O})$	$\mathrm{Bun}_G(C)$ moduli of G-bundles
$G(\mathcal{O}) \setminus G(K)/G(\mathcal{O})$	Gr_G Grassmannian
$I\mathbb{G}(K)/I$	Fl_G affine flag variety
Function	$\mathcal{O}$-module coherent sheaf

The notion of a smooth function that is locally constant is a D-module.
First theorem on the subject, showing where the dictionary leads, is the following.

Theorem 13.1. *We have $C_c(G(\mathcal{O}) \setminus G(K)/G(\mathcal{O})) = K(\mathrm{Rep}G)$ (commutative rings) on the arithmetic side; on the geometric side it is $D - \mathrm{mod}(G(\mathcal{O}) \setminus G(K)/G(\mathcal{O})) \simeq \mathrm{Rep}(G^v)$.*

Remark 13.2. *All constants implicitly defined. Higher homotopical data. Example: $H_G(\mathrm{pt}) = \mathrm{Sym}(\mathfrak{G}^*[-Z])^G$. Arthur parameters in geometric setting.*

The way you capture families in geometry is to talk about deformations.

13.2. **Primer on D-modules.**

Definition 13.3. *A D-module on X equals $\mathcal{O}$-module on de Rham space X_{dR}:*

$$\widehat{\Delta}_X \text{ ' } X \;\to\; X_{\mathrm{dR}}. \tag{13.1}$$

D stands for differential operator, can explain in terms of descent. A D-module on X equals an $\mathcal{O}$-module plus a complex structure of D_X-module, where D_X is dists on $\widehat{\Delta}_X$.

Universal form of special functions. Imagine $P(f) = 0$ and P is a linear Partial Differential Equation. For example, $P = \partial_x - 1$ with solution $f(x) = e^x$. The D-module $M = D_X/D_X(P)$. The solution of M are

$$\mathrm{Sol}(M) \;=\; \mathrm{Hom}_{D_X}(M, \mathrm{fn\ space}). \tag{13.2}$$

It recovers f, captures all the key parts of the differential equations.

Infinite geom: Smooth functions have values and derivatives think of D_X-modules as objects on T^*X.

Singular support: D_X has a natural filtration with $\mathrm{gr}D_X = \mathcal{O}_{T^*X}$.

M D_X-module filt $\mathrm{ss}(M) \subset T^*X$.

For example, take $X = \mathbb{A}$, $D_X = \mathbb{C}\langle x, \partial x\rangle$ with $\partial_x x - x\partial x = 1$.

(1) Favorite example is $\mathcal{O}_X = D_X/(\partial_x)$. In $x - \partial_x$ plane, this is just the x-axis.
(2) Take $\partial_0 = D_X/(x)$: y-axis now.
(3) $\exp_\lambda = D_X/(\partial_x - \lambda)$: line at height λ.
(4) $\partial_a = D_X/(x-a)$: vertical line $x = a$.

First three above have singular support equal to the zero section.

Example: $X = \mathbb{G}_m$, $D_X = \mathbb{C}\langle x, x^{-1}, x\partial_x\rangle$. For every $\lambda \in t^*$, $M_\lambda = D)X(x\partial x - \lambda)$. What is the character sheaf? This is a piece of the Mellin transform. D-mod$(T) = \mathcal{O}$-mod(t^*/Λ_T^v). ss is zero section $\leftrightarrow$ finite support. Character sheaf of T is D-module with singular support equal to zero section.

13.3. **What is a trace?** Take $n \times n$ matrices, have a map to the ground field by taking the trace: $\mathrm{tr}: M_{n\times n} \to k$, and satisfies $\mathrm{tr}(fg) = \mathrm{tr}(gf)$. It is the "universal coabelianization". Given a finite dim module / vector space, any endomorphism has a trace.

Question: what is k? If just give an algebra and want to develop a theory and ant to recover k, how do we do this?

Example: G formal group, $A = \mathbb{C}[G]$, M a finite dimensional A-module, $\mathrm{tr}: \mathrm{End}(M) \to \mathbb{C}(\frac{G}{G})$ classifies. Why $\mathbb{C}(\frac{G}{G})$?

Abstract: A algebra $hh(A) = A \otimes A = \int_{S^1} A$.

Let's do an example of this formalism. $A = \mathrm{Rep}(G^v)$ $hh(A) - \mathrm{Coh}(\frac{G}{G})$. So geometric Satake parameters live in $hh(A)$.

Example: $\mathcal{H} = D - \mathrm{mod}(B \setminus G/B)$ $jj(\mathcal{H})$ miracle that equals character sheaves on G. Is $D - \mathrm{mod}(\frac{G}{G})_{\text{nilp ss}}$.

Do get interesting things and get them through interesting constructions.

13.4. **Traces in Geometric Langlands.** Now consider traces for spherical and affine Hecke operators. Try to ask first the same question: what kind of objects are they? One answer that you can imagine is that they should be class functions. Morally D-mod$(\frac{G(K)}{G(K)})$. Escape: Field Theory: predicts characters should be D-modules on $\mathrm{Bun}_G(E_q)$.

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14. Rigid inner forms and endoscopy (Kaletha)

Lecture notes online.

15. Cohomology growth on $U(3)$ (Marshall)

Take $\Gamma \subset \mathrm{SO}(d,1)$, take $\Gamma(n) \subset \Gamma$, look at $h^i(\Gamma(n) \setminus \mathbb{H}^d/\mathbb{C})$ as $n \to \infty$. Have $L^2(\Gamma(n) \setminus \mathrm{SO}(d,1)) = \oplus_{\pi \in \widehat{G}} m(\pi, \Gamma(n))\pi$.

Consider π_1 with $0 < i < d/2$, $i = d/2$, $\pi^+_{d/2}$.

$h^i(\Gamma(n) \setminus \mathbb{H}^\phi, \mathbb{C}) = m(\pi_i, \Gamma(n)) m(\pi^+, \Gamma(n)) + m(\pi^-, \Gamma(n))$.

$V(n) = |\Gamma : \Gamma(n)|$.

$\lim \frac{m(\pi, \Gamma(n))}{V(n)} = 0$ if $i \neq d/2$.

$p(\pi) = \inf\{p \geq 2 : \langle \pi(g)v, u\rangle \in L^p(G)\}$

π discrete series implies $p(\pi) = 2$.

Conjecture 2 (Sarnak-Xue). $m(\pi, \Gamma(n)) \ll_\epsilon V(n)^{2/p(\pi)+\epsilon}$.

Sarnak and Xue reduced this to counting $\Gamma(n) \cap B_G(T)$. If $\uparrow \mathrm{rk}_{\mathbb{R}} G = 1$, Γ co-compact, counting implies result. Proved for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$. Partial bound for $\mathrm{SU}(2,1)$.

Will switch to unitary groups to talk about endoscopy. A bit easier to talk about in this case, this is the case I've worked out in details.

E/F CM, $G^* = U(n)$ for $\mathrm{anti}-\mathrm{diag}(1,\ldots,1)(\times\xi)$.

$L^2_{\mathrm{disc}}(G^*(F) \setminus G^*(\mathbb{A}))$.

$\psi = \oplus \mathrm{Sym}^{d_i} \otimes \mu_i$.

μ_i CSD on GL_{n_i}/E.

$\sum (d_i + 1) m_i = n$.

$L^2_d(G^*(F) \setminus G^*(\mathbb{A})) = \sum_\psi \sum_{\pi \in \Pi_\psi} m(\pi)\pi$.

$m(\pi) \leq 1$.

G inner form, $G_V \simeq U(p,q)$.

Parameter ψ, global packet Π_ψ, $\pi_v \in \Pi_\psi$.

$H^i(g, K; \pi_v) \neq 0$.

$\psi_{\mathbb{C}^\times}, \mathbb{C}^\times = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \subset \mathrm{SL}(2,\mathbb{C})$, $\psi : \mathbb{C}^\times \to \mathrm{GL}(n,\mathbb{C})$.

$\Lambda^p \otimes \psi|_{\mathbb{C}^\times} : \mathbb{C}^\times \to \mathrm{GL}(\binom{n}{p}, \mathbb{C})$ by $z \mapsto \mathrm{diag}(z^{k_1}, \ldots, z^{k_{\binom{n}{p}}})$; ψ should contribute only in degrees $pq + k_i$.

Take $G - U(3)$, $E/F = \mathbb{Q}$.

Contributing to H^2: μ, $\mu \oplus \chi$, $\chi_1 \oplus \chi_2 \oplus \chi_3$.

Contributing to H^1, H^2, H^3: $\mathrm{sym}^1 \oplus \chi_1 \oplus \chi_2$.

Contributing to H^0, H^2, H^4: $\mathrm{sym}^2 \oplus \chi$.

$H = U(2) \times U(1)$.

$\pi_{\psi_v} = \{\pi^n(\xi), \pi^S(\xi)\}$, $f \in C_0^\infty(G_v)$.

$\mathrm{tr}(\pi^n(\xi)(f)) + \mathrm{tr}(\pi^s(\xi)(f)) = \xi(f^H)$.

$\mathrm{tr}(\pi^n(\xi)(f)) - \mathrm{tr}(\pi^s(\xi)(f)) = \mathrm{tr}(\widetilde{\pi}_{\psi_q} \circ \epsilon(\widetilde{f}))$, $\widetilde{f}$ on GL_3/E_2.

$m(\pi) = 1$ if and only if $(-1)^{n_s} = \epsilon$.

$s \in S_\psi$ $\widehat{H} = Z_{\widehat{G}}(s)$.

For example: $U(3)$: $S_\psi = I$, $\mathrm{diag}(-1,-1,1) \in \mathrm{GL}_3(\mathbb{C})$. Pretty reasonable endoscopic group is $U(2) \times U(1)$ (in the $\mathrm{diag}(-1,-1,1)$), while in I get the relation with GL_3/E. There is a special element that shows up here. $s_\psi = \psi(1,-1)$, $s_\psi \simeq (\mathbb{Z}/2)^2$.

$s_\psi = \mathrm{diag}(-1,-1,1)$.

$\langle , \rangle : s_\psi \times \Pi_{\psi_v}$.

Each $s \in S_\psi$, $\langle \pi, \cdot \rangle \in \widehat{S}_\psi$.

$\sum_{\pi \in \Pi_{\psi_i}} \langle \pi, s s_\psi \rangle \mathrm{tr}(\pi(f)) = f^H(\psi^H)$.

$U(3)$, $E/F = \mathbb{Q}$, $\Gamma(n) \subset \Gamma$.

$h^{(1,0)}_{(2)}(\Gamma(n) \setminus \mathbb{H}^3_{\mathbb{C}}, \mathbb{C}) = m(\pi_{1,0}, \Gamma(n))$, which Sarnak-Xue predict is at most $V(n)^{1/2}$ (other methods can get / predict exponents of 7/12 and 3/4).

Theorem 15.1 (Rogawski). *If $\pi \in L^2_{\mathrm{disc}}$ has $\pi_\infty \simeq \pi_{1,0}$, there exists $\psi = \mathrm{sym}^1 \otimes \chi_1 \oplus \chi_2$ such that $\pi \in \Pi_\psi$. There exists ξ on $U(2) \times U(1)$ such that $\pi \in \Pi_\xi$, ξ_∞ fixed.*

$$\begin{aligned} h^{(1,0)}_{(2)} &\leq \sum_{\xi : \xi_\infty \text{ fixed}} \sum_{\pi \in \Pi_3} \dim \pi_f^{K(n)} \\ &= (\mathrm{vol}\,(K(n)))^{-1} \sum_{\xi} \sum_{\pi \in \Pi_\xi} \mathrm{tr}(\pi_f(1_{K(n)})) \\ &= (\mathrm{vol}\,(K(n)))^{-1} \sum_{\xi} \xi(1^H_{K(n)}). \end{aligned} \tag{15.1}$$

Can choose $1^H_{K(n)} = n^{-2} 1_{K_H(n)}$.

$$h^{(1,0)}_{(2)} \ll n^9 n^{-2} n^2 n^{-5} = n^4. \tag{15.2}$$

How might this look for $U(4)$? Take $G^* = U(4)$, $G_\infty = U(3,1)$, H^2.

Expect $H^2(\Gamma(n) \setminus \mathbb{H}^3_{\mathbb{C}}, \mathbb{C}) \ll V(n)^{8/15}$ versus $V(n)^{2/3}$. Main term $\psi = \chi \otimes \mathrm{sym}^1 \oplus \mu$.

Most error: $\mathrm{sym}^1 \otimes \mu$.

$s_\psi = \mathrm{diag}(1,-1,1,1) \in \mathrm{GL}_4(\mathbb{C})$.

$\sum \mathrm{tr}(\pi(f))$ $U(2) \times U(2)$ ($\chi \circ \det \times \mu$) equals $\mathrm{tr}(\chi \circ \det \oplus \mu(f^H))$..

16. On local constancy of characters (Kim)

Joint work with G. Lusztig.

16.1. **Applications.** Characters are locally constant.

- One application is computing character as a locally constant function.
- Another application is finding the support of distributions.
- Find estimate of characters.

Won't talk about the first aspect.

k a nonarchimedian field with residue field $\mathbb{F}_q$.
$\mathbb{G}$ a connected semi-simple group over k.
$G = \mathbb{G}(k)$.
Irr $(\pi\ V_\pi) \in R(G)$: category of smooth representations.
$\Theta_\pi : C_c^\infty(G) \to \mathbb{C}$ with $\Theta_\pi(f) = \int_G f(y)\Theta_\pi(y)dy$.

Example: Let π be supercuspidal. Harish-Chandra integral formula: $\gamma \in G^{\text{r.s.s.}}$:

$$\Theta_\pi(\gamma) \;=\; \frac{\deg\pi}{\Theta(1)} \int_{G/Z_G} \int_K \theta\left(\begin{pmatrix} gk & \\ & \gamma \end{pmatrix}\right) dkdg, \tag{16.1}$$

with θ the matrix coefficient of π, $\pi = c - \operatorname{Ind}_J^G \rho$, $\theta = \theta_\rho$ (Sally's ??).

For example, $\pi = \operatorname{Ind}_B^G \chi$,

$$\Theta_\pi(\gamma) \;=\; \sum_{w \in W} \chi(\gamma^w) D_{G/T}(\gamma^w)^{-1/2}. \tag{16.2}$$

Example: Steinberg subquotient representations.

The aim: computing characters of noncuspidal unipotent representations at very regular elements. For example, Iwahari spherical representations.

$T_\infty|_s$: old (1): Casselman: $(\pi\ V_\pi)$, $r \in GG^{\text{r.s.s.}}$, $P_\gamma = M_\gamma N_\gamma$. $N_\gamma = \{u \in G : \gamma^n u \gamma^{-n} \to 1\}$. $\overline{N}_\gamma = \{\overline{u} \in G : \gamma^{-n} u \gamma^n \to 1\}$. Implies $\gamma \in M_\gamma$ compact mod center in M_K. $\Theta_\pi(\gamma) = \Theta_{\pi_{N_\gamma}}(\gamma)$, $\pi_{N_\gamma} \in R(M_\gamma)$.

new (2): Local constancy of characters at r.s.s.

Adler-Korman, Meyer-Solleveld: determine the size of neighborhood of γ where Θ_γ is constant. This neighborhood depends on the depth of the representation π and γ.

(3) Hecke representations via categorical equivalence.

$\mathcal{B}(G) = \{(M,\sigma)\}/\sim G\text{conj} \;\to\; R^{\mathfrak{s}}(G) \;\simeq\; \operatorname{Mod} - \mathcal{H}(G//J,\rho)$, with $\mathfrak{s} = [M,\sigma]$ and (J,ρ) with ρ irred f on rep.

Example: $[T,1]$, Iwahari spherical $R^{\mathfrak{s}}(G)$, $\operatorname{Mod} - \mathcal{H}(G//I,1)$ with Iwahari subgroup I, $\mathfrak{s} = [M,\sigma]$ with σ unipotent cuspidal, $R^{\mathfrak{s}}(G)$ unipotent.

Definition 16.1. *(1) A split r.s.s. γ is very regular if $D_{M_\gamma}(\gamma) = 1$. (2) A $\gamma \in G^{\text{r.s.s.}}$ is very regular if it splits over an unramified extension k'/k and γ is a very regular split element.*

Remark 16.2. *If $M_\gamma = G_\gamma$, γ is a lift of r.s.s. element of finite reductive group.*

Exmaple: Steinberg representation $T \subset B \subset G$,

$$\mathrm{St} \;=\; \sum_{B \subset P} (-1)\mathrm{INd}_P^G \delta_P^{-1/2}. \tag{16.3}$$

The left hand side is polynomial in q. van Dijk and Casselman. Implies

$$\Theta_{\mathrm{St}}(\gamma) \;=\; \delta_{P_\gamma}(\gamma)(-1)^{\dim T - \dim A_\gamma}. \tag{16.4}$$

$\pi < \mathrm{Ind}_{MN}^G \sigma$, depth zero, σ: depth zero super-cuspidal.
$[M\ \sigma] \sim (P_0\ \sigma_0)$.
γ compact very regular (i.e., $M_\gamma \in G$).
Let Q be a smallest parabolic subgroup such that $\gamma \in Q \supset Q^+$.

Theorem 16.3. *We have $\Theta_\pi(\gamma) = \mathrm{Tr}(\pi(\gamma)|V_\pi^{Q^+})$.*

AK. and [MS]: Have Θ_π is constant $(\gamma T_\gamma^+)^G \supset Q^+$.
$\Theta_\pi(\gamma) = \Theta_\pi(\delta * 1_{Q^+}/\mathrm{vol}\,(Q^+))$ (last fraction is proj operator onto V^{Q^+}. □

Corollary 16.4. *If ${}^gP_0 \nsubseteq Q$ for all $g \in G$ then $\Theta_\pi(\gamma) = 0$.*

Proof. $V^{Q^+} = 0$. Let $(\pi\ V_\pi)$ be unipotent. γ very regular. Q is as above.
Then $\Theta_\pi(\gamma) = [V_\pi^{Q^+} : R_{\mathrm{ctr}}^{\overline{Q}}]_{\overline{Q}}$ with $\overline{Q} = Q/Q^+$. □

Here $Q_\pi(\gamma) \in \mathbb{Z}$ and constant on $(T_\gamma)^{\mathrm{cvr}}$. C_{T_γ} conjugacy class in W: Weyl group of $\overline{Q}$.
$C_{T_\gamma}\ \ R_{\mathrm{ctr}}$.
There exists a 1-1 correspondence: $\{T$: max torus split over $k^m\}$ and $\{w \in W' : \mathrm{ord}(w) < \infty\}/\sim$ by $T\ \ C_T$.
By corollary $P_0 \subset Q$.
Have $R^{[M\ \sigma]}(G) \supset R^{\sigma_?}(\overline{Q})$. Send $V_\pi \to V^{Q^+}$.
Left hand side $\simeq \mathrm{Mod} - \mathcal{H}(G//P_0\ \sigma_0)$, which contains $\mathrm{Mod} - \mathcal{H}(\overline{G}//\overline{P_0 \cap Q}\ \sigma_0)$ (which is $\simeq$ to the right hand side). Continue along these lines.

Remark 16.5. *γ noncompact very regular. $\Theta_\pi(\gamma) = \Theta_{\pi_{N_\gamma}}(\gamma)$.*
π_{N_γ}? (Classical groups Tadic). Use Hecke algebra and theory of type to find π_{N_γ}.

17. SUBCONVEXITY BOUNDS FOR RANKIN-SELBERG L-FUNCTIONS (HOLOWINSKY)

Basic object of study: π an irreducible automorphic representation, think of it as having some data (parameters) associated to it that make it belong to a natural family (spectrally complete, trace formula). Consider $L(s, \pi)$. We have the analytic conductor $Q(s, \pi)$, measures the complexity of the L-function. The reason we're introducing this is that if we establish non-trivial estimates, then we end up proving something non-trivial about the associated arithmetic object.

Often want non-trivial estimates at $\mathrm{Re}(s) = 1/2$, where we will restrict for the rest of the talk. Have the convexity bound, follows trivially from complex analysis, functional equation, Phragmen-Lindelof. Gives $L(s, \pi) \ll_\epsilon Q(s, \pi)^{1/4+\epsilon}$. The sub-convexity problem is to beat this by some power of the conductor: $L(s, \pi) \ll_\epsilon Q(s, \pi)^{1/4-\delta}$ (for some positive δ). The Riemann hypothesis implies $L(s, \pi) \ll_\epsilon Q(s, \pi)^\epsilon$. In many applications (such as equidistribution) all we need is some $\delta > 0$; the larger the δ the better results we have on the *rate* of convergence.

17.1. **Standard tools and methods. Spectral Formula:**

- Orthogonality of Dirichlet characters.
- Petersson / Kuznetsov trace formula.

Reduces complexity / conductor.

Dual summation formula:

- Poisson
- Voronoi

Change arithmetic structure, change in length of summation.

Tricks, locally:

- Reciprocity, etc.

17.2. **The Moment Method.** Choose $\mathcal{F}$, $Q(s, \pi) = Q(s, \mathcal{F})$.

Choose a moment r, study

$$\frac{1}{|\mathcal{F}|} \sum_{\pi \in \mathcal{F}} |L(1/2, \pi)|^r \ \ll \ Q(1/2, \mathcal{F})^{r/4}/|\mathcal{F}|; \tag{17.1}$$

goal is to have a better bound that this. In the past these methods give just the bound above; we need to do additional work (amplification) to go further.

Will study $\pi = \pi_1 \times \pi_2$, both representations varying, will use this to our advantage. The first will be $\mathrm{GL}(d)$ and the second $\mathrm{GL}(e)$. We have associated L-functions $L(s, \pi_i)$ with conductors $Q(s, \pi_i)$. The Rankin-Selberg convolution $L(s, \pi_1 \times \pi_2)$ has conductor $Q(s, \pi_1 \times \pi_2) = Q(s, \pi_1)^e Q(s, \pi_2)^d$.

Advantages:

- Have a large conductor, but know something about π (factorization).
- Extra degree in analytic freedom. Choices in family to average over.
- Know that certain co-primality conditions exist. Think $Q(s, \pi_1) = p$ and $Q(s, \pi_2) = q$ where these are two different primes.

Motivation: In QUE want to study $L(1/2, \mathrm{sym}^2 f \times g)$ where f is varying and g is fixed. Establishing sub-convexity here is difficult and hasn't been done. Best is work by Sound saving a logarithm.

Cases: π_1 varying, π_2 varying. Holomorphic weight k-fixed level p, trivial nebentypus newform, $\pi_2 \in B_K^*(p)$. For π_1, consider $\mathrm{GL}(1)$, $\chi \bmod q$, $\mathrm{GL}(2)$, level q, $\mathrm{GL}(3)$, $\mathrm{sym}^2 f$ with f of weight k.

Use A for coefficients of π_1 and λ (averaging) for π_2. Let $Q = Q(1/2, \pi_1 \times \pi_2)$.

Method: A.F.E.

$$L(1/2, \pi_1 \times \pi_2) = \sum_{n \leq X} \frac{A(n)\lambda(n)}{\sqrt{n}} + \epsilon(1/2, \pi_1 \times \pi_2) \sum_{n \leq Q/X} \frac{\overline{A(n)}\lambda(n)}{\sqrt{n}}. \tag{17.2}$$

Average over π_2 using Petersson's trace formula. Get

$$\sum_{g \in B_k^*(p)} w_g^{-1} \lambda_g(n) \overline{\lambda_g(m)} = \delta(n,m) + 2\pi i^{-k} \sum_{\substack{c \equiv 0 \ (p) \\ c > 0}} \frac{S(n,m;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right), \tag{17.3}$$

with $w_g \cong KP$.

Dual summation in π_2: Poisson, Voronoi, $(a,c) = 1$.

(a)

$$\sum_{n \leq X} \frac{A(n)}{\sqrt{n}} e(na/c) \to \sum_{n \leq c^2 q_2/X} \frac{A(n)}{\sqrt{n}} e(-n\overline{aq_2}/c). \tag{17.4}$$

(b)

$$\sum_{n \leq X} \frac{A(n)}{\sqrt{n}} e(na/c) \to \sum_{n \leq c^3 k^2/X} \frac{A(n)}{\sqrt{n}} \frac{S(\overline{a}, n; c)}{\sqrt{c}}. \tag{17.5}$$

17.3. **Results.**

- (with Templier):

$$\sum_{g \in B_k^*(p)} w_g^{-1} L(1/2, \chi \times g) \ll \left(1 + \frac{q^{1/2}}{p}\right)(pq)^\epsilon. \tag{17.6}$$

  Have $Q = Q(1/2, g \times \chi) = q^2 p$. Convexity: $Q^{1/4}/p = q^{1/2}/p^{3/4}$. So $1 < q^{1/2}/p^{3/4}$, need $p < q^{2/3}$, $q^{1/2}/p < q^{1/2}/p^{3/4}$, $p > 1/$.

Subconvexity when $q^\eta < p < q^{2/3-\eta}$ for all $\eta > 0$.

Lindelöf on average: $q^{1/2}/p < 1$, $q^{1/2} < p$.

If we amplify in this range, we get subconvexity for $q^\eta < p < q^{4/5-\eta}$ for all $\eta > 0$.

Another result with Templier: f dihedral of level q:

$$\sum_{g \in B_k^*(p)} w_g^{-1} L(1/2, f \times g) \ll \left(1 + \frac{\sqrt{q}}{p}\right)(qp)^\epsilon. \tag{17.7}$$

$Q = q^2 p^2$, convexity $q^{1/2} p^{1/2}/p = q^{1/2}/p^{1/2}$.

Open Discussion Session Notes

18. Open Discussion Session: I (Calegari)

Let F be a number field, $\mathcal{O} = \mathcal{O}_F$, $\mathfrak{p} \subset \mathcal{O}_F$ prime of res. char p.

Problem: Is $\pi_1(\mathrm{Spec}(\mathcal{O}_F \setminus \mathfrak{p})$ infinite? Let $F_\mathfrak{p}/F$ be the max. extension unm out $\mathfrak{p}$. Is $F_\mathfrak{p}/F$ infinite? Analogous questions: Is $\pi_1(\mathrm{Spec}(\mathcal{O}_F))$ finite or infinite? Sometimes each. For example, Minkowski showed there are no unramified extensions of $\mathbb{Q}$; take $F = \mathbb{Q}$ and $\mathbb{Q}(\sqrt{-1})$: finite. Sometimes it is infinite. The first example is Golod-Shava.

Easier questions: Does there exist a (smooth, proper) curve C/F with good reduction outside $\mathfrak{p}$ with $g \geq 7$ (or A/F of dimension at least 1). Just asking for existence, not infinitely many.

Consider $\mathbb{Q}$, $X_0(p)$, F totally real, D/F ramified at all but all ∞-place.

Does there exist a motive M/F with the property that it has good reduction outside $\mathfrak{p}$ and (2) either it has bad reduction at $\mathfrak{p}$ over any finite extension of F? Want to exclude trivial examples, like $\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$.

Let $F = \mathbb{Q}(\sqrt{-1})$, let $p = \pi\overline{\pi}$ be a prime congruent to 1 modulo 4 that splits. Goal is to find $A/\mathcal{O}_F[1/\pi]$.

Step 1: does there exist an elliptic curve $E/\mathcal{O}_F[1/\pi]$. Try to write down an equation with discriminant, maybe good reduction outside 2, ... Try $\Delta = 16\pi$. If it has rational points you win, else have to do something else. We're just looking for an abelian variety, so we can do the following.

First compute the maximal abelian extension of F unramified outside π: $\mu_4 \setminus \mathcal{O}^*_{F,\pi} = \mu_4 \setminus \mathbb{Z}_p^\times = \mathbb{Z}_p \times \mathbb{F}_p^\times/\mu_4$. Can take $F - -F_1 - -F_2 - -F_2 \cdots$. Try to create an elliptic curve over F_n. Can ask whether or not there exists an elliptic curve $E/O_{F_n}[1/\pi]$. If yes take $A = \mathrm{Res}^F_{F_n}(E)$. Have a tower. Not only do we have our original curve (does it have a rational point over the tower), but we have many more elliptic curves we can write down. Bigger and bigger fields, maybe one eventually has a rational point.

Weight 2 modular forms. Level N, $(N, \ell) = 1$. $\Gamma_0(N)$. Distribution of a_ℓ in $\overline{\mathbb{F}}_p$. Naive: given $\alpha \in \overline{\mathbb{F}}_p$, does there exist an f with $a_\ell(f) = \alpha$? Does there exist a Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ that is irreducible, continuous, odd, has determinant equal to ϵ, unramified at ℓ, $x^2 - \alpha x + \ell = (x - \beta_1)(x - \beta_2)$. One would believe that if you choose a Galois group and a fixed number of primes then there exists field with what want at here. Supped up version of Inverse Galois. Same would be true here. Want elements that contain this characteristic polynomial.... Is there a trace formula-ish method to extract this? If think about weight 1 modular forms then trace of Frobenius and try and count things that are close.

19. Open Discussion Session: II (Urban)

$K = K^P I$, $G/\mathbb{Q}$, $G/\mathbb{Q}_p$ split. λ weight in $\mathcal{X} = \mathrm{Hom}(T(\mathbb{Z}_p), \overline{\mathbb{Q}}_p)$, θ equals a system of Hecke eigenvalues. We have $x = (\theta, \lambda) \in \mathcal{E}$ if and only if θ shows up in $H^0(S_G(K), \mathcal{D}_\lambda)$.

We have $(\theta, \lambda) \in \mathcal{E}_K \supset \mathcal{E}_K^{\mathrm{EP}}$, where the right is a union of the irreducible components of dimension $\dim H$ and the left maps to $\mathcal{X}$.

Question: What are the dimensions of the irreducible components inside $\mathcal{E} \setminus \mathcal{E}_K^{\mathrm{EP}}$? $x \in \mathcal{E}_K$, x belongs to exactly one irreducible component C (smooth). $x = (\theta_x, \lambda_x)$.

Conjecture: There exists integers a, b such that $H^i(S_G(K), \mathcal{D}_{\lambda_x})_{\theta_x} \neq 0$ if and only if $i \in [a, b]$ and the projection of C onto $\mathcal{X}$ has codimension $c = b - a$ (which we denote ℓ_0).

Example: F totally real field, $G = \mathrm{SL}(2)/F$, $d = [F : \mathbb{Q}]$, λ parallel weight. $\mathcal{X}$ is dimension d. Construct Eisenstein classes $\mathcal{F}$ Hecke characters of $F^\times$ of weight λ in $H^i(\Gamma, W_\lambda)$ for $i = d, d+1, \ldots, 2d-1$.

Conjecture: The projection of C_{Eis} onto $\mathcal{X}$ is codimension $d - 1$ implies the dimension is 1. Implies correspond to a p-adic family of Hecke characters of F of dimension $1 + \delta$ with δ the Leopold defect. This means

$$0 \;\to\; \text{kernel} \;\to\; \overline{O}_P^x \;\to\; \mathcal{O}_{F,p}^\times \;\to\; \mathrm{Cl}_{F,p^\infty} \;\to\; 0. \tag{19.1}$$

The rank of the kernel is δ.

Remark: It is not sufficient to show that the codimension of the projection is at most $b - 1$ implies the dimension of the component is at least $\dim \mathcal{H} - (b - a)$. Special case of the conjecture is where $b - a = 1$ $\Longrightarrow$ the conjecture holds for trivial reasons.

$H^i(T, \mathcal{D}_u)$ $\quad A(u)$
$x_1, \ldots, x_\nu$ $\quad H^{i_0}(T, \mathcal{D}_u) \neq 0$. Implies $H^i(\mathcal{D}_\lambda)_\theta \neq 0$ for $i = i_0, i_0 - 1, \ldots, i_o - r$.

(2) Let D be a quaternion algebra over an imaginary quadratic field K. Have H^1, H^2. Eigenvarieties have to be dimension 3. Don't have discrete series.

(3) Now take $G = \mathrm{PGSp}_4/\mathbb{Q}$. π rep of $\mathrm{PGL}(2)/\mathbb{Q}$ eigenform of weight $2k - 2$ for $k \geq 2$. Assume that $L(\pi, 1/2) = 0$. In this case we have two situations. $\mathrm{SK}(\pi)$ Saito-Kurakowe type.

$$L(\mathrm{SK}(\pi), s) \;=\; L(\pi, s)\zeta(s + 1/2)\zeta(s - 1/2). \tag{19.2}$$

Have two cases. If $\epsilon = 1$ then $\mathrm{SK}(\pi)_\infty$ nontempered and H^2, H^4, and if $\epsilon = -1$ then discrete series and H^3. Classes in the cuspidal cohomology (k, k).

$L(\pi, 1/2) = 0$ implies there exists Eisenstein classes in degree 2, 3 (Harole).

If $\epsilon = -1$ then we have cohomology in degree 3 only $(H^i(\mathcal{D}_\lambda))$, Here C maximal dimension.

If $\epsilon = 1$ we have cohomology in degree ?, 3, 4. Conjecture: dimension should be 0, the point is isolated.

20. Open Discussion Session: III (Cluckers, Gordon)

Families: Sets in $G(\mathbb{Q}_p)$. Function on $G(\mathbb{Q}_p)$. Indexed by integers, points on projections of varieties over the residue field, parametrized from $X(\mathbb{Q}_p)$ variety.

Example: μ (in $\mathbb{Z}^n$) goes to τ_μ (function on $G(\mathbb{Q}_p)$) goes to orbital integral $O_\gamma(\tau_\mu)$ (family indexed by μ, γ).

Nilpotent orbits in $G(\mathbb{Q}_p)$. (1) Combinatorial (partitions). (2) Res. field parameters (in SL_n with $x^n = 1$). Family of automorphic forms??? Not sure how to answer. If however want to talk about a family of elliptic curves then maybe not so bad as defined by algebraic equations and back in business.

Question: E_λ with $\lambda \in \mathcal{O}_F$. If F is local fine, but if F is global...? Here $\lambda = j(E)$. What we can do, which may or may not be useful, is we can write down some formulas (need to figure out if j-invariant is definable) that ensure if global field $F = \mathbb{Q}$ and (E_p) such that for each p the j-invariant of E_p is [–] modulo p. Might get a lot more objects than want, might end up getting it for the one you want.

Igusa (homogeneous): $f(x_1, \ldots, x_n)$ over $\mathbb{Z}$, $N > 1$, study

$$S(N) \;=\; \sum_{\substack{x_i=1\\ i=1,\ldots,n}}^{N-1} \exp(2\pi i f(x)/N)\frac{1}{N^m}. \tag{20.1}$$

Have

$$|S(p^m)|_{\mathbb{C}} \;\leq\; c_\epsilon p^{-m_0\alpha_p(f)+\epsilon}, \tag{20.2}$$

where $\alpha(f) = \inf_p \alpha_p(f)$ in $\mathbb{Q}$. Don't want $C_{\epsilon,p} \leq c + \epsilon p^a$; this is trivial. Conjecture can bound independently of p if homogeneous. If $\alpha(f) \leq 1$ then $\alpha(f)$ is logcanonical ??

Let Ψ be an additive real character on $\mathbb{A}_{\mathbb{Q}}$, finite. Let ϕ S.B. on $\mathbb{A}_{\mathbb{Q}}^n$. For $\lambda \in \mathbb{A}$,

$$I(\lambda) \;=\; \int_{x\in\mathbb{A}_p^n} \phi(x)\Psi(f(x)\lambda)(dx). \tag{20.3}$$

Fir f non-homogeneous: using $q < \alpha(f)$ and finite $S(p)$ should have a bound in terms of p^{-a}.

21. Open Discussion Session: IV: Slopes of Modular Forms (Gee)

Fix $p > 2$ that doesn't divide N, let $k \geq 2$, $f - \sum a_n q^n$.

$S_k(\Gamma_0(N_p), \overline{\mathbb{Q}}_p)$ U_p.

Newforms $\leftrightarrow$ St at p.

Oldforms: T_p-eigenvector in $S_k(\Gamma_0(N), \overline{\mathbb{Q}}_p)$. Two eigenvectors in $S_k(\Gamma_0(N_p), \overline{\mathbb{Q}}_p)$. U_p-eigenvalues α, β roots of $x^2 \ -a_p x + p^{k-1}$.

U_p-eigenvalues: p-adic valuations = slopes are in $[0, k-1]$. Symmetry in slopes of oldforms: $\nu(\alpha) + \nu(\beta) = \nu(p^{k-1}) = k-1$.

Oldforms: normalize so that slopes are in $[0, 1]$. What happens to the slopes as $k \to \infty$? Do we get a limiting distribution?

Conjecture (Gouvêa): Converges to the distribution which is uniform on $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$ and zero elsewhere.

As $a_p = \alpha + \beta$ it suggests that its valuation $\nu(a_p)$ is usually at most $\frac{k-1}{p+1}$; this is not known. We do know it is at most $\frac{k-1}{p-1}$. This should be provable using p-adic Hodge theory.

Let's assume that N, p are such that every mod p Galois representation associated to a form in $S_k(\Gamma_0(N_p), \overline{\mathbb{Q}}_p)$ is ordinary in the sense of being reducible when restricted to a decomposition group at p. If $k \leq p+1$, oldforms $S_k(\Gamma_0(N_p), \overline{\mathbb{Q}}_p)$ have T_p-eigenvalues which are p-adic units. For example, $N = 1, p < 100$ and $p \notin \{59, 79\}$.

Can consider local question. Consider local Galois representations.

f $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$. $\rho_f|_{G_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

If $a_p = 0$: $\overline{\rho}_f|_{G_p}$ is irreducible.

If a_p is p-adically close to zero, i.e., $\nu(a_p)$ is large, $\overline{\rho}_f|_{a_p}$ is irreducible.

Assume $\Gamma_0(N)$-regular.

Conjecture (Buzzard): Explicit algorithm with input the list of slopes in weight at most $p+1$ and outputs the list of slopes in any weight.

Consequence of algorithm is that all slopes are in $\mathbb{Z}$ (would be false if we didn't have $\Gamma_0(N)$ regular). At the heart is a pairing algorithm.

Additional comments available online.

22. Open Discussion Session: IV: Growth of field of rationality (Shin)

Consider $f \in S_k(N)$, the set of weight k cusp forms of level $\Gamma_0(N)$, let's normalize so that $a_1 = 1$ in $f = \sum a_n q^n$ and set $\mathbb{Q}(f) = \mathbb{Q}(a_n)$.

Define for integer A the set $S_k(N)^{\leq A}$ to be all $f \in S_k(N)$ such that $[\mathbb{Q}(f) : \mathbb{Q}] \leq A$.

Problem (Serre): Consider $|S_k(N)^{\leq A}|/|S_k(N)|$ in limiting cases (fix one of k, N and let the other tend to infinity).

Consider the level aspect first.

Theorem 22.1 (Serre)**.** *There exists a prime* p, $(p, N_m) = 1$ *for all* m*, such that* $\lim_{m\to\infty} |S_k(N)^{\leq A}|/|S_k(N)| = 0$.

Proof. The crucial ingredient is the proof of Ramanujan transferred to modular forms. Consider $\cup_{m\geq 1}\{a_p(f) : f \in S_k(N_m)^{\leq A}\}$; is finite. Each $a_p(f)$ is an algebraic integer, so a root of a monic polynomial in $\mathbb{Z}[x]$. Bounded degree, bounded coefficient in p, A, finitely many such monic polynomials. $\square$

Equidistribution: $\lim |S_k(N_m)^{\leq A}|/|S_k(N_m)|$.

Example: $\{N_m\}$ equals $2, 2\cdot 3, 2\cdot 3\cdot 5, \ldots$. Or $2, 2^2\cdot 3^2, 2^3\cdot 3^3\cdot 5^3, \ldots$.

Question: Can you prove without this hypothesis?

The case $A = 1$ is known (counting elliptic curves).

In the weight aspect: No prediction in general.

Maeda's Conjecture: Let $N = 1$. For all $f \in S_k(1)$, $[\mathbb{Q}(f) : \mathbb{Q}] = |S_k(1)|$ (the left hand side is $[\mathbb{Q}(a_2) : \mathbb{Q}]$).

Question: For all $A \geq 1$, does there exists a prime q and an $f \in S_k(q)$ such that $[\mathbb{Q}_p(f) : \mathbb{Q}_p] \geq A$? Calegari believes 'yes' while Hida believes 'no'.

Additional comments available online.

23. Open Discussion Session: V: (Hoffman)

Slides are online.

24. Open Discussion Session: VI: Closed-form moments in elliptic curve families (Miller)

Slides online:

http://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/SimonsTalk_EllC

25. Open Discussion Session: VII: Geometric Langlands (Nadler)

Discussed modules and their connections.

26. Problem Session: I (Miller)

Slides online:

http://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/SimonsProblems_

26.1. Introduction. I study n-level density of zeros of families of L-functions

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, g_k an even Schwartz function: $D_{n,\mathcal{F}}(g) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i\neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, n-level density converges to

$$\int g(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} \quad = \quad \int \widehat{g}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

Conjecture (Katz-Sarnak) (In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Results / Applications

- Results:
  - ◇ Agreement: Many families, small support.
  - ◇ Extending support: Related to arithmetic.
- Applications:
  - ◇ Class number: Bounds on growth rate.
  - ◇ Average rank: Vanishing at central point.

Techniques

- Explicit Formula: Convert sums over zeros to sums over Satake parameter moments.
- Averaging: Dirichlet, Petersson, Kuznetsov, ...
- Combinatorics: Showing agreement b/w NT and RMT.

26.2. Problem 1: Lower Order Terms. Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake params $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n \left(1-\alpha_{\pi,i}(p)p^{-s}\right)^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu}\widehat{g}\left(\frac{\nu\log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

1-**Level Density**
Assuming conductors constant in family $\mathcal{F}$, have to study

$$\begin{aligned}\lambda_f(p^\nu) &= \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu \\ S_1(\mathcal{F}) &= -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right] \\ S_2(\mathcal{F}) &= -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]\end{aligned}$$

Corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.}\end{cases}$$

Open Problem: Lower order terms
Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.

- First moment zero save for families of elliptic curves.

- Higher moments control convergence and can depend on arithmetic of family.

Open Problem: Develop a theory of lower order terms to split the universality and see the arithmetic.

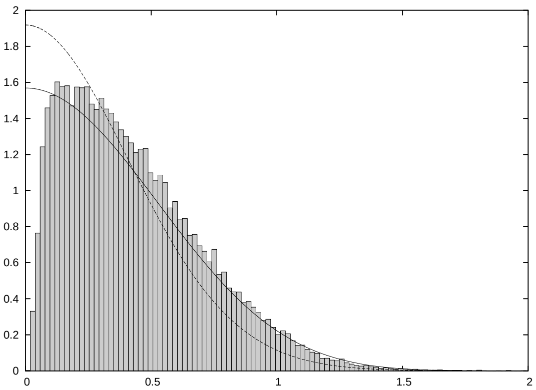
26.3. **Problem 2: Repulsion at the Central Point. Behavior of zeros near central point**
For *one* L-function: good theory high up critical line.

For a *family* of L-functions: good theory as conductors tend to infinity.

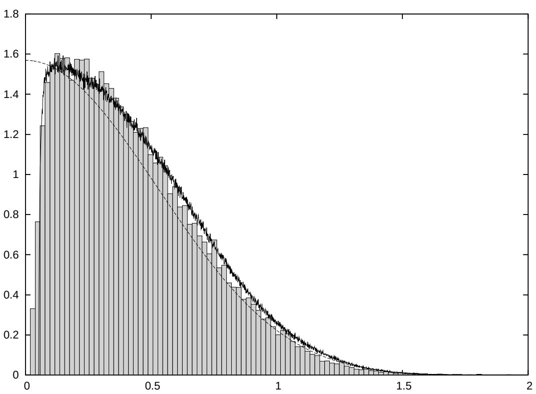
Goal is to understand behavior **at central point** for finite conductors.
Questions (Elliptic Curve Families)
Excess rank: Expected vanishing at central point.

Repulsion: First zero above central point.

Open Problem Model the observed behavior here (done) and extend to other families (in progress).
Modeling lowest zero of $L_{E_{11}}(s,\chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

26.4. **Problem 3: Combinatorics. Background**
Different techniques to compute Number Theory and Random Matrix Theory.

Challenge is showing the two quantities are the same.

n-Level Density: Determinant Expansions from RMT

- $\mathrm{U(N)}$, $\mathrm{U}_k(N)$: $\det\Big(K_0(x_j, x_k)\Big)_{1 \leqslant j,k \leqslant n}$
- $\mathrm{USp(N)}$: $\det\Big(K_{-1}(x_j, x_k)\Big)_{1 \leqslant j,k \leqslant n}$
- $\mathrm{SO(even)}$: $\det\Big(K_1(x_j, x_k)\Big)_{1 \leqslant j,k \leqslant n}$
- $\mathrm{SO(odd)}$: $\det\left(K_{-1}(x_j, x_k)\right)_{1 \leqslant j,k \leqslant n} + \sum_{\nu=1}^{n} \delta(x_\nu) \det\Big(K_{-1}(x_j, x_k)\Big)_{1 \leqslant j,k \neq \nu \leqslant n}$

where

$$K_\epsilon(x,y) = \frac{\sin\Big(\pi(x-y)\Big)}{\pi(x-y)} + \epsilon\frac{\sin\Big(\pi(x+y)\Big)}{\pi(x+y)}.$$

Alternative to Determinant Expansion
Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\underset{\cdot}{\mathrm{x}}}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from Iwaniec-Luo-Sarnak.

Problems

Open Problem: Further develop alternatives to the Katz-Sarnak determinant expansions.

Open Problem: Directly prove agreement for quadratic Dirichlet families (compare with Entin, Roddity-Gershon and Rudnick).

26.5. **References. References to my work on these problems**

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- *Moment Formulas for Ensembles of Classical Compact Groups* (with Alan Chang, Geoffrey Iyer, Kyle Pratt, Nicholas Triantafillou and Minh-Tam Trinh), preprint.

27. PROBLEM SESSION: II (HIDA)

Slides available online.

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