A Symplectic Test of the L-Functions Ratios Conjecture.

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History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\,dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} \;=\; \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ◇ n-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point;

- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) \;=\; \sum_{m\leq x}\frac{a_m}{m^s}+\epsilon\mathbb{X}_L(s)\sum_{n\leq y}\frac{a_n}{n^{1-s}};$$

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- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}}\omega_f\sum_{\gamma} g\left(\gamma\frac{\log N_f}{2\pi}\right) \;=\; \frac{1}{2\pi i}\int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\cdots)g(\cdots)$$

◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha}R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

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where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Main Results

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  - ◇ even (resp., odd) if $d > 0$ (resp., $d < 0$).

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Will study following families:

◇ even fundamental discriminants at most X;
◇ $\{8d : 0 < d \leq X, \; d$ an odd, positive square-free fundamental discriminant$\}$.

Prediction from Ratios Conjecture

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right) = \frac{1}{X^*\log X}\int_{-\infty}^{\infty} g(\tau)\sum_{d\le X}\left[\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\pm\frac{i\pi\tau}{\log X}\right)\right]d\tau$$
$$+\frac{2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau)\left[\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)+A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right.$$
$$\left.-e^{-2\pi i\tau\log(d/\pi)/\log X}\frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right]d\tau + O(X^{-\frac{1}{2}+\epsilon}),$$

with

$$A_D(-r,r) = \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}$$
$$A_D'(r;r) = \sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}.$$

Prediction from Ratios Conjecture

Main term is

$$\frac{1}{X^*}\sum_{d\leq X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right) = \int_{-\infty}^{\infty} g(x)\left(1-\frac{\sin(2\pi x)}{2\pi x}\right)dx$$

$$+\, O\left(\frac{1}{\log X}\right),$$

which is the 1-level density for the scaling limit of $\mathrm{USp}(2N)$. If $\mathrm{supp}(\widehat{g}) \subset (-1,1)$, then the integral of $g(x)$ against $-\sin(2\pi x)/2\pi x$ is $-g(0)/2$.

Prediction from Ratios Conjecture

Assuming RH for $\zeta(s)$, for $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$:

$$\frac{-2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau)\, e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau$$

$$= -\frac{g(0)}{2} + O(X^{-\frac{3}{4}(1-\sigma)+\epsilon});$$

the error term may be absorbed into the $O(X^{-1/2+\epsilon})$ error if $\sigma < 1/3$.

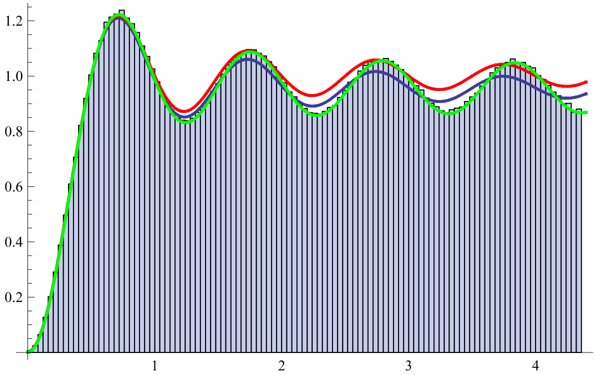
Main Results

Theorem (M– '07)

Let $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma)$*, assume RH for* $\zeta(s)$*. 1-Level Density agrees with prediction from Ratios Conjecture*

- *up to* $O(X^{-(1-\sigma)/2+\epsilon})$ *for the family of quadratic Dirichlet characters with even fundamental discriminants at most* X*;*
- *up to* $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$ *for our sub-family. If* $\sigma < 1/3$ *then agrees up to* $O(X^{-1/2+\epsilon})$*.*

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).
$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.
$\diamond$ Green: all lower order terms.

Sketch of Proofs

Ratios Calculation

Hardest piece to analyze is

$$R(g;X) = -\frac{2}{X^* \log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau) e^{-2\pi i\tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)} \cdot \zeta\left(1-\frac{4\pi i\tau}{\log X}\right) A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right) d\tau,$$

$$A_D(-r,r) = \prod_p \left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}.$$

Proof: shift contours, keep track of poles of ratios of Γ and zeta functions.

Ratios Calculation: Weaker result for $\mathrm{supp}(\widehat{g}) \subset (-1,1)$.

- d-sum is $X^* e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} \left(1 - \frac{2\pi i \tau}{\log X}\right)^{-1} + O(X^{1/2})$;

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- decay of g restricts τ-sum to $|\tau| \le \log X$, Taylor expand everything but g: small error term and

$$\int_{|\tau|\le \log X} g(\tau) \sum_{n=-1}^{N} \frac{a_n}{\log^n X} (2\pi i \tau)^n e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau$$

$$= \sum_{n=-1}^{N} \frac{a_n}{\log^n X} \int_{|\tau|\le \log X} (2\pi i \tau)^n g(\tau) e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau;$$

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- from decay of g can extend the τ-integral to $\mathbb{R}$ (essential that N is fixed and finite!), for $n \ge 0$ get the Fourier transform of $g^{(n)}$ (the n^{th} derivative of g) at $1 - \frac{\pi}{\log X}$, vanishes if $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$.

Number Theory Sums

$$S_{\text{even}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p}\frac{\chi_d(p)^2\log p}{p^\ell \log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right)$$

$$S_{\text{odd}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\chi_d(p)\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right).$$

Number Theory Sums

Lemma

Let $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$*. Then*

$$\begin{aligned} S_{\mathrm{even}} &= -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1 + \frac{4\pi i\tau}{\log X}\right) d\tau \\ &\quad + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau) A_D'\left(\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}) \\ S_{\mathrm{odd}} &= O(X^{-\frac{1-\sigma}{2}} \log^6 X). \end{aligned}$$

If instead we consider the family of characters χ_{8d} *for odd, positive square-free* $d \in (0, X)$ *(*d *a fundamental discriminant), then*

$$S_{\mathrm{odd}} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}).$$

Analysis of S_{even}

$\chi_d(p)^2 = 1$ except when $p|d$. Replace $\chi_d(p)^2$ with 1, and subtract off the contribution from when $p|d$:

$$\begin{aligned} S_{\text{even}} &= -2\sum_{\ell=1}^{\infty}\sum_{p}\frac{\log p}{p^\ell \log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right) \\ &\quad + \frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p|d}\frac{\log p}{p^\ell \log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right) \\ &= S_{\text{even};1} + S_{\text{even};2}. \end{aligned}$$

Lemma (Perron's Formula)

$$S_{\text{even};1} = -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)d\tau.$$

Analysis of S_{even}: $S_{\mathrm{even};2}$

This piece gives us $\int g(\tau) A'_D(-\cdots,\cdots)$.

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  - ⋄ For $p < X^{1/2}$: $\sum_{d \leq X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.

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- Main ideas:
  - ◇ Restrict to $p \le X^{1/2}$.
  - ◇ For $p < X^{1/2}$: $\sum_{d \le X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.
  - ◇ Use Fourier Transform to expand $\widehat{g}$.

Analysis of $S_{\rm odd}$

$$S_{\rm odd} \quad = \quad -\frac{2}{X^*}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right)\sum_{d\le X}\chi_d(p).$$

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Jutila's bound

$$\sum_{\substack{1<n\le N\\ n\ \text{non-square}}}\left|\sum_{\substack{0<d\le X\\ d\ \text{fund. disc.}}}\chi_d(n)\right|^2 \ll NX\log^{10}N.$$

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Proof: Cauchy-Schwarz and Jutila: $p^{2\ell+1}$ non-square:

$$\left(\sum_{\ell=0}^{\infty}\sum_{p^{(2\ell+1)/2}\le X^\sigma}\left|\sum_{d\le X}\chi_d(p)\right|^2\right)^{1/2} \ll X^{\frac{1+\sigma}{2}}\log^5 X.$$

Analysis of $S_{\rm odd}$: Extending Support

More technical, replace Jutila's bound by applying Poisson Summation to character sums.

Lemma

Let $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. *For family* $\{8d : 0 < d \leq X,$ *d an odd, positive square-free fundamental discriminant*$\}$, $S_{\rm odd} = O(X^{-\frac{1}{2}+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$. *In particular, if* $\sigma < 1/3$ *then* $S_{\rm odd} = O(X^{-1/2+\epsilon})$.

Conclusions

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- Ratios Conjecture gives detailed predictions (up to $X^{1/2+\epsilon}$).
- Number Theory agrees with predictions for suitably restricted test functions.
- Numerics quite good.

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