VC Dimension and Distance Chains in $\mathbb{F}_q^d$

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Notation

- We let $\mathbb{F}_q^d$ denote the d-dimensional vector space over the finite field $\mathbb{F}_q$.
- We write $||x|| = x_1^2 + x_2^2 ... \in \mathbb{F}_q$, where $x = (x_1, ..., x_d) \in \mathbb{F}_q^d$.
- For the remainder of the talk we let t be a fixed, nonzero element of the finite field $\mathbb{F}_q$ (i.e. this is not a variable).

VC-Dimension

Fix a set E and a collection of functions (hypothesis class) $\mathcal{H}$ from E to $\{0, 1\}$.

Definition

Say $\mathcal{H}$ *shatters* a finite subset $A \subset E$ if restricting functions in $\mathcal{H}$ to A yields all $2^{|A|}$ functions from A to $\{0, 1\}$.

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Definition (Vapnik and Chervonenkis, 1968)

The *Vapnik–Chervonenkis dimension* (VCD) of $\mathcal{H}$ is the maximal cardinality of sets $A \subset E$ that are shattered by $\mathcal{H}$.

Explicitly, $VCD(\mathcal{H}) = n$ if there exists $A \subset E$, $|A| = n$ such that A is shattered by $\mathcal{H}$, but there is no such subset of size $n + 1$.

Prior Work: Spheres in $\mathbb{F}_q^2$

Spheres in $\mathbb{F}_q^2$: Fix $t \neq 0$. For $E \subset \mathbb{F}_q^2$, Fitzpatrick, Iosevich, McDonald, and Wyman defined the class of functions $\mathcal{H}_t^2(E) = \{h_y : y \in E\}$, where $h_y : E \to \{0,1\}$ is the indicator function for the sphere of radius t centered at y:

$$h_y(x) = \begin{cases} 1 & ||y - x|| = t \\ 0 & ||y - x|| \neq t. \end{cases}$$

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Theorem (Fitzpatrick, Iosevich, McDonald, and Wyman, 2021)

If $|E| \geq Cq^{15/8}$ for C large, then $VCD\left(\mathcal{H}_t^2(E)\right) = 3$, the largest possible value.

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One can define the analogous hypothesis class $\mathcal{H}_t^d$ for higher dimensions. However no analogous theorem is known beyond $d = 2$.

New Classifiers: Chains in $\mathbb{F}_q^d$

2-Chains in $\mathbb{F}_q^d$: Fix $t \neq 0$, $d \geq 3$, and $E \subset \mathbb{F}_q^d$. Define the collection of functions $\mathcal{H}_t^d(E) = \{h_{y,z} : y, z \in E, y \neq z\}$, where $h_{y,z} : E \to \{0, 1\}$ is the indicator function for the intersection of spheres of radius t centered at y and z:

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If $|E| \geq \begin{cases} Cq^{\frac{7}{4}} & d = 2 \\ Cq^{\frac{7}{3}} & d = 3 \\ Cq^{d-\frac{1}{d-1}} & d \geq 4 \end{cases}$

Where C depends on d but not q, then $VCD\left(\mathcal{H}_t^d(E)\right) = d$, the largest possible value.

Chains in $\mathbb{F}_q^d$ (cont.)

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- Thus each $h_{y,z}$ takes on the value 1 at most d times, and so $VCD(\mathcal{H}_t^d) \leq d$. And so it only remains to establish the lower bound.

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- Finally, we only consider the $d \geq 3$ case here: the $d = 2$ case follows immediately from the techniques in (Fitzpatrick, et al., 2021)

Definitions

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The $(d+2)$-tuple $P = (y, z, x^1, \ldots, x^d) \in (\mathbb{F}_q^d)^{d+2}$ is an *d-prism* if $y, z \subset \text{Pole}(\{x^1, x^2, ..., x^d\})$. The superscripts here should be read as indices.

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Prisms: Distance Graph

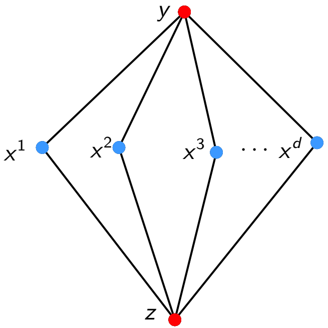

Figure: A nondegenerate d-prism $P = (y, z, x^1, \ldots, x^d)$ with tail $\mathcal{T}(P) = \{y, z\}$ and center $\mathcal{C}(P) = \{x^1, \ldots, x^d\}$.

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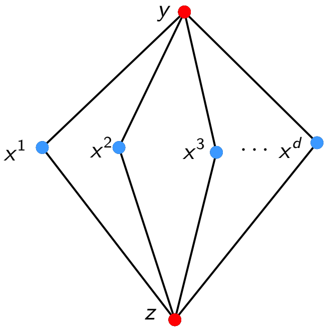

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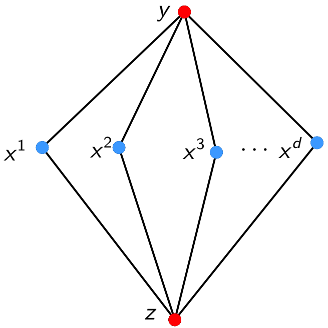

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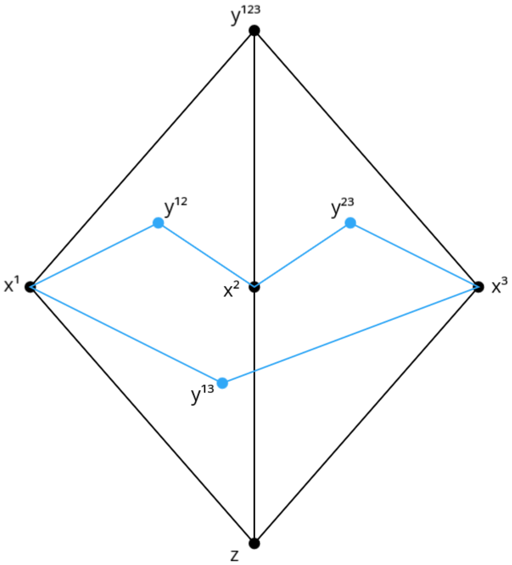
Henceforth, the term prism refers to a nondegenerate d-prism.

Using Prisms to Shatter

Observation

Let P be a prism, and z be one of its tails. Suppose that for every subset $A \subset \mathcal{C}(P)$ we can find a point $y_A \in \text{Pole}(A)$ that isn't a pole of any point in $\mathcal{C}(P) \setminus A$. Then by choosing the classifiers $\{h_{y_A,z}\}$ we can shatter $\mathcal{C}(P)$, a set of size d.

Example of Shattering



Bad Sets

Definition (SMALL 2022++)

Fix a prism P. A subset $A \subset \mathcal{C}(P)$ is P-bad if every $x \in \text{Pole}(A)$ is also in $\text{Pole}(y)$ for some $y \in \mathcal{C} \setminus A$. We say that a set is bad of if it is P-bad for some P. We say that P *admits a bad set* if some $A \subset \mathcal{C}$ is P-bad.

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To shatter d points, it suffices to find a prism that admits no bad sets.

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Note that there are two ways for a set $A \subset \mathcal{C}(P)$ to be P-bad:

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The latter case is impossible for pairs in $d = 3$, leading to a simpler proof and a stronger bound.

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- Count the number of prisms that can admit a bad set by counting the number of prisms a set can be bad in.
- Show that if $|E|$ is large, there must be some prism for which no subset of its center is bad.

Prisms, Pairs, and 2-Paths

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Up to ordering a prism corresponds to a choice of pair (y, z) and a choice of d 2-paths between them—(y, z) are the tails and the set of midpoints of these paths is the center.

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Theorem (Iosevich et al., 2018)

Let $E \subset \mathbb{F}_q^d$ with $|E| > \frac{4}{\ln(2)} q^{\frac{d+1}{2}}$ and let $\Gamma_2(E)$ be the number of 2-paths in E. Then $\Gamma_2(E) = \frac{|E|^3}{q^2} + \mathcal{D}_2(E)$ where $\mathcal{D}_2(E) \leq C \frac{|E|^2}{q}$.

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$$N_d(E) = d! \sum_{(x,y)\in E\times E} \binom{k_{(x,y)}}{d}$$

We also have

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- From here it is straightforward to apply Hölder's Inequality (proof omitted) to obtain $N_d(E) \geq C\frac{|E|^{d+2}}{q^{2d}}$

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- Now observe that there are only $1/2|E|^2$ pairs total, and so at most $C|E|^2q$ prisms admit a P-bad pair. But we just showed there are at least $|E|^5q^{-6}$ prisms total so if $E > Cq^{7/3}$, one must admit no bad pairs.

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- Finally, note that we can specify the singleton sets since $\{x, y\} \cap \{x, z\} = \{x\}$. Thus we obtain our result.

Lemmas

Lemma 1

If $|E| \geq Cq^{d-\frac{1}{d-1}}$ then a positive proportion of prisms have affinely independent centers.

Lemma 2

Suppose the k points $a_i \in \mathbb{F}_q^d$, are affinely independent. Then $\text{Pole}(\{a_i\}) \leq 2q^{d-k}$

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- Since there are $C|E|^k$ sets of size k, this gives us the bound $|E| > Cq^{d-\frac{2}{3}}$. However this is weaker than the $|E| \geq Cq^{1-\frac{1}{d-1}}$ bound required for Lemma 1.

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- There are $\approx k_{(x,y)}^{d-1}$ ways to choose the first $d - 1$ points, and $\leq Cq^{d-3}$ ways to choose the last point.

Proof of Lemma 1

Lemma 1

If $|E| \geq Cq^{d-\frac{1}{d-1}}$ then a positive proportion of prisms have affinely independent centers.

Sketch of Proof

- We show the equivalent statement that the proportion of prisms with affinely dependent centers is less than 1.
- Given a pair $(x, y) \in E \times E$, any such prism P with $\mathcal{T}(P) = (x, y)$ can be chosen by choosing $d - 1$ center points, arbitrarily, then choosing the last point to be in the affine subspace generated by those points.
- There are $\approx k_{(x,y)}^{d-1}$ ways to choose the first $d - 1$ points, and $\leq Cq^{d-3}$ ways to choose the last point.
- From here, we can apply similar techniques as in our prism count to obtain our result.

Proof of Lemma 2

Lemma 2

Lemma 2: Suppose the k points $a_i \in \mathbb{F}_q^d$, are affinely independent. Then $\text{Pole}(\{a_i\}) \leq 2q^{d-k}$

Sketch of Proof

Proof of Lemma 2

Lemma 2

Lemma 2: Suppose the k points $a_i \in \mathbb{F}_q^d$, are affinely independent. Then $\text{Pole}(\{a_i\}) \leq 2q^{d-k}$

Sketch of Proof

- The condition that $y \in \text{Pole}(\{a_i\})$ corresponds to the conditions $y \in E$ and $\forall i : ||y - a_i|| = t$.

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- The condition that $y \in \text{Pole}(\{a_i\})$ corresponds to the conditions $y \in E$ and $\forall i : ||y - a_i|| = t$.
- Let $x = y - a_1$ and $a_i' = a_i - a_1$. Then ignoring the first condition, this reduces to a linear system of equations in the coordinates of x.
- Affine independence means the linear system has full rank, so its solution space has dimension $d - k + 1$. The restriction $||x|| = t$ reduces this to $\approx q^{d-k}$ solutions.

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