Five College Number Theory Seminar

Random Matrix Theory and Families of Elliptic Curves: Evidence for the Underlying Group Symmetries

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Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Info by shooting high-energy neutrons into nucleus.

Fundamental Equation:Quantum Mechanics

$$H\psi_n \ = \ E_n\psi_n$$

Similar to stat mech, leads to considering eigenvalues of ensembles of matrices.

Real Symmetric (GOE), Complex Hermitian (GUE), Classical Compact Groups.

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ be an increasing sequence of numbers, $B \subset \mathbf{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i\neq j_k\leq N\right\}}{N}$$

Results on Zeros (Assuming GRH):

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak)
4. n-level correlations for the classical compact groups (Katz-Sarnak)
5. **insensitive to any finite set of zeros**

Measures of Spacings: n-Level Density and Families

Let $\phi(x) = \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions, $\widehat{\phi}$ compactly supported.

$$D_{n,f}(\phi) \;=\; \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

$L_f =$ Conductor, Scale factor for low zeros.

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

$$D_{n,\mathcal{F}}(\phi) \;=\; \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} D_{n,f}(\phi).$$

Limiting Behavior

As $N \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi)$$

$$= \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\gamma_f^{(j_i)} \log L_f}{2\pi} \right)$$

$$\to \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx$$

$$\to \int \cdots \int \widehat{\phi}(y) \widehat{W}_{n,\mathcal{G}(\mathcal{F})}(y) dy.$$

Conj: Distribution of Low Zeros agrees with a classical compact group.

Correspondences

Similarities b/w Nuclei and L-Fns:

$$\text{Zeros} \longleftrightarrow \text{Energy Levels}$$

$$\text{Support} \longleftrightarrow \text{Neutron Energy.}$$

Conjecture: Zeros near central point in a family of L-functions behave like eigenvalues near 1 of a classical compact group (Unitary, Symplectic, Orthogonal).

Some Number Theory Results

- Orthogonal:

  Iwaniec-Luo-Sarnak: 1-level density for $H_k^{\pm}(N)$, N square-free;

  Dueñez-Miller: 1, 2-level for $\{\phi \times f^2 : f \in H_k(1)\}$, ϕ even Maass;

  Miller: One-parameter families of elliptic curves.

- Symplectic:

  Rubinstein: n-level densities for $L(s, \chi_d)$;

  Dueñez-Miller: 1-level for $\{\phi \times f : f \in H_k(1)\}$, ϕ even Maass.

- Unitary: Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.

Main Tools

- Explicit Formula: Relates sums over zeros to sums over primes.

- Averaging Formulas: Orthogonality of characters, Petersson formula.

- Control of conductors: Monotone.

1-Level Densities

Fourier Transforms for 1-level densities:

$$\begin{aligned}
\widehat{W}_{1,SO(\text{even})}(u) &= \delta(u) + \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,SO}(u) &= \delta(u) + \tfrac{1}{2} \\
\widehat{W}_{1,SO(\text{odd})}(u) &= \delta(u) - \tfrac{1}{2}\eta(u) + 1 \\
\widehat{W}_{1,Sp}(u) &= \delta(u) - \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,U}(u) &= \delta(u)
\end{aligned}$$

where $\delta(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Dirichlet Characters: m Prime

$\{\chi_0\} \cup \{\chi_l\}_{l \le m-2}$ are all the characters mod m.

Consider the family of primitive characters mod a prime m ($m-2$ characters):

$$\begin{aligned}
&\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{\gamma_\chi} \phi\left(\gamma_\chi \frac{\log(\frac{m}{\pi})}{2\pi}\right) \\
=& \int_{-\infty}^{\infty} \phi(y) dy \\
-& \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \\
-& \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1} \\
+& O\Big(\frac{1}{\log m}\Big).
\end{aligned}$$

Can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise} \end{cases}$$

For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \; = \begin{cases} m-1-1 & p \equiv 1(m) \\ -1 & \text{otherwise} \end{cases}$$

Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}$$

First Sum

$$
\begin{aligned}
&\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}\\
+\;& 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}\\
\ll\;& \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}} + \sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}}\\
\ll\;& \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} + \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac{1}{2}}\\
\ll\;& \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} + \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}\\
\ll\;& \frac{1}{m}m^{\sigma/2}.
\end{aligned}
$$

No contribution if $\sigma < 2$.

Results

Theorem [Hughes-Rudnick 2002]
$\mathcal{F}_N$ all primitive characters with prime conductor N.

If supp$(\widehat{\phi}) < 2$, as $N \to \infty$, agrees with Unitary.

Theorem [Miller 2002]
$\mathcal{F}_N$ all primitive characters with conductor odd square-free integer in $[N, 2N]$.

If supp$(\widehat{\phi}) < 2$, as $N \to \infty$, agrees with Unitary.

Elliptic Curves

Conductors grow rapidly.

Results are for small support, where Orthogonal densities indistinguishable.

Study 2-Level Density.

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

$$D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} D_{n,f}(\phi).$$

2-Level Densities

$$c(\mathcal{G}) \ = \ \begin{cases} 0 & \text{if } \mathcal{G} \ = \ \text{SO(even)} \\ \frac{1}{2} & \text{if } \mathcal{G} \ = \ \text{SO} \\ 1 & \text{if } \mathcal{G} \ = \ \text{SO(odd)} \end{cases}$$

For $\mathcal{G} = \text{SO(even)}$, SO or SO(odd):

$$\begin{aligned} & \int \int \widehat{\phi_1}(u_1)\widehat{\phi_2}(u_2)\widehat{W_{2,\mathcal{G}}}(u)du_1du_2 \\ = \ & \left[\widehat{\phi_1}(0) + \frac{1}{2}\phi_1(0)\right]\left[\widehat{f_2}(0) + \frac{1}{2}\phi_2(0)\right] \\ & + 2\int |u|\widehat{\phi_1}(u)\widehat{\phi_2}(u)du \\ & -2\widehat{\phi_1\phi_2}(0) - \phi_1(0)\phi_2(0) \\ & + c(\mathcal{G})\phi_1(0)\phi_2(0). \end{aligned}$$

Elliptic Curves

$$E : y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6, \;\; a_i \in \mathbb{Q}$$

Often can write $E : y^2 = x^3 + Ax + B$.

Let N_p be the number of solns mod p:

$$N_p \;=\; \sum_{x(p)} \left[1 + \left(\frac{x^3+Ax+B}{p}\right)\right] \;=\; p + \sum_{x(p)} \left(\frac{x^3+Ax+B}{p}\right)$$

Local data: $a_E(p) = p - N_p$. Use to build the L-function:

$$a_E(p) \;=\; -\sum_{x \bmod p} \left(\frac{x^3+Ax+B}{p}\right)$$

Elliptic Curves: Arithmetic Progression

One-parameter families:

$$E_t : y^2 \ = \ x^3 + A(t)x + B(t), \quad A(t), B(t) \in \mathbb{Z}(t).$$

We have

$$a_t(p) \ = \ -\sum_{x \bmod p} \left(\frac{x^3 + A(t)x + B(t)}{p}\right) \ = \ a_{t+mp}(p)$$

Can handle sums of $a_t(p)$ for t in arithmetic progression.

Elliptic Curves (cont)

$$L(E,s) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \;=\; \prod_p L_p(E,s).$$

By GRH: All zeros on the critical line.

Rational solutions: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$.

Birch and Swinnerton-Dyer Conjecture:
Geometric rank r equals the analytic rank (order of vanishing at central point).

Comments on Previous Results

- explicit formula relating zeros and Fourier coeffs;
- averaging formulas for the family;
- conductors easy to control (constant or monotone)

Elliptic curve E_t: discriminant $\Delta(t)$, conductor $N_{E_t} = C(t)$ is

$$C(t) \;=\; \prod_{p|\Delta(t)} p^{f_p(t)}$$

Normalization of Zeros

Local (hard) vs Global (easy).
As $N \to \infty$:

$$\frac{1}{|\mathcal{F}_N|}\sum_{E\in\mathcal{F}} D_{n,E}(\phi)$$

$$= \frac{1}{|\mathcal{F}_N|}\sum_{E\in\mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i\neq\pm j_k}} \prod_i \phi_i\left(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx$$

$$\to \int\cdots\int \widehat{\phi}(y)\widehat{W}_{n,\mathcal{G}(\mathcal{F})}(y)dy.$$

Conj: Distribution of Low Zeros agrees with Orthogonal Densities.

1-Level Expansion

$$
\begin{aligned}
D_{1,\mathcal{F}}(\phi) &= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{j}\phi\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right)\\
&= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\widehat{\phi}(0)+\phi_i(0)\\
&\quad -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log N_E}\right)a_E(p)\\
&\quad -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{\phi}\left(2\frac{\log p}{\log N_E}\right)a_E^2(p)\\
&\quad +O\left(\frac{\log\log N_E}{\log N_E}\right)
\end{aligned}
$$

Want to move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$, Leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t \bmod p} a_t^r(p), \quad r = 1 \text{ or } 2.$$

2-Level Expansion

Need to evaluate terms like

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\prod_{i=1}^{2}\frac{1}{p_i^{r_i}}g_i\left(\frac{\log p_i}{\log N_E}\right)a_E^{r_i}(p_i).$$

Analogue of Petersson / Orthogonality:
If $p_1,\dots,p_n$ are distinct primes

$$\sum_{t \bmod p_1\cdots p_n} a_{t_1}^{r_1}(p_1)\cdots a_{t_n}^{r_n}(p_n)$$
$$= A_{r_1,\mathcal{F}}(p_1)\cdots A_{r_n,\mathcal{F}}(p_n).$$

Input

For many families

$$(1) : A_{1,\mathcal{F}}(p) \;=\; -rp \;+\; O(1)$$
$$(2) : A_{2,\mathcal{F}}(p) \;=\; p^2 \;+\; O(p^{3/2})$$

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(t)$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} -\frac{A_{1,\mathcal{F}}(p)\log p}{p} = r$$

Surfaces with $j(t)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p^2 + O\left(p^{3/2}\right).$$

DEFINITIONS

$$D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|} \sum_{E\in\mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i\left(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$D_{n,\mathcal{F}}^{(r)}(\phi)$: n-level density with contribution of r zeros at central point removed.

$\mathcal{F}_N$: Rational one-parameter family, $t \in [N, 2N]$, conductors monotone.

ASSUMPTIONS

1-parameter family of Ell Curves, rank r over $\mathbb{Q}(t)$, rational surface. Assume

- GRH;
- $j(t)$ non-constant;
- Sq-Free Sieve if $\Delta(t)$ has irr poly factor of deg ≥ 4.

Pass to positive percent sub-seq where conductors polynomial of degree m.

ϕ_i even Schwartz, support σ_i:

- $\sigma_1 < \min\left(\frac{1}{2}, \frac{2}{3m}\right)$ for 1-level
- $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for 2-level.

MAIN RESULT

Theorem (Miller 2004): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D_{n,\mathcal{F}_N}^{(r)}(\phi) \longrightarrow \int \phi(x) W_{\mathcal{G}}(x) dx,$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

1 and 2-level densities confirm Katz-Sarnak, B-SD predictions for small support.

Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$,
   $9t+1$ Square-Free: all even.

2. $y^2 = x^3 \pm 4(4t+2)x$,
   $4t+2$ Square-Free:
   $+$ all odd, $-$ all even.

3. $y^2 = x^3 + tx^2 - (t+3)x + 1$,
   t^2+3t+9 Square-Free: all odd.

First two rank 0 over $\mathbb{Q}(t)$, third is rank 1.

Without 2-Level Density, couldn't say *which* orthogonal group.

Examples (cont)

Rational Surface of Rank 6 over $\mathbf{Q}(t)$:

$$\begin{aligned} y^2 &= x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ &\quad + (2ct - D)(t^2 + 2t - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8,916,100,448,256,000,000 \\ B &= -811,365,140,824,616,222,208 \\ C &= 26,497,490,347,321,493,520,384 \\ D &= -343,107,594,345,448,813,363,200 \\ a &= 16,660,111,104 \\ b &= -1,603,174,809,600 \\ c &= 2,149,908,480,000 \end{aligned}$$

Need GRH, Sq-Free Sieve to handle sieving.

Sketch of Proof

1. Sieving (Arithmetic Progressions)

2. Partial Summation (Complete Sums)

3. Controlling Conductors (Monotone).

Sieving

$$\sum_{\substack{t=N\\D(t)\\sqfree}}^{2N} S(t) = \sum_{d=1}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\t\in[N,2N]}} S(t)$$

$$= \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\t\in[N,2N]}} S(t) + \sum_{d\geq \log^l N}^{N^{k/2}} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\t\in[N,2N]}} S(t).$$

Handle first by progressions.

Handle second by Cauchy-Schwartz: The number of t in the second sum (by Sq-Free Sieve Conj) is $o(N)$:

$$\sum_{t\in\mathcal{T}} S(t) \ll \left(\sum_{t\in\mathcal{T}} S^2(t)\right)^{\frac{1}{2}} \cdot \left(\sum_{t\in\mathcal{T}} 1\right)^{\frac{1}{2}}$$

$$\ll \left(\sum_{t\in[N,2N]} S^2(t)\right)^{\frac{1}{2}} \cdot o\left(\sqrt{N}\right).$$

Sieving (cont)

$$\sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t)\equiv 0(d^2)\\ t\in[N,2N]}} S(t)$$

$t_i(d)$ roots of $D(t) \equiv 0 \bmod d^2$.

$t_i(d), t_i(d) + d^2, \ldots, t_i(d) + \left[\frac{N}{d^2}\right] d^2$.

If $(d, p_1p_2) = 1$, go through complete set of residue classes $\frac{N/d^2}{p_1p_2}$ times.

Partial Summation

$\widetilde{a}_{d,i,p}(t') = a_{t(d,i,t')}(p)$, $G_{d,i,P}(u)$ is related to the test functions, d and i from progressions.

Applying Partial Summation

$$S(d,i,r,p) = \sum_{t'=0}^{[N/d^2]} \widetilde{a}^r_{d,i,p}(t') G_{d,i,p}(t')$$

$$= \left(\frac{[N/d^2]}{p} A_{r,\mathcal{F}}(p) + O\left(p^R\right) \right) G_{d,i,p}([N/d^2])$$

$$- \sum_{u=0}^{[N/d^2]-1} \left(\frac{u}{p} A_{r,\mathcal{F}}(p) + O\left(p^R\right) \right) \left(G_{d,i,p}(u) - G_{d,i,p}(u+1) \right)$$

Difficult Piece: Fourth Sum I

$$\sum_{u=0}^{[N/d^2]-1} O(P^R)\Big(G_{d,i,P}(u) - G_{d,i,P}(u+1)\Big)$$

Taylor $G_{d,i,P}(u) - G_{d,i,P}(u+1)$ gives $P^R \frac{N}{d^2} \frac{1}{P^r \log N}$.

$\frac{1}{|\mathcal{F}|} \sum_{i,d}$ gives $O(\frac{P^R}{P^r \log N})$.

Problem is in summing over the primes, as we no longer have $\frac{1}{|\mathcal{F}|}$.

Fourth Sum: II

If exactly one of the r_j's is non-zero, then

$$\sum_{u=0}^{[N/d^2]-1} \left| G_{d,i,P}(u) - G_{d,i,P}(u+1) \right|$$
$$= \sum_{u=0}^{[N/d^2]-1} \left| g\left(\frac{\log p}{\log C(t_i(d) + ud^2)}\right) - g\left(\frac{\log p}{\log C(t_i(d) + (u+1)d^2)}\right) \right|$$

If the conductors are monotone, for fixed i, d and p, small independent of N.

If two of the r_j's are non-zero:

$$\begin{aligned} |a_1a_2 - b_1b_2| &= |a_1a_2 - b_1a_2 + b_1a_2 - b_1b_2| \\ &\leq |a_1a_2 - b_1a_2| + |b_1a_2 - b_1b_2| \\ &= |a_2| \cdot |a_1 - b_1| + |b_1| \cdot |a_2 - b_2| \end{aligned}$$

Handling the Conductors: I

$$y^2+a_1(t)xy+a_3(t)y = x^3+a_2(t)x^2+a_4(t)x+a_6(t)$$

$$C(t) = \prod_{p|\Delta(t)} p^{f_p(t)}$$

$D_1(t)$ = primitive irred. poly. factors $\Delta(t)$ and $c_4(t)$ share

$D_2(t)$ = remaining primitive irred. poly. factors of $\Delta(t)$

$$D(t) = D_1(t)D_2(t)$$

$D(t)$ sq-free, $C(t)$ like $D_1^2(t)D_2(t)$ except for a finite set of bad primes.

Careful: $t(t+1)(t+2)(t+3)$.

Handling the Conductors: II

$$y^2+a_1(t)xy+a_3(t)y \; = \; x^3+a_2(t)x^2+a_4(t)x+a_6(t)$$

Let P be the product of the bad primes.

Tate's Algorithm gives $f_p(t)$, depend only on $a_i(t)$ mod powers of p.

Apply Tate's Algorithm to E_{t_1}. Get $f_p(t_1)$ for $p|P$. For m large, $p|P$,

$$f_p(\tau) = f_p(P^m t + t_1) = f_p(t_1),$$

and order of p dividing $D(P^m t + t_1)$ is independent of t.

Get integers st $C(\tau) = c_{bad} \frac{D_1^2(\tau)}{c_1} \frac{D_2(\tau)}{c_2}$, $D(\tau)$ sq-free.

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(t)$.

RMT $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r = 32%
Percent with rank r+1 = 48%
Percent with rank r+2 = 18%
Percent with rank r+3 = 2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$.

Data on Excess Rank

$$y^2+a_1xy+a_3y \;=\; x^3+a_2x^2+a_4x+a6$$

Family: $a_1 : 0$ to 10, rest -10 to 10.

Percent with rank 0 = 28.60%
Percent with rank 1 = 47.56%
Percent with rank 2 = 20.97%
Percent with rank 3 = 2.79%
Percent with rank 4 = .08%

14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).

Data on Excess Rank

$$y^2 + y = x^3 + tx$$

Each data set 2000 curves from start.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{11}, but on logarithmic scale still small.

Excess Rank Calculations
Families with $y^2 = f_t(x)$; $D(t)$ SqFree

Family	t Range	Num t	r	r	$r+1$	$r+2$	$r+3$
$+4(4t+2)$	$[2, 2002]$	1622	0		95.44		4.56
$-4(4t+2)$	$[2, 2002]$	1622	0	70.53		29.35	
$9t+1$	$[2, 247]$	169	0	71.01		28.99	
t^2+9t+1	$[2, 272]$	169	1	71.60		27.81	
$t(t-1)$	$[2, 2002]$	643	0	40.44	48.68	10.26	0.62
$(6t+1)x^2$	$[2, 101]$	93	1	34.41	47.31	17.20	1.08
$(6t+1)x$	$[2, 77]$	66	2	30.30	50.00	16.67	3.03

1. $x^3 + 4(4t+2)x$, $4t+2$ Sq-Free, odd.
2. $x^3 - 4(4t+2)x$, $4t+2$ Sq-Free, even.
3. $x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free, even.
4. $x^3 + tx^2 - (t+3)x + 1$, t^2+3t+9 Sq-Free, odd.
5. $x^3 + (t+1)x^2 + tx$, $t(t-1)$ Sq-Free, rank 0.
6. $x^3 + (6t+1)x^2 + 1$, $4(6t+1)^3 + 27$ Sq-Free, rank 1.
7. $x^3 - (6t+1)^2 x + (6t+1)^2$, $(6t+1)[4(6t+1)^2 - 27]$ Sq-Free, rank 2.

Excess Rank Calculations
Families with $y^2 = f_t(x)$; All $D(t)$

Family	t Range	Num t	r	r	$r+1$	$r+2$	$r+3$
$+4(4t+2)$	$[2, 2002]$	2001	0	6.45	85.76	3.95	3.85
$-4(4t+2)$	$[2, 2002]$	2001	0	63.52	9.90	25.99	.50
$9t+1$	$[2, 247]$	247	0	55.28	23.98	20.73	
t^2+9t+1	$[2, 272]$	271	1	73.80		25.83	
$t(t-1)$	$[2, 2002]$	2001	0	42.03	48.43	9.25	0.30
$(6t+1)x^2$	$[2, 101]$	100	1	32.00	50.00	17.00	1.00
$(6t+1)x$	$[2, 77]$	76	2	32.89	50.00	14.47	2.63

1. $x^3 + 4(4t+2)x$, $4t+2$ Sq-Free, odd.
2. $x^3 - 4(4t+2)x$, $4t+2$ Sq-Free, even.
3. $x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free, even.
4. $x^3 + tx^2 - (t+3)x + 1$, t^2+3t+9 Sq-Free, odd.
5. $x^3 + (t+1)x^2 + tx$, $t(t-1)$ Sq-Free, rank 0.
6. $x^3 + (6t+1)x^2 + 1$, $4(6t+1)^3 + 27$ Sq-Free, rank 1.
7. $x^3 - (6t+1)^2x + (6t+1)^2$, $(6t+1)[4(6t+1)^2 - 27]$ Sq-Free, rank 2.

Additional Experiments

RMT predicts the effect of multiple eigenvalues at 1 on nearby eigenvalues. (Dueñez).

Extensive numerical investigation of zeros near central point underway (Dueñez, Lint, Miller).

Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j$$

First Model:

Sub-ensemble of $SO(2N)$ with the last $2n$ of the $2N$ eigenvalues equal $+1$:

$$d\varepsilon_{2n}(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2n} \prod_j d\theta_j,$$

with $1 \le j,k \le N-n$.

Second Model:

$$\mathcal{A}_{2N,2n} = \left\{ \begin{pmatrix} g & \\ & I_{2n} \end{pmatrix} : g \in SO(2N-2n) \right\}$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2,\mathrm{Ind}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\mathrm{Int}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Testing RMT Models

For small support, 1-level densities for Elliptic Curves agree with $\rho_{r,\text{Indep}}$ and not $\rho_{r,\text{Interaction}}$.

Curve E, conductor N_E, expect first zero $\frac{1}{2} + i\gamma_E^{(1)}$ with $\gamma_E^{(1)} \approx \frac{1}{\log N_E}$.

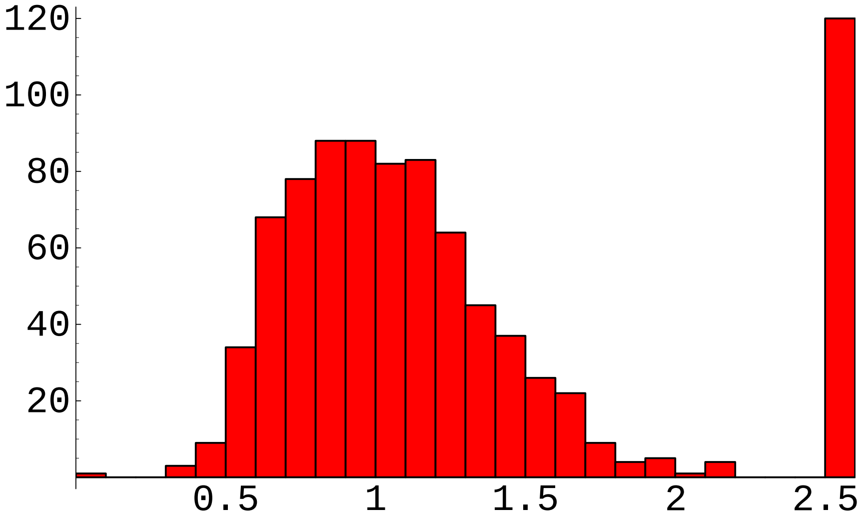
If r zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log N_E}$, might detect in 1-level density:

$$\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_E^{(j)} \log N_E}{2\pi}\right).$$

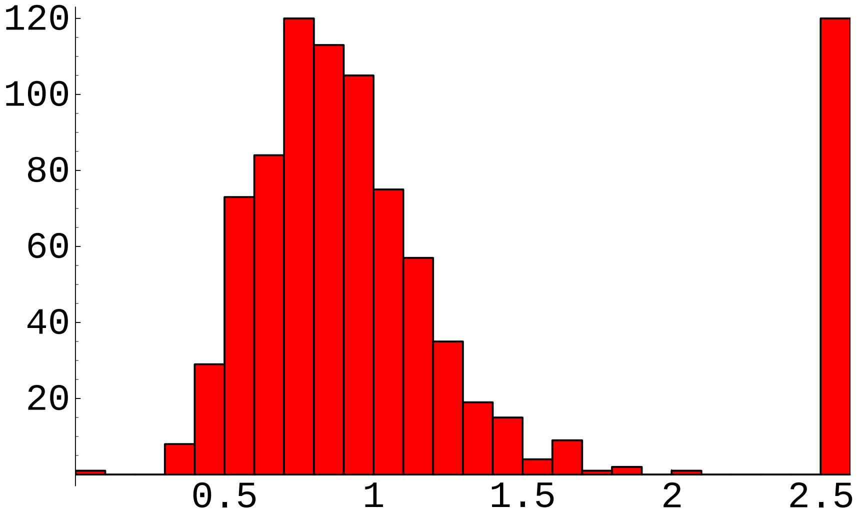
Corrections of size

$$\phi\left(x_0 + c_r\right) - \phi(x_0) \approx \phi'\left(x(x_0, c_r)\right) \cdot c_r.$$

Rank 0 Curves: 1st Normalized Zero



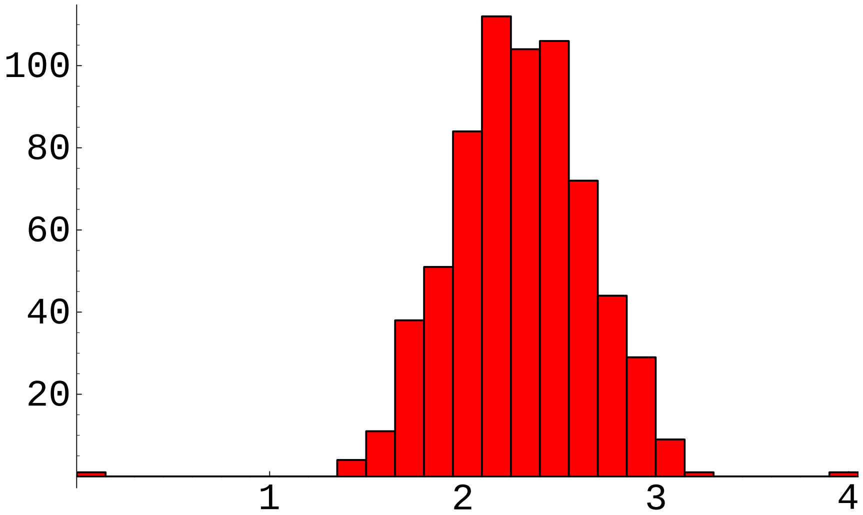
$750 \text{curves}, \log(\text{cond}) \in [3.2, 12.6]; \text{mean} = 1.04$



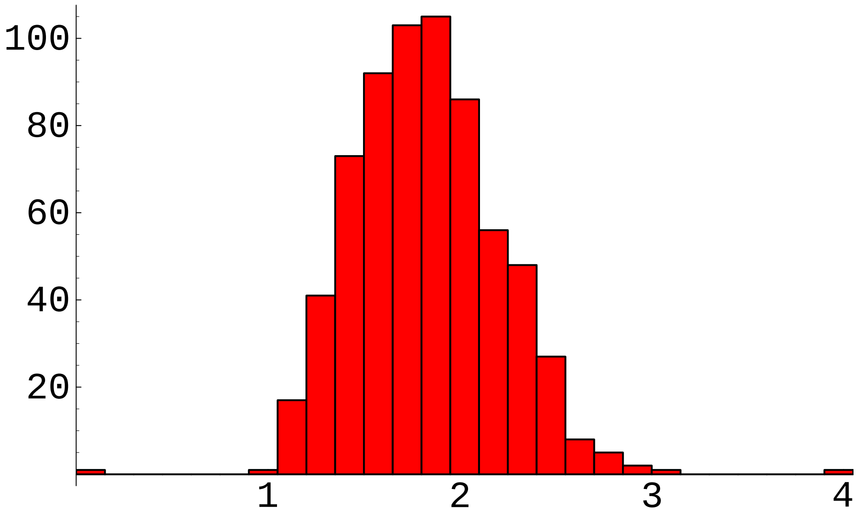
$750 \text{curves}, \log(\text{cond}) \in [12.6, 14.9]; \text{mean} = .88$

(Far left and right bins just for formatting)

Rank 2 Curves: 1st Normalized Zero

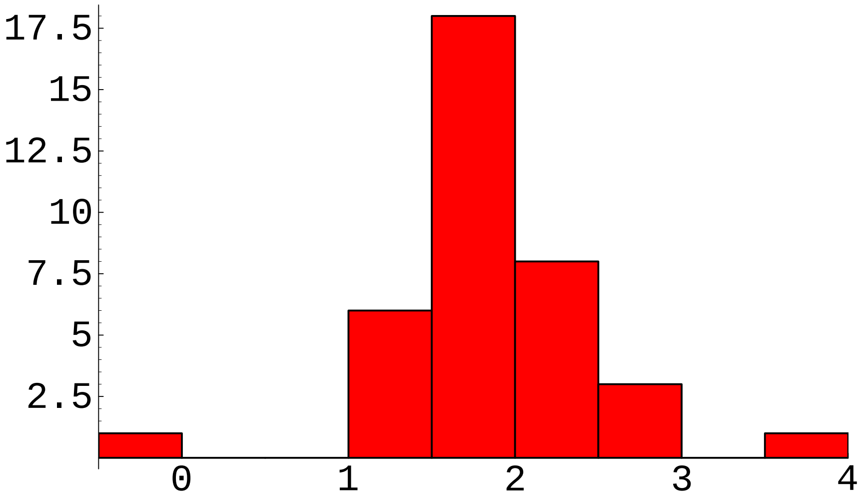


665 curves, $\log(\text{cond}) \in [10, 10.3125]$;
mean $= 2.30$

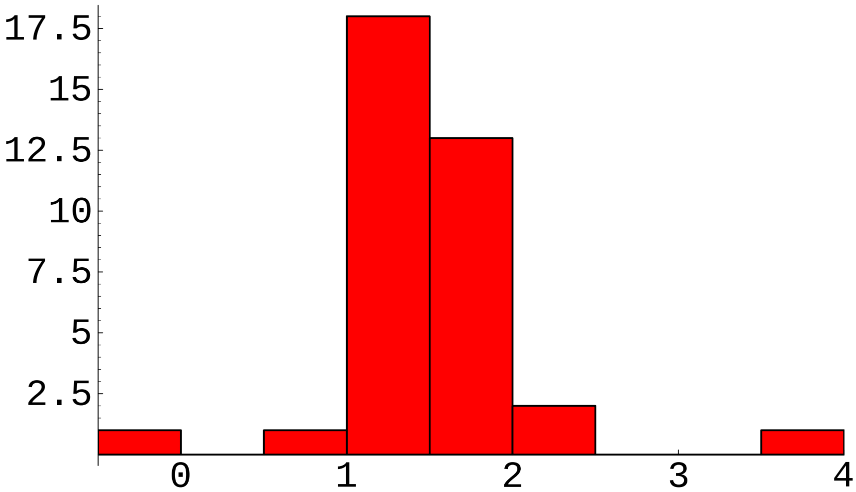


665 curves, $\log(\text{cond}) \in [16, 16.5]$;
mean $= 1.82$

Rank 2 Curves: $[0,0,0,-t^2,t^2]$
1st Normalized Zero



35 curves, $\log(\text{cond}) \in [7.8, 16.1]$; mean $= 2.24$



34curves, $\log(\text{cond}) \in [16.2, 23.3]$; mean $= 2.00$

Summary

- Similar behavior in different systems.
- Find correct scale.
- Average over similar elements.
- Need an Explicit Formula.
- Different statistics tell different stories.
- Evidence for B-SD, RMT interpretation of zeros
- Need more data.

Appendices

The first two appendices list various standard conjectures. The second provides (at least conjecturally) when a family should have equidistribution of signs of functional equations. Experimental evidence is provided in the third appendix, which is on the distribution of signs of elliptic curves in a one-parameter family. Testing whether or not a generic family is equidistributed in sign. We looked at 1000 consecutive elliptic curves, and calculated the excess of positive over negative. We did this many times, and created a histogram plot. The fluctuations look Gaussian! The third appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor N_E, one needs about $\sqrt{N_E}\log N_E$ Fourier coefficients. The fourth appendix gives some estimates on bounding the number of curves in a family with given rank.

Appendix I: Standard Conjectures

Generalized Riemann Hypothesis (for Elliptic Curves) *Let $L(s, E)$ be the (normalized) L-function of the elliptic curve E. Then the non-trivial zeros of $L(s, E)$ satisfy Re$(s) = \frac{1}{2}$.*

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let E be an elliptic curve of geometric rank r over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$, T is the subset of torsion points). Then the analytic rank (the order of vanishing of the L-function at the central point) is also r.*

Tate's Conjecture for Elliptic Surfaces [Ta] *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to **C** and satisfies $-$ord$_{s=2}L_2(\mathcal{E}, s) =$ rank $NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line Re$(s) = 2$.*

Most of the 1-param families we investigate are rational surfaces, where Tate's conjecture is known. See [RSi].

Appendix II: Equidistribution of Signs

ABC Conjecture *Fix $\epsilon > 0$. For co-prime positive integers a, b and c with $c = a + b$ and $N(a,b,c) = \prod_{p|abc} p$, $c \ll_\epsilon N(a,b,c)^{1+\epsilon}$.*

The full strength of ABC is never needed; rather, we need a consequence of ABC, the Square-Free Sieve (see [Gr]):

Square-Free Sieve Conjecture *Fix an irreducible polynomial $f(t)$ of degree at least 4. As $N \to \infty$, the number of $t \in [N, 2N]$ with $f(t)$ divisible by p^2 for some $p > \log N$ is $o(N)$.*

For irreducible polynomials of degree at most 3, the above is known, complete with a better error than $o(N)$ ([Ho], chapter 4).

Restricted Sign Conjecture (for the Family $\mathcal{F}$) *Consider a one-parameter family $\mathcal{F}$ of elliptic curves. As $N \to \infty$, the signs of the curves E_t are equidistributed for $t \in [N, 2N]$.*

The Restricted Sign conjecture often fails. First, there are families with constant $j(E_t)$ where all curves have the same sign. Helfgott [He] has recently related the Restricted Sign conjecture to the Square-Free Sieve conjecture and standard conjectures on sums of Moebius:

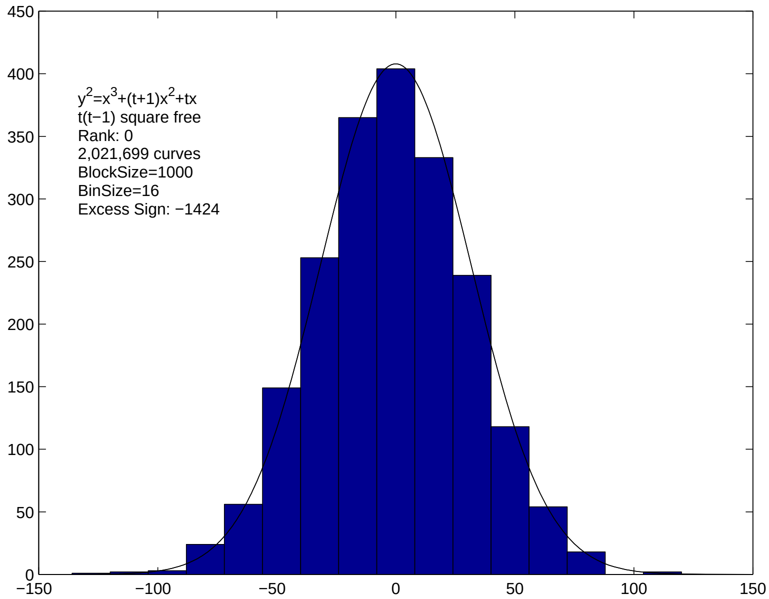
Polynomial Moebius *Let $f(t)$ be a non-constant polynomial such that no fixed square divides $f(t)$ for all t. Then $\sum_{t=N}^{2N} \mu(f(t)) = o(N)$.*

The Polynomial Moebius conjecture is known for linear $f(t)$.

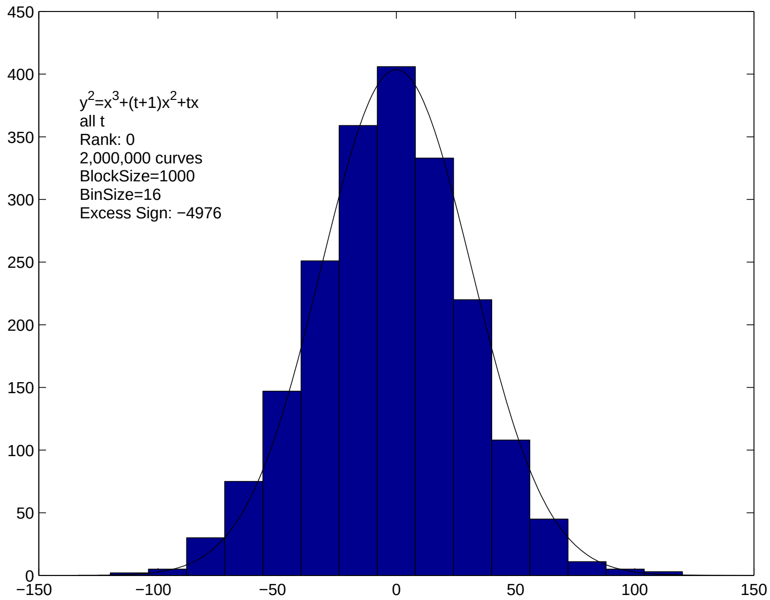
Helfgott shows the Square-Free Sieve and Polynomial Moebius imply the Restricted Sign conjecture for many families. More precisely, let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$.

Theorem: Equidistribution of Sign in a Family [He]: *Let $\mathcal{F}$ be a one-parameter family with $a_i(t) \in \mathbb{Z}[t]$. If $j(E_t)$ and $M(t)$ are non-constant, then the signs of E_t, $t \in [N, 2N]$, are equidistributed as $N \to \infty$. Further, if we restrict to good t, $t \in [N, 2N]$ such that $D(t)$ is good (usually square-free), the signs are still equidistributed in the limit.*

Distribution of Signs: $y^2 = x^3 + (t+1)x^2 + tx$

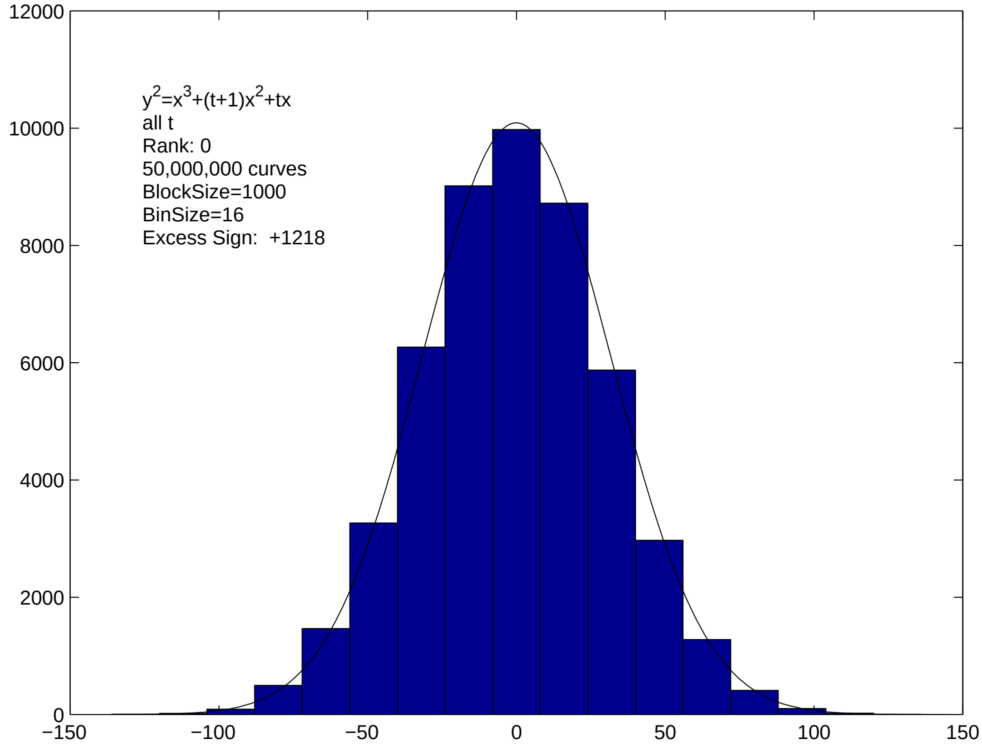


Histogram plot: $D(t)$ sq-free, first $2 \cdot 10^6$ such t.



Histogram plot: All $t \in [2, 2 \cdot 10^6]$.

Distribution of signs: $y^2 = x^3 + (t+1)x^2 + tx$



Histogram plot: All $t \in [2, 5 \cdot 10^7]$

The observed behavior agrees with the predicted behavior. Note as the number of curves increase (comparing the plot of $5 \cdot 10^7$ points to $2 \cdot 10^6$ points), the fit to the Gaussian improves.

Graphs by Atul Pokharel

Appendix III: Numerically Approximating Ranks: Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f,2-s), \quad \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

Algorithm for $L^r(s, E)$: I

$$\begin{aligned}\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_1^\infty f(iy/\sqrt{N})(y^{s-1} + \epsilon y^{1-s})dy.\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1} + \epsilon(-1)^k y^{1-s})dy.$$

At $s = 1$,

$$\Lambda^{(k)}(E,1) = (1 + \epsilon(-1)^k) \int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s, E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy \\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-r}}{n^r \cdot n!}x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t; \\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12}; \\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3}; \\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160}; \\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\,e^{-2\pi n y/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

Appendix IV: Bounding Excess Rank

$$D_{1,\mathcal{F}}(\phi_1) = \widehat{\phi}_1(0) + \frac{1}{2}\phi_1(0) + r\phi_1(0).$$

To estimate the percent with rank at least $r + R$, P_R, we get

$$R\phi_1(0)P_R \leq \widehat{\phi}_1(0) + \frac{1}{2}\phi_1(0), \;\; R > 1.$$

Note the family rank r has been cancelled from both sides.

The 2-level density gives ***squares*** of the rank on the left, get a cross term rR.

The disadvantage is our support is smaller.

Once R is large, the 2-level density yields better results. We now give more details.

n-Level Density and Excess Rank Bounds

For $n = 1$ and 2, consider the test functions

$$\begin{aligned}\widehat{f_i}(u) &= \frac{1}{2}\Big(\frac{1}{2}\sigma_n - \frac{1}{2}|u|\Big),\ |u| \le \sigma \\ f_i(x) &= \frac{\sin^2(2\pi\frac{1}{2}\sigma_n x)}{(2\pi x)^2}.\end{aligned}$$

Expect $\sigma_2 = \frac{\sigma_1}{2}$; only able to prove for $\sigma_2 = \frac{\sigma_1}{4}$.

Note $f_i(0) = \frac{\sigma_n^2}{4}$, $\widehat{f_i}(0) = f_i(0)\frac{1}{\sigma_n}$.

Assume B-SD, Equidistribution of Sign

Notation

Family with rank r, $D_{1,\mathcal{F}}(f) = \widehat{f}(0) + \frac{1}{2}f(0) + rf(0)$.

By even (odd) we mean a curve whose rank r_E has $r_E - r$ even (odd).

P_0: probability even curve has rank $\geq r + 2a_0$.

P_1: probability odd curve has rank $\geq r + 1 + 2b_0$.

$$D_{1,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_{\gamma_E} f\left(\frac{\log N_E}{2\pi}\gamma_E\right),$$

γ_E is the imaginary part of the zeros.

Average Rank: 1-Level Bounds

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E f(0) \ \le\ \widehat{f_1}(0) + \frac{1}{2}f_1(0) + r f_1(0)$$
$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E \ \le\ \frac{1}{\sigma_1} + \frac{1}{2} + r.$$

- All Curves: $r = 0$, $\sigma = \frac{4}{7}$, giving 2.25 (Brumer, Heath-Brown: [Br], [BHB3], [BHB5])
- 1-Parameter Families: $\left(\deg(N(t)) + r + \frac{1}{2}\right)\cdot(1 + o(1))$ (Silverman [Si3]).

Hope 1-Level Density true for $\sigma \to \infty$.

Would yield average rank is $r + \frac{1}{2}$.

Excess Rank: 1-Level Bounds

Assume half even, half odd.

Even curves: $1 - P_0$ have rank $\le r + 2a_0 - 2$; replace ranks with r. P_0 have rank $\ge r+2a_0$; replace with $r+2a_0$.

Odd curves: $1 - P_1$ contributing $r+1$. P_1 contributing $r + 1 + 2b_0$.

$$\begin{aligned} \frac{1}{\sigma_1} + \frac{1}{2} + r \ \ge \ & \frac{1}{2}\Big[(1 - P_0)r + P_0(r + 2a_0)\Big] \\ & + \frac{1}{2}\Big[(1 - P_1)(r + 1) + P_1(r + 1 + 2b_0)\Big] \\ \frac{1}{\sigma_1} \ \ge \ & a_0 P_0 + b_0 P_1. \end{aligned}$$

1-Level Density Bounds for Excess Rank

$$\begin{aligned} P_0 & \le \frac{1}{a_0\sigma_1} \\ P_1 & \le \frac{1}{b_0\sigma_1} \\ \text{Prob}\{\text{rank} \ge r + 2a_0\} & \le \frac{1}{a_0\sigma_1}. \end{aligned}$$

2-Level Bounds:

$$
\begin{aligned}
D_{2,\mathcal{F}}(f) &= D^*_{2,\mathcal{F}}(f) - 2D_{1,\mathcal{F}}(f_1 f_2) + f_1(0) f_2(0) N(\mathcal{F}, -1) \\
D^*_{2,\mathcal{F}}(f) &= \prod_{i=1}^{2} \left[\widehat{f_i}(0) + \frac{1}{2} f_i(0) \right] + 2 \int |u| \widehat{f_1}(u) \widehat{f_2}(u) du \\
& \quad + r \widehat{f_1}(0) f_2(0) + r f_1(0) \widehat{f_2}(0) + (r^2 + r) f_1(0) f_2(0) \\
D_{1,\mathcal{F}}(f) &= \widehat{f}(0) + \frac{1}{2} f(0) + r f(0).
\end{aligned}
$$

$D^*_{2,\mathcal{F}}(f)$ is over all zeros. Gives

$$
\begin{aligned}
\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} r_E^2 &\leq \frac{1}{\sigma_2^2} + \frac{1}{\sigma_2} + \frac{1}{4} + \frac{1}{3} + \frac{2r}{\sigma_2} + r^2 + r \\
&= \frac{1}{\sigma_2^2} + \frac{2r + 1}{\sigma_2} + \frac{1}{12} + r^2 + r + \frac{1}{2}.
\end{aligned}
$$

Excess Rank: 2-Level Bounds: I

Similar proof yields

Theorem: First 2-Level Density Bounds

$$P_0 \le \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r+\frac{1}{2}}{\sigma_2}}{a_0(a_0+r)}$$

$$P_1 \le \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r+\frac{1}{2}}{\sigma_2}}{b_0(b_0+r+1)}.$$

For $\sigma_2 = \frac{\sigma_1}{4}$, $r = 0$, $a_1 = 1$: ***worse*** than 1-level density.

For fixed $\sigma_2 = \frac{\sigma_1}{4}$ and r, as we increase a_0 we eventually do get a better bound.

Proportional to $\frac{1}{(a_0\sigma_1)^2}$ instead of $\frac{1}{a_0\sigma_1}$.

Excess Rank: 2-Level Bounds: II

Use $D_{2,\mathcal{F}}(f)$ instead of $D^*_{2,\mathcal{F}}(f)$.

r_E = number of zeros of curve E. Sum over $j_1 \neq j_2$.

r_E even, get $r_E(r_E - 2)$ (each zero matched with $r_E - 2$ others).

r_E odd: $(r_E - 1)(r_E - 2) + (r_E - 1) = r_E(r_E - 2) + 1$.

Theorem: Second 2-Level Density Bounds

$$P_0 \leq \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r}{\sigma_2} - \frac{1}{6\sigma_2}}{a_0(a_0 + r - 1)}$$
$$P_1 \leq \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r}{\sigma_2} - \frac{1}{6\sigma_2}}{b_0(b_0 + r)},$$

where $a_0 \neq 1$ if $r = 0$.

$\sigma_2 = \frac{\sigma_1}{4}$ and $r = 0$, better for $a_0 > \frac{\sigma_1^2 + 8\sigma_1 + 192}{24\sigma_1}$.
$r = 1$, better for $a_0 > \frac{\sigma_1^2 + 80\sigma_1 + 192}{24\sigma_1}$.

Decay is proportional to $\frac{1}{(a_0\sigma_1)^2}$.
Note the numerator is never negative; at least $\frac{1}{18}$.

Excess Rank: 2-Level Bounds: IIIa

$$r_E \;=\; r + z_E.$$

$\sum_{j_1} \sum_{j_2} f_1(L\gamma_{E_{j_1}}) f_2(L\gamma_{E_{j_2}})$. Let j_1 be one of the r family zeros, varying j_2 gives $f_1(0)D_{1,E}(f_2)$. Interchanging j_1 and j_2 we get a contribution of $D_{1,E}(f_1)f_2(0)$ for each of the r family.

Only double counting when j_1 and j_2 are both a family zero. Subtract off $r^2 f_1(0)f_2(0)$.

For the other z_E zeros: already taken into account contribution from j_1 one of the z_E zeros and j_2 one of the r family zeros (and vice-versa).

Thus, for a given curve, a lower bound of the contribution from all pairs (j_1, j_2) is

$$r f_1(0)D_{1,E}(f_2) + rD_{1,E}(f_1)f_2(0) - r^2 f_1(0)f_2(0) + z_E^2.$$

Excess Rank: 2-Level Bounds: IIIb

Summing over all $E \in \mathcal{F}$ and simplifying gives

$$\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} z_E^2 \ \leq \ \frac{1}{\sigma_2^2} + \frac{1}{\sigma_2} + \frac{1}{12} + \frac{1}{2}.$$

Similar calculation gives

Theorem: Third 2-Level Density Bounds

$$P_0 \ \leq \ \frac{\frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_2} + \frac{1}{24}}{a_0^2}$$
$$P_1 \ \leq \ \frac{\frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_2} + \frac{1}{24}}{b_0 + b_0^2}$$

$\sigma_2 = \frac{\sigma_1}{4}$: beats 1-level for $a_0 > \frac{\sigma_1^2 + 48\sigma_1 + 192}{24\sigma_1}$.
$r \neq 0$: beats first 2-level once $a_0 > \frac{\sigma_1^2 + 48\sigma_1 + 192}{96\sigma_1}$.

$r \geq 1$: beats second 2-level once $a_0 > \frac{3(r-1)}{3r-2} \frac{\sigma_1^2 + 48\sigma_1 + 192}{96\sigma_1}$.

Heath-Brown & Brumer

Family of all elliptic curves $E_{a,b}$:

$$\mathcal{F}_T = \{y^2 = x^3 + ax + b; |a| \leq T^{\frac{1}{3}}, |b| \leq T^{\frac{1}{2}}.$$

From 1-Level Expansion, get

$$r(E_{a,b}) \leq 2 + \frac{\log T}{\log X} - 2 \sum_{p \leq X} a_P(E_{a,b}) h\left(\frac{\log p}{\log X}\right) + O\left(\frac{1}{\log X}\right).$$

If $r(E_{a,b}) \geq r \geq 3 + 2\frac{\log T}{\log X}$, then $|U(E_{a,b}, X)| \geq \frac{\log T}{2}$.

Led to

$$\#\{E_{a,b} \in \mathcal{F}_T : r(E_{a,b}) \geq r\} \cdot \left(\frac{\log T}{2}\right)^{2k} \leq \sum_{E_{a,b} \in \mathcal{F}} |U(E_{a,b}, X)|^{2k}.$$

Find $X = T^{\frac{1}{10k}}$, $k = \left[\frac{r-3}{20}\right]$. Yields

$$\begin{aligned} \text{Prob}\,(\text{rank}(E_{a,b}) \geq r) &\ll (11r)^{-\frac{r}{20}} \\ \text{rank}(E_{a,b}) &\leq 17\frac{\log T}{\log\log T}. \end{aligned}$$

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