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FROM FIBONACCI NUMBERS TO CENTRAL LIMIT TYPE THEOREMS

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Abstract

A beautiful theorem of Zeckendorf states that every integer can be written uniquely as a sum of non-consecutive Fibonacci numbers $\{F_n\}_{n=1}^{\infty}$. Lekkerkerker proved that the average number of summands for integers in $[F_n, F_{n+1})$ is $n/(\varphi^2+1)$, with φ the golden mean. We prove the following massive generalization: given nonnegative integers $c_1, c_2, \ldots, c_L$ with $c_1, c_L > 0$ and recursive sequence $\{H_n\}_{n=1}^{\infty}$ with $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ $(n \geq L)$,$H_1 = 1$ and $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1$ $(1 \leq n < L)$, every positive integer can be written uniquely as $\sum a_i H_i$ under natural constraints on the a_i's, the mean and the variance of the numbers of summands for integers in $[H_n, H_{n+1})$ are of size n, and the distribution of the numbers of summands converges to a Gaussian as n goes to the infinity. Previous approaches were number theoretic, involving continued fractions, and were limited to results on existence and, in some cases, the mean. By recasting as a combinatorial problem and using generating functions and differentiating identities, we surmount the limitations inherent in the previous approaches.

Our method generalizes to a multitude of other problems. For example, every integer can be written uniquely as a sum of the $\pm F_n$'s, such that every two terms of the same (opposite) sign differ in index by at least 4 (3). We prove similar results as above; for instance, the distribution of the numbers of positive and negative summands converges to a bivariate normal with computable, negative correlation, namely $-(21-2\varphi)/(29+2\varphi)$.

Contents

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1. Introduction

1.1. History.

1.2. Main Results.

[Not only we prove the Gaussian behavior of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ as n goes to infinity, we also extend Zeckendorf's Theorem, Lekkerkerker's Theorem and Gaussian behavior for a large class of recursive sequences defined as follows.]

Definition 1.1. *We say a sequence $\{H_n\}_{n=1}^{\infty}$ of positive integers is a* ***good recurrence relation*** *if the following properties hold:*

- Recurrence relation: *There are non-negative integers $L, c_1, \dots, c_L$ such that*
$$H_{n+1} \;=\; c_1 H_n + \cdots + c_L H_{n+1-L},$$
with L, c_1 and c_L positive.
- Initial conditions: *$H_1 = 1$, and for $1 \leq n < L$ we have*
$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1.$$

We call a decomposition $\sum_{i=1}^{m} a_i H_{m+1-i}$ of a positive integer N (and the sequence $\{a_i\}_{i=1}^{m}$) ***legal*** *if $a_1 > 0$, the other $a_i \geq 0$, and one of the following two conditions holds:*
Condition 1. *We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.*
Condition 2. *There exists $s \in \{0, \dots, L\}$ such that*
$$a_1 = c_1,\ a_2 = c_2,\ \cdots,\ a_{s-1} = c_{s-1} \text{ and } a_s < c_s, \tag{1.1}$$
$a_{s+1}, \dots, a_{s+\ell} = 0$ for some $\ell > 0$, and $\{b_i\}_{i=1}^{m-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is legal.

We define the unique legal decomposition of $N = 0$ to be the empty one, i.e., $m = 0$. If $\sum_{i=1}^{m} a_i H_{m+1-i}$ is a legal decomposition of N, we define the ***number of summands*** *(of this decomposition of N) to be $a_1 + \cdots + a_m$. Our first result is the following.*

Theorem 1.2. (Generalized Zeckendorf's Theorem) *If $\{H_n\}_{n=1}^{\infty}$ is a* good recurrence relation, *then*
(a) *There is a unique legal decomposition for each integer $N \geq 0$.*
(b) *There is a bijection between the set S_n of integers in $[H_n, H_{n+1})$ and the set D_n of legal decompositions $\sum_{i=1}^{n} a_i H_{n+1-i}$.*

We also prove the generalized Lekkerkerker's Theorem for good recurrence relation $\{H_n\}$. We need set some definitions before formally stating the theorem. Define $p_{n,k}$ as the number of integers in $[H_n, H_{n+1})$ that have exactly k summands in their legal decompositions, and let K_n be the random variable associated to k for integers in $[H_n, H_{n+1})$. Then the probability of having an exact k-summand legal decomposition for an integer in $[H_n, H_{n+1})$ is $\text{Prob}(n,k) := p_{n,k}/\Delta_n$ where $\Delta_n := H_{n+1} - H_n$.

Theorem 1.3. (Generalized Lekkerkerker's Theorem) *Let μ_n be the mean of K_n, then as $n \to \infty$,*

$$\mu_n = Cn + d + o(1), \tag{1.2}$$

where C and d are constants depending only on L and the c_i's.

A natural question to ask is how the number of summands are distributed. We prove that it is a Gaussian.

Theorem 1.4. *As $n \to \infty$, the distribution of K_n converges to a Gaussian.*

Our method generalizes to a multitude of other problems. For example, the analogue of Zeckendorf's Theorem was recently proved for the far-difference representation defined below.

Definition 1.5. *We call a sum of the $\pm F_n$'s a* ***far-difference representation*** *if every two terms of the same sign differ in index by at least 4, and every two terms of opposite sign differ in index by at least 3.*

Alpert [1] proved the following.

Theorem 1.6. (Analogue of Zeckendorf's Theorem) *Every integer has a unique far-difference representation. Further, if $S_n = \sum_{0<n-4i\leq n} F_{n-4i}$ for positive n and 0 otherwise, then for each $N \in (S_{n-1} = F_n - S_{n-3} - 1, S_n]$, the first term in its far-difference representation is F_n. Note that the unique far-difference representation of 0 is the empty representation.*

We prove the Lekkerkerker's Theorem and Gaussian behavior for far-difference representation, stated as follows.

Theorem 1.7. *Let $\mathcal{K}_n$ and $\mathcal{L}_n$ be the corresponding random variables denoting the number of positive summands and the number of negative summands in the far-difference representation for integers in $(S_{n-1}, S_n]$. As n goes to infinity, the expected value of $\mathcal{K}_n$, denoted by $\mathbb{E}[\mathcal{K}_n]$, is $\frac{1}{10}n + \frac{371-113\sqrt{5}}{40}$ and one greater than $\mathbb{E}[\mathcal{L}_n]$; the variance of both is of size $\frac{15+21\sqrt{5}}{1000}n$; the joint density of $\mathcal{K}_n$ and $\mathcal{L}_n$ is a bivariate Gaussian with negative correlation $\frac{10\sqrt{5}-121}{179} = -\frac{21-2\varphi}{29+2\varphi} \approx -0.551$; and $\mathcal{K}_n + \mathcal{L}_n$ and $\mathcal{K}_n - \mathcal{L}_n$ are independent.*

1.3. **Approach.**

Previous investigations in Lekkerkerker's Theorem were number theoretic, involving continued fractions, and were limited to results in some special cases, e.g., the Fibonacci numbers, and on the mean. By recasting as a combinatorial problem and using generating functions, we surmount the limitations inherent in the previous approaches. The key techniques in our proof are generating functions, partial fractional expansions, differentiating identities and the method of moments.

We look at the special case of the Fibonacci numbers, as this highlights the main ideas of the method ... of the technicalities.

We first derive a recurrence relation for the $p_{n,k}$'s, which is $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$ in this case. Multiplying both sides of this equation by $x^k y^n$, summing over $n, k > 0$, and calculating the initial values of the $p_{n,k}$'s, namely $p_{1,1}$, $p_{2,1}$ and $p_{2,2}$, we can obtain a formula for the generating function $\sum_{n,k>0} p_{n,k}x^k y^n$:

$$g(x) := \sum_{n,k>0} p_{n,k}x^k y^n = \frac{y}{1-y-xy^2}. \tag{1.3}$$

By partial fraction expansion, we expand the right-hand side as

$$-\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)}-\frac{1}{y-y_2(x)}\right),$$

where $y_1(x)$ and $y_2(x)$ are the roots of $1-y-xy^2=0$. Rewriting $\frac{1}{y-y_i(x)}$ as $-\frac{1}{1-\frac{y}{y_i(x)}}$ and using a power series expansion, we are able to compare the coefficients of y^n of both sides of (1.3). This gives an explicit closed formula for $g(x) = \sum_{n,k>0} p_{n,k}x^k$.

Note that $g(1) = \sum_{k>0} p_{n,k}$ which is $F_{n+1} - F_n$ by the definition. Further, we have $g'(1) = \sum_{n,k>0} kp_{n,k} = \mathbb{E}[K_n](F_{n+1} - F_n) = \mathbb{E}[K_n]g(1)$. Therefore, once we determine $g(1)$ and $g'(1)$, we know $\mathbb{E}[K_n]$.

Given the value of $\mathbb{E}[K_n]$ (denoted by μ_n), we let $h(x) = x^{-\mu_n}g(x)$ and random variable $K_n' = K_n - \mu_n$. Then we have an explicit and closed formula for $h(x)$ and similarly, we have $h(1) = F_{n+1} - F_n$ and $h'(1) = \mathbb{E}[K_n']h(1)$. Furthermore, we get $(xh'(x))' = \mathbb{E}[{K_n'}^2]h(1)$, $\left(x\left(xh'(x)\right)'\right)' = \mathbb{E}[{K_n'}^3]h(1), \dots$. Thus we can compute the moments of K_n'.

To show that K_n is Gaussian, it suffices to show that the normalized K_n, namely $K_n'/\sigma(K_n')$ is Gaussian, where $\sigma(K_n')$ is the standard deviation of K_n'. By method of moments, we only need to verify that the moments of $K_n'/\sigma(K_n')$ tends to the those of the standard normal distribution as $n \to \infty$, which are known as $(2m-1)!!$ for the $2m^{\text{th}}$ moment and 0 for the $(2m-1)^{\text{th}}$ moment. This is tractable since we have the formula for the moments of K_n' and therefore for $K_n'/\sigma(K_n')$ as well.

We begin the paper by generalizing Zeckendorf's Theorem in Section 2. In section 3, we derive the formula for the generating function of the probability density. Then we prove the generalized Lekerkerker's Theorem in Section 4 and the Gaussian behavior in Section 5. Finally, we prove the results for the far-difference representation.

2. Proof of Theorem 1.2 (Generalized Zeckendorf)

We need the following lemma about the legality in our proof.

Lemma 2.1. *For $m \geq 1$, if $N = \sum_{i=1}^{m} a_{m+1-i}H_i$ is legal, then $N < H_{m+1}$.*

Proof. We proceed by induction on m.

When $m = 1$, $N = a_1H_1 = a_1 \leq c_1 < H_2$.

Suppose the lemma holds for any $m' < m$ ($m \geq 2$). From Definition 1.1, we see that there exists $1 \leq j \leq L$ such that $a_j < c_j$. Let j be the smallest number such that $a_j < c_j$. Since $\sum_{i=1}^{m-j-\ell+1} a_{m+1-i}H_i$ is legal for some $\ell > 0$, by the induction hypothesis

$$\sum_{i=1}^{m-j} a_{m+1-i}H_i = \sum_{i=1}^{m-j-\ell+1} a_{m+1-i}H_i < H_{m+1-j}.$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{m} a_{m+1-i}H_i &= \sum_{i=1}^{m-j} a_{m+1-i}H_i + \sum_{i=m-j+1}^{m} a_{m+1-i}H_i \\
&= \sum_{i=1}^{m-j} a_{m+1-i}H_i + a_j H_{m+1-j} + \sum_{i=1}^{j-1} c_i H_{m+1-i} \\
&< H_{m+1-j} + (c_j-1)H_{m+1-j} + \sum_{i=1}^{j-1} c_i H_{m+1-i} \\
&= \sum_{i=1}^{j} c_i H_{m+1-i} \le \sum_{i=1}^{L} c_i H_{m+1-i} \le H_{m+1},
\end{aligned}$$

where the last inequality comes from (2.1). □

The following result immediately follows from Lemma 2.1.

Corollary 2.2. *If $N \in [H_n, H_{n+1})$, then the legal decomposition of N must be of the form $\sum a_i H_{n+1-i}$ with $a_1 > 0$.*

Let us return to the proof of Theorem 1.2.

The case of $L = 1$ is clearly true, since the legal decomposition is just the base c_1 decomposition.

Assume that $L \ge 2$. Define $H_i = 0$ for $i < 1$, then for $1 \le n < L$,

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1} + 1.$$

Hence by Definition 1.1, for any $n \ge 1$, we have

$$\begin{aligned}
&c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1} \\
&\le H_{n+1} \le c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1} + 1. \qquad (2.1)
\end{aligned}$$

We call a legal decomposition *Type 1* if it satisfies Condition 1 in Definition 1.1 and *Type 2* if it satisfies Condition 2. Note that Conditions 1 and 2 cannot hold at the same time.

If $N = 0$, then it has a unique decomposition by the definition.

Note to prove Theorem 1.2(a), it suffices to show that there is a unique legal decomposition for every integer $N \in [H_n, H_{n+1})$ for all n. We proceed by induction on n.

For $n = 1$, recall the definitions that $H_1 = 1$ and $H_2 = 1 + c_1$. For any $N \in [H_1, H_2) = [1, 1 + c_1)$,

$$N = N \cdot 1 = N \cdot H_1. \qquad (2.2)$$

Since $0 < N \le c_1$, (2.2) is a legal decomposition of N. On the other hand, since $N < H_2$, (2.2) is the only legal decomposition of N. Therefore, there is a unique legal decomposition for every integer $N \in [H_1, H_2)$.

Assume that the statement holds for any $n' < n$ ($n \ge 2$). We first prove the existence for $N \in [H_n, H_{n+1})$.

If $n \ge L$, then $N < H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}$. Thus there exists a unique $s \in \{0, \dots, L-1\}$ such that

$$c_1 H_n + c_2 H_{n-1} + \cdots + c_s H_{n-s+1} \le N < c_1 H_n + c_2 H_{n-1} + \cdots + c_{s+1} H_{n-s} \qquad (2.3)$$

(if $s = 0$, then the left-hand side is zero.)

Let a_{s+1} be the unique integer such that

$$a_{s+1}H_{n-s} \leq N - \sum_{i=1}^{s} c_i H_{n-i+1} < (a_{s+1}+1)H_{n-s},$$

then $a_{s+1} < c_{s+1}$ and

$$N' := N - \sum_{i=1}^{s} c_i H_{n-i+1} - a_{s+1}H_{n-s} < H_{n-s}.$$

By the induction hypothesis, there exists a unique legal decomposition $\sum_{i=1}^{m} b_i H_{m+1-i}$ ($m < n-s$) of N'. Hence

$$\sum_{i=1}^{s} c_i H_{n-i+1} + a_{s+1}H_{n-s} + \sum_{i=1}^{m} b_i H_{m+1-i}$$

is a legal decomposition of N.

If $n < L$ and there exists s satisfying (2.3), then we can prove the existence in the same way. If there does not exist such s, then since $N < H_{n+1} = c_1H_n + c_2H_{n-1} + \cdots + c_nH_1 + 1$, i.e., $N \leq c_1H_n + c_2H_{n-1} + \cdots + c_nH_1$, the equality must be achieved. Thus $\sum_{i=1}^{n} c_i H_{n-i+1}$ is a legal decomposition of N as $n < L$. This completes the proof of existence.

We prove the uniqueness by contradiction. Assume there exists two distinct legal decompositions of N: $\sum_{i=1}^{m} a_i H_{m+1-i}$ and $\sum_{i=1}^{m'} a'_i H_{m'+1-i}$.

First, since $0 < H_n \leq N < H_{n+1}$, we have $m, m' \leq n$. On the other hand, by Lemma 2.1 we have $m, m' \geq n$. Hence $m = m' = n$.

We have three cases in terms of the types of the above two decompositions.

Case 1. If both decompositions are of Type 1, i.e., satisfy Condition 1, then they are the same since $m = m'$.

Case 2. If both decompositions are of Type 2, let s and s' be the corresponding integers that satisfy Condition 2. We want to show that $s = s'$. Otherwise, we assume $s > s'$ without loss of generality. Thus $a_i = c_i$ ($1 \leq i < s$), $a_{s'} < c_{s'}$, $a'_i = c_i$ ($1 \leq i < s'$), $\sum_{i=s+\ell}^{n} a_i H_{n+1-i}$ and $\sum_{i=s'+\ell'}^{n} a'_i H_{n+1-i}$ are legal for some positive ℓ and ℓ'. By Lemma 2.1, we have $\sum_{i=s'+1}^{n} a'_i H_{n+1-i} = \sum_{i=s'+\ell'}^{n} a'_i H_{n+1-i} < H_{n-s'+1}$, thus

$$\begin{aligned}
\sum_{i=1}^{s-1} c_i H_{n+1-i} &\leq \sum_{i=1}^{n} a_i H_{n+1-i} = N = \sum_{i=1}^{n} a'_i H_{n+1-i} \\
&\leq \sum_{i=1}^{s'-1} c_i H_{n+1-i} + (c_{s'}-1)H_{n-s'+1} + \sum_{i=s'+1}^{n} a'_i H_{n+1-i} \\
&< \sum_{i=1}^{s'-1} c_i H_{n+1-i} + (c_{s'}-1)H_{n-s'+1} + H_{n-s'+1} \\
&= \sum_{i=1}^{s'} c_i H_{n+1-i} \leq \sum_{i=1}^{s} c_i H_{n+1-i}, \qquad (2.4)
\end{aligned}$$

contradiction. Hence $s = s'$. As a result, $a_i = c_i = a'_i$ $(1 \le i < s)$. Thus

$$a_s H_{n-s+1} + \sum_{i=s+\ell}^{n} a_i H_{n+1-i} = a'_s H_{n-s+1} + \sum_{i=s+\ell'}^{n} a'_i H_{n+1-i}. \tag{2.5}$$

Since $\sum_{i=s+\ell}^{n} a_i H_{n+1-i}$ and $\sum_{i=s+\ell'}^{n} a'_i H_{n+1-i}$ are legal, they are less than H_{n-s+1} by Lemma 2.1. Let N'' be the value of both sides of (2.5), then there exist unique integers $q \ge 0$ and $r \in [0, H_{n-s+1})$, such that $N'' = qH_{n-s+1} + r$. Therefore $a_s = q = a'_s$ and

$$\sum_{i=s+\ell}^{n} a_i H_{n+1-i} = r = \sum_{i=s+\ell'}^{n} a'_i H_{n+1-i}.$$

Since $r < H_{n-s+1}$, there is a unique legal decomposition of r. Hence $a_i = a'_i$ $(s+1 \le i \le n)$. Thus we have $a_i = a'_i$ for any i, which leads to a contradiction that the two decompositions of N are equal.

Case 3. If one of the decompositions is of Type 1 and the other one is of Type 2, without loss of generality we can assume that $\sum_{i=1}^{n} a'_i H_{n+1-i}$ is of Type 1 and $\sum_{i=1}^{n} a_i H_{n+1-i}$ is of Type 2 with the corresponding s satisfying (1.1). From (2.4), we see that

$$\sum_{i=1}^{n} a_i H_{n+1-i} < \sum_{i=1}^{s} c_i H_{n+1-i} \le \sum_{i=1}^{n} c_i H_{n+1-i} = N,$$

contradiction. This completes the proof of (a).

For (b), in the proof of (a) we showed that each N has a unique legal decomposition of the form $\sum_{i=1}^{n} a_i H_{n+1-i}$, which induces an injective map σ from S_n to D_n.

On the other hand, by Lemma 2.1, $H_n \le \sum_{i=1}^{n} a_i H_{n+1-i} < H_{n+1}$, therefore $|D_n| \le H_{n+1} - H_n = |S_n|$. Hence σ is a bijective map.

3. Generating Function of the Probability Density

By Theorem 1.2(b), $p_{n,k}$ is exactly the number of legal k-summand decompositions of the form $\sum_{i=1}^{n} a_i H_{n+1-i}$, i.e., $k = a_1 + a_2 + \cdots + a_n$, with $a_1 > 0$. In this section, we will derive a recurrence relation for the $p_{n,k}$'s and then get the following formula for the generating function $\mathscr{G}(x,y) = \sum_{n,k>0} p_{n,k} x^k y^n$.

Proposition 3.1. *The generating function $\mathscr{G}(x,y) = \sum_{n,k>0} p_{n,k} x^k y^n$ is of the form:*

$$\mathscr{G}(x,y) = \frac{\mathscr{B}(x,y)}{\mathscr{A}(x,y)},$$

where

$$\mathscr{A}(x,y) = 1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \tag{3.1}$$

and

$$\mathscr{B}(x,y) = \sum_{n \le L} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}, \sum_{n<L-m} p_{n,k} x^k y^n. \tag{3.2}$$

Proof. The initial values of $p_{n,k}$'s, namely with $n < L$, can be calculated directly. Assume $n \geq L$.

Case 1. If $a_1 < c_1$, let i_2 be the smallest integer greater than 1 such that $a_{i_2} > 0$, then $H_n \leq \sum_{i=1}^n a_i H_{n+1-i}$ is legal if and only if $\sum_{i=i_2}^n a_i H_{n+1-i}$ is. Since the the number of legal $(k - a_1)$-summand decompositions of the form $\sum_{i=i_2}^n a_i H_{n+1-i}$ is $p_{n+1-i_2,k-a_1}$, the number of legal k-summand decompositions of the form $\sum_{i=1}^n a_i H_{n+1-i}$ with $a_1 < c_1$ is

$$\sum_{a_1=1}^{c_1-1} \sum_{i_2=2}^{n} p_{n+1-i_2,k-a_1} = \sum_{j=1}^{c_1-1} \sum_{i=1}^{n-1} p_{i,k-j},$$

where $p_{n,k} = 0$ if $k \leq 0$.

If $a_1 = c_1$, then $a_2 \leq c_2$ by Definition 1.1.

Case 2. If $a_1 = c_1$ and $a_2 < c_2$, let i_3 be the smallest integer greater than 2 such that $a_{i_3} > 0$, then $\sum_{i=1}^n a_i H_{n+1-i}$ is legal if and only if $\sum_{i=i_3}^n a_i H_{n+1-i}$ is. Note that $a_1 = c_1$ and $a_2 = c_2$. Since the the number of legal $(k - c_1 - a_2)$-summand decompositions of the form $\sum_{i=i_3}^n a_i H_{n+1-i}$ is $p_{n+1-i_3,k-c_1-a_2}$, the number of legal k-summand decompositions of the form $\sum_{i=1}^n a_i H_{n+1-i}$ with $a_1 = c_1$ and $a_2 < c_2$ is

$$\sum_{a_2=0}^{c_2-1} \sum_{i_3=3}^{n} p_{n+1-i_3,k-c_1-a_2} = \sum_{j=c_1}^{c_1+c_2-1} \sum_{i=1}^{n-2} p_{i,k-j}.$$

If $a_i = c_i$ for $1 \leq i \leq m < L$, we can repeat the above procedure. By Definition 1.1, we have $a_{m+1} \leq c_{m+1}$.

Case $\boldsymbol{m+1}$ $\boldsymbol{(m \geq 1)}$. If $a_i = c_i$ for $1 \leq i \leq m < L$ and $a_{m+1} < c_{m+1}$, let i_{m+2} be the smallest integer greater than $m+1$ such that $a_{i_{m+2}} > 0$, then $\sum_{i=1}^n a_i H_{n+1-i}$ is legal if and only if $\sum_{i=i_{m+2}}^n a_i H_{n+1-i}$ is. Define

$$s_0 = 0, \; s_0' = 1 \text{ and } s_m' = s_m = c_1 + c_2 + \cdots + c_m, \; 1 \leq m \leq L. \tag{3.3}$$

Note that $a_i = c_i$ for $1 \leq i \leq m < L$. Since the the number of legal $(k - s_m - a_{m+1})$-summand decompositions of the form $\sum_{i=i_{m+2}}^n a_i H_{n+1-i}$ is $p_{n+1-i_{m+2},k-s_m-a_{m+1}}$, the number of legal k-summand decompositions of the form $\sum_{i=1}^n a_i H_{n+1-i}$ with $a_i = c_i$ for $1 \leq i \leq m < L$ and $a_{m+1} < c_{m+1}$ is

$$\sum_{a_{m+1}=0}^{c_{m+1}-1} \sum_{i_3=3}^{n} p_{n+1-i_{m+2},k-s_m-a_{m+1}} = \sum_{j=s_m}^{s_{m+1}-1} \sum_{i=1}^{n-m-1} p_{i,k-j}.$$

Every legal decomposition belongs to exactly one of Cases $1, 2, \ldots, L$ by Definition 1.1, hence for $n \geq L$,

$$\begin{aligned} p_{n,k} &= \sum_{j=1}^{c_1-1} \sum_{i=1}^{n-1} p_{i,k-j} + \sum_{m=1}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \sum_{i=1}^{n-m-1} p_{i,k-j} \\ &= \sum_{m=0}^{L-1} \sum_{j=s_m'}^{s_{m+1}'-1} \sum_{i=1}^{n-m-1} p_{i,k-j}. \end{aligned} \tag{3.4}$$

Replacing n with $n+1$ yields

$$p_{n+1,k} = \sum_{m=0}^{L-1} \sum_{j=s'_m}^{s'_{m+1}-1} \sum_{i=1}^{n-m} p_{i,k-j}. \tag{3.5}$$

Subtracting (3.4) from (3.5), we get

$$p_{n+1,k} - p_{n,k} = \sum_{m=0}^{L-1} \sum_{j=s'_m}^{s'_{m+1}-1} p_{n-m,k-j}.$$

Thus we obtain a recurrence relation for the $p_{n,k}$'s:

$$p_{n+1,k} = p_{n,k} + \sum_{m=0}^{L-1} \sum_{j=s'_m}^{s'_{m+1}-1} p_{n-m,k-j} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} p_{n-m,k-j}. \tag{3.6}$$

Multiplying both sides of (3.6) by $x^k y^{n+1}$ yields

$$p_{n+1,k} x^k y^{n+1} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} p_{n-m,k-j} x^{k-j} y^{n-m}. \tag{3.7}$$

Summing both sides of (3.7) for $n \geq L$ and $k \geq M := s = c_1 + c_2 + \cdots + c_L$, we get

$$\sum_{\substack{n > L \\ k \geq M}} p_{n,k} x^k y^n = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{\substack{n \geq L-m \\ k \geq M-j}} p_{n,k} x^k y^n. \tag{3.8}$$

Using the definition $\mathscr{G}(x,y) = \sum_{n,k>0} p_{n,k} x^n y^k$, we can write (3.8) in the following form (where n and k are always positive):

$$\begin{aligned} \mathscr{G}(x,y) - \sum_{\substack{n \leq L \\ \text{or } k < M}} p_{n,k} x^k y^n \\ = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \left[\mathscr{G}(x,y) - \sum_{\substack{n < L-m \\ \text{or } k < M-j}} p_{n,k} x^k y^n \right]. \end{aligned} \tag{3.9}$$

Rearranging the terms of (3.9), we get

$$\begin{aligned} &\mathscr{G}(x,y) \left(1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \right) \\ &= \sum_{\substack{n \leq L \\ \text{or } k < M}} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{\substack{n < L-m \\ \text{or } k < M-j}} p_{n,k} x^k y^n \\ &= \sum_{n \leq L} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n < L-m} p_{n,k} x^k y^n \end{aligned}$$

$$+\left[\sum_{\substack{n>L\\k<M}} p_{n,k}x^k y^n - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{\substack{n\geq L-m\\k<M-j}} p_{n,k}x^k y^n\right]. \tag{3.10}$$

Let $D(L,M)$ be the parenthesized part in (3.10).Then

$$\begin{aligned} D(L,M) &= \sum_{\substack{n>L\\k<M}} p_{n,k}x^k y^n - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} \sum_{\substack{n>L\\k<M}} p_{n-m-1,k-j}x^k y^n \\ &= \sum_{\substack{n>L\\k<M}} x^k y^n \left(p_{n,k} - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} p_{n-m-1,k-j}\right) \\ &= 0, \end{aligned}$$

where the last equality follows by (3.6) with n replaced by $n-1$.

As $D(L,M)=0$, we can simplify the right-hand side of (3.10) to

$$\mathscr{B}(x,y) = \sum_{n\leq L} p_{n,k}x^k y^n - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n<L-m} p_{n,k}x^k y^n, \tag{3.11}$$

which completes the proof with (3.10).

□

Remark 3.1. *Since $H_n \geq 1$, $p_{n,k}=0$ if $k>n$. Therefore, to find the explicit expression for $\mathscr{B}(x,y)$ of a given sequence H_n, we only need to find the initial values of the $p_{n,k}$'s, namely those with $0<k\leq n\leq L$, which is tractable.*

4. Proof of Theorem 1.3 (Generalized Lekkerkerker)

Proof. Set $A(y)$ and $B(y)$ be the polynomials of y defined in (3.1) and (3.11). Define

$$G(y) = \frac{B(y)}{A(y)}. \tag{4.1}$$

From (3.11), we see that B is of degree at most L, thus we can write

$$B(y) = \sum_{m=1}^{L} b_m(x)y^m, \tag{4.2}$$

where the $b_i(x)$'s are polynomials of x.

Letting $g(x)$ be the coefficient of y^n in $G(y)$, denoted by $\langle y^n\rangle G(y)$, then we have

$$g(x) = \sum_{k>0} p_{n,k}x^k. \tag{4.3}$$

For a fixed n, taking $x = 1$ in (4.3) gives us the sum of the $p_{n,k}$'s, which should be Δ_n according to the definition of the $p_{n,k}$'s, i.e.,

$$g(1) = \sum_{k>0} p_{n,k} = \Delta_n. \tag{4.4}$$

Moreover, taking the derivative of both sides of (4.3) gives

$$g'(1) = \sum_{k>0} kp_{n,k} = \Delta_n \sum_{k>0} k\mathrm{Prob}(n,k) = \Delta_n \mu_n,$$

therefore

$$\mu_n = \frac{g'(1)}{g(1)}. \tag{4.5}$$

Thus the problem reduces to finding g and g' at $x = 1$.

Recall that $A(y)$ is the polynomial of y defined in (3.1), i.e.,

$$A(y) = 1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}. \tag{4.6}$$

Let $y_1(x), y_2(x), \ldots, y_L(x)$ be the roots of $A(y)$ (i.e., regarding A as function of y). We want to write $\frac{1}{A(y)}$ as a linear combination of the $\frac{1}{y-y_i(x)}$'s, i.e., the partial fraction expansion, as we can use power series expansion to find the coefficient of y^n in $\frac{B(y)}{A(y)}$.

To achieve this goal, we need to show that the $y_i(x)$'s are pairwise distinct, specifically, $A(y)$ has no multiple roots for x in some neighborhood of 1 excluding 1, i.e., $I_\varepsilon := (1-\varepsilon, 1+\varepsilon)\backslash\{1\}$. This result is formally stated in Theorem 4.1(a) and proved in Appendix B.

Here is a sketch of the proof.

If $x > 0$ and $L = 1$, then $A(y) = 1 - \sum_{j=0}^{c_1-1} x^j y$ has a unique root $y_1(x) = \left(\sum_{j=0}^{c_1-1} x^j\right)^{-1}$ and $y_1(x) \in (0,1)$ since $c_1 > 1$ (see the assumption of Theorem 1.2). Note that if $x > 0$, then $y_1(x)$ is continuous and ℓ-times differentiable for all $\ell > 0$. Thus in this case, ε can be 1.

For $L \geq 2$, there is an easy proof for non-increasing c_i's (see Appendix C), but the proof for general cases (see Appendix B) is much more complicated, which involves continuity and the range of the $|y_i(x)|$'s. The main idea is to first show that there exists $x > 0$ such that $A(y)$ has no multiple roots and then prove that there are finitely many $x > 0$ such that $A(y)$ has multiple roots.

In this section, we repeatedly use the continuity of the $y_i(x)$'s, which follows from the fact that the roots of a polynomial with continuous coefficients are continuous (for completeness, see Appendix A for the formal statement and the proof). Since for any $x > 0$, the coefficients of $A(y)$ are continuous functions of x and the leading coefficient is nonzero, the roots of $A(y)$ are continuous at x.

Now we can prove that $A(y)$ has no multiple roots for $x \in I_\varepsilon$ for some ε and then apply partial fraction expansion.

Proposition 4.1. *There exists $\varepsilon \in (0,1)$ with the following properties.*

(a) *For any $x \in I_\varepsilon$, $A(y)$ as polynomial of y has no multiple roots, i.e.,*

$$A'(y_i(x)) = -\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1) x^j y_i^m(x) \neq 0, \tag{4.7}$$

where $A'(y)$ is the derivative with respect to y.

(b) *If $x = 1$, then $A(y)$ has a unique positive real root. Let it be $y_1(1)$ without loss of generality, then $0 < y_1(1) < 1$ and $|y_i(1)| > y_1(1)$ for $i > 1$. and $|y_i(1)| > y_1(1)$ for $i > 1$.*

(c) *For any $x \in I_\varepsilon$, $A(y)$ has a unique positive real root. Let it be $y_1(x)$ without loss of generality, then $0 < y_1(x) < 1$ and $|y_i(x)/y_1(x)| > \sqrt{|y_i(1)/y_1(1)|} > 1$ for $i > 1$.*

If ε satisfies the above properties, then for any $x \in I_\varepsilon$, we have

$$\frac{1}{A(y)} = -\frac{1}{\sum_{j=s_{L-1}}^{s_L-1} x^j} \sum_{i=1}^{L} \frac{1}{(y - y_i(x)) \prod_{j \neq i} (y_j(x) - y_i(x))}. \tag{4.8}$$

Proof. We prove in Appendix B that there exists $\epsilon \in (0,1)$ such that for any $x \in I_\epsilon$, $A(y)$ has no multiple roots.

For (b), when $x = 1$, $A(y)$ is strictly decreasing on $(0, \infty)$ and $A(0) = 1 > 0 > A(1)$. Thus $A(y)$ has a unique positive root $y_1(1)$ and $y_1(1) \in (0,1)$. Since $A'(y_1(1)) < 0$, $y_1(1)$ is not a multiple root of $A(y)$.

For any other root $y_i(1)$ $(i > 1)$, if $|y_i(x)| \leq y_1(x)$, then

$$\begin{aligned} 0 = |A(y_i(1))| &= |1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} y_i^{m+1}(1)| \geq 1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} |y_i^{m+1}(1)| \\ &\geq 1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} |y_1^{m+1}(1)| = 0. \end{aligned}$$

Hence the equalities holds. Thus each $y_i^{m+1}(1)$ is nonnegative, i.e., $y_i(1)$ is nonnegative. Since $A(0) \neq 0$, $y_i(1) \neq 0$, Thus $y_i(1) > 0$. However, $A(y)$ only has one positive root $y_1(1)$ and it is not a multiple root, contradiction.

For (c), denote $\lambda = \min_{i>1}\{\sqrt{|y_i(1)/y_1(1)|}\} > 1$. By the continuity of the $y_i(x)$'s, there exists $\varepsilon \in (0, \epsilon)$ such that for all $x \in I_\varepsilon$,

$$y_1(x) < (1+\kappa) y_1(1) \text{ and } y_i(x) > (1-\kappa) y_i(1) \text{ for } 1 < i \leq L,$$

where $\kappa = (\lambda - 1)/2(1+\lambda) \in (0,1)$. Thus

$$\frac{y_i(x)}{y_1(x)} > \frac{1-\kappa}{1+\kappa} \frac{y_i(1)}{y_1(1)} = \frac{3+\lambda}{1+3\lambda} \frac{y_i(1)}{y_1(1)} > \frac{3+\lambda}{\lambda^2+3\lambda} \frac{y_i(1)}{y_1(1)} = \frac{1}{\lambda} \frac{y_i(1)}{y_1(1)}.$$

Since $\lambda = \min_{i>1}\{\sqrt{|y_i(1)/y_1(1)|}\} \leq \sqrt{|y_i(1)/y_1(1)|}$,

$$\frac{y_i(x)}{y_1(x)} > \frac{1}{\lambda} \frac{y_i(1)}{y_1(1)} \geq \sqrt{\frac{y_i(1)}{y_1(1)}},$$

as desired.

Now suppose ε satisfies (a), (b) and (c). Since the leading coefficient of $A(y)$ is $-\sum_{j=s_{L-1}}^{s_L-1} x^j$ and the roots of $A(y)$ are $y_1(x), y_2(x), \cdots, y_L(x)$,

$$A(y) = -\sum_{j=s_{L-1}}^{s_L-1} x^j \prod_{i=1}^{L} (y - y_i(x)). \tag{4.9}$$

For any $x \in I_\varepsilon$, the $y_i(x)$'s are distinct, thus we can interpolate the Lagrange polynomial of $\mathscr{L}(y) = 1$ at $y_1(x)$, $y_2(x)$, $\dots$, $y_L(x)$:

$$\sum_{i=1}^{L} \frac{\prod_{j\neq i}(y - y_i(x))}{(y-y_i(x))\prod_{j\neq i}(y_j(x)-y_i(x))} = 1.$$

Dividing both sides by $\prod_{i=1}^{L}(y-y_i(x))$ and combining with (4.9) yields (4.8). $\square$

Proposition 4.2. *For any $x > 0$, if $y_i(x)$ is not a multiple root of $A(y)$, then $y_i(x)$ is ℓ-times differentiable for any $\ell \geq 1$. In particular, given ε as in Proposition 4.1, for any $x \in I_\varepsilon$ and each $1 \leq i \leq L$, we have $y_i(x)$ is ℓ-times differentiable for any $\ell \geq 1$. Additionally, note that $y_1(x)$ is not a multiple root of $A(y)$ when $x = 1$ since $A'(y_1(1)) < 0$, thus $y_1(x)$ is ℓ-times differentiable at 1 for any $\ell \geq 1$. If $y_i(x)$ is differentiable at x, then its derivative*

$$y_i'(x) = -\frac{\sum_{m=0}^{L-1}\sum_{j=s_m'}^{s_{m+1}'-1} j y_i^{m+1}(x)x^{j-1}}{\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1}(m+1)x^j y_i^m(x)}. \tag{4.10}$$

Proof. We prove the differentiability by induction on ℓ. For the derivative, we differentiate $A(y)$ at $y_i(x)$ to get (4.10). See Appendix D for the detailed proof. $\square$

Let us return to finding g (with $L \geq 1$). From now on, we assume that $x \in I_\varepsilon$. Plugging (4.2) and (4.8) into (4.1), we get

$$\begin{aligned}
&\sum_{j=s_{L-1}}^{s_L-1} x^j G(y)\\
=\ & -\sum_{m=1}^{L} b_m(x)y^m \sum_{i=1}^{L}\frac{1}{(y-y_i(x))\prod_{j\neq i}(y_j(x)-y_i(x))}\\
=\ & \sum_{m=1}^{L} b_m(x)y^m \sum_{i=1}^{L}\frac{1}{(1-\frac{y}{y_i(x)})y_i(x)\prod_{j\neq i}(y_j(x)-y_i(x))}\\
=\ & \sum_{m=1}^{L} b_m(x)y^m \sum_{i=1}^{L}\frac{1}{y_i(x)\prod_{j\neq i}(y_j(x)-y_i(x))}\sum_{l\geq 0}\left(\frac{y}{y_i(x)}\right)^l.
\end{aligned}$$

Thus for $n \geq L$, by looking at the coefficient of y^n, we obtain

$$g(x) = \frac{1}{\sum_{j=s_{L-1}}^{s_L-1} x^j}\sum_{i=1}^{L}\frac{1}{y_i(x)\prod_{j\neq i}(y_j(x)-y_i(x))}\sum_{m=1}^{L}\frac{b_m(x)}{y_i^{n-m}(x)}.$$

Define

$$q_i(x) = \frac{\sum_{m=1}^{L} b_m(x)y_i^m(x)}{\sum_{j=s_{L-1}+1}^{s_L} x^j y_i(x)\prod_{j\neq i}(y_j(x)-y_i(x))}, \tag{4.11}$$

then

$$g(x) = \sum_{i=1}^{L} x q_i(x) y_i^{-n}(x). \tag{4.12}$$

Note that the $q_i(x)$'s are independent of n.

Define

$$\mathcal{A}(y) = y^L A\left(\frac{1}{y}\right) = y^L - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{L-1-m}. \tag{4.13}$$

Since $A(0) \neq 0$, the roots of $\mathcal{A}(y)$ are $\alpha_i(x) := (y_i(x))^{-1}$. Therefore, by Proposition 4.1, $\alpha_1(x)$ is real, and

$$\alpha_1(x) > 1, \text{ and } |\alpha_i(x)/\alpha_1(x)| < \sqrt{|\alpha_i(1)/\alpha_1(1)|} < 1 \text{ for } i > 1. \tag{4.14}$$

Plugging $\alpha_i(x) = (y_i(x))^{-1}$ into (4.12), we get

$$g(x) = \sum_{i=1}^{L} x q_i(x) \alpha_i^n(x). \tag{4.15}$$

Since $g(x)$ is a polynomial of x, we have

$$g^{(\ell)}(1) = \lim_{x\to 1} g^{(\ell)}(x) = \lim_{x\to 1} \left[\sum_{i=1}^{L} x q_i(x) \alpha_i^n(x)\right]^{(\ell)}, \ \forall\ \ell \geq 0. \tag{4.16}$$

We want the main term of $g^{(\ell)}(x)$ to be $[xq_1(x)\alpha_1^n(x)]^{(\ell)}$ for $x \in (x-\varepsilon, x+\varepsilon$. Since $g(x)$ is ℓ-times differentiable at 1, by (4.16) it suffices to prove the following two claims.

Claim 4.3. *For any $\ell \geq 1$ and any $i \in \{1, 2, \ldots, L\}$, we have $\alpha_i(x)$ and $q_i(x)$ are ℓ-times differentiable at $x \in I_\varepsilon$ and $\alpha_1(x)$ and $q_1(x)$ are ℓ-times differentiable at 1.*

Claim 4.4. *For any $x \in I_\varepsilon$ and $\ell \geq 0$, we have*

$$\left[\sum_{i=2}^{L} x q_i(x) \alpha_i^n(x)\right]^{(\ell)} = o(1)\alpha_1^n(x), \tag{4.17}$$

where o(1) vanishes exponentially at ∞ with respect to n, namely for sufficiently large n, $|o(1)| < \gamma^n$ for some $\gamma \in (0,1)$ which might be dependent on ℓ but is independent of x, n and i.

With the result and (4.16), we see that

$$g^{(\ell)}(1) = [q_1(1)\alpha_1^n(1)]^{(\ell)} + o(1)\alpha_1^n(1), \tag{4.18}$$

Remark 4.1. *In the equations afterwards, $o(1)$ may be different in different equations, but at each time, it represents a function that vanishes exponentially at ∞ in terms of n.*

Proof. There is an easy proof if $A(y)$ has no multiple roots when $x = 1$. In this case, all $y_i(x)$'s, $\alpha_i(x)$'s and $q_i(x)$'s are ℓ-times differentiable for all ℓ at $x = 1$. Therefore Claim 4.4 follows immediately by Proposition 4.1 and Proposition 4.1 follows directly from the continuity of the $y_i(x)$'s.

The situation becomes totally different and harder if $A(y)$ has multiple roots when $x = 1$. See Appendix F for the proof. □

Recall from (4.4) that $g(1) = \Delta_n = H_{n+1} - H_n$, thus by Claim 4.4 with $\ell = 0$, we get

$$\Delta_n = g(1) = (q_1(1) + o(1))\alpha_1^n(1). \tag{4.19}$$

Since Δ_n is positive and unbounded, we have $q_1(1) > 0$.

Define $g_i(x) = xq_i(x)\alpha_i^n(x)$. According to (4.15), we have $g(x) = \sum_{i=1}^L g_i(x)$. Since

$$g_i'(x) = nxq_i(x)\alpha_i'(x)\alpha_i^{n-1}(x) + (xq_i(x))'\alpha_i^n(x),$$

$$\begin{aligned} g'(x) &= \sum_{i=1}^L g_i'(x) = \sum_{i=1}^L \left(nxq_i(x)\alpha_i'(x)\alpha_i^{n-1}(x) + (xq_i(x))'\alpha_i^n(x)\right) \\ &= nxq_i(x)\alpha_1'(x)\alpha_1^{n-1}(x) + (xq_i(x))'\alpha_1^n(x) + o(1)\alpha_1^n(x). \end{aligned}$$

Letting $x \to 1$ and using (4.19), we get

$$\begin{aligned} \frac{g'(1)}{g(1)} &= \frac{nq_1(1)\alpha_1'(1)\alpha_1^{n-1}(1) + (q_1(1) + q_1'(1))\alpha_1^n(1) + o(1)\alpha_1^n(1)}{q_1(1)\alpha_1^n(1) + o(1)\alpha_1^n(1)} \\ &= \frac{nq_1(1)\alpha_1'(1)(\alpha_1(1))^{-1} + (q_1(1) + q_1'(1)) + o(1)}{q_1(1) + o(1)} \\ &= \frac{\alpha_1'(1)}{\alpha_1(1)}n + \frac{q_1(1) + q_1'(1)}{q_1(1)} + o(1). \end{aligned}$$

Therefore, by (4.5) μ_n is of the form (1.2): $\mu_n = Cn + d + o(1)$, where

$$C = \frac{\alpha_1'(1)}{\alpha_1(1)} \text{ and } d = 1 + \frac{q_1'(1)}{q_1(1)}. \tag{4.20}$$

□

Remark 4.2. (a) *A formula for C:*
Note that C can be computed as follows:

$$C = \left.\frac{\alpha_1'(x)}{\alpha_1(x)}\right|_{x=1} = \left.\frac{((y_1(x))^{-1})'}{(y_1(x))^{-1}}\right|_{x=1} = -\left.\frac{y_1'(x)}{y_1(x)}\right|_{x=1} = -\frac{y_1'(1)}{y_1(1)}, \tag{4.21}$$

where $y_1'(1)$ is given by (4.10). Then we get

$$C = -\frac{y_1'(1)}{y_1(1)} = \frac{\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} jy_1^m(1)}{\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1}(m+1)y_1^m(1)} \tag{4.22}$$

$$= \frac{\sum_{m=0}^{L-1}\frac{1}{2}(s_m + s_{m+1} - 1)(s_{m+1} - s_m)y_1^m(1)}{\sum_{m=0}^{L-1}(m+1)(s_{m+1} - s_m)y_1^m(1)}. \tag{4.23}$$

(b) *Upper and lower bounds for C.*
Applying (4.23) with some approximations, we get

$$\min\{\frac{c_1 - 1}{2},\ \frac{c_1 - 2}{L} + 1\} \le C \le \frac{(2L-1)c_1 - 1}{2L} < c_1$$

(see Appendix G for the detailed proof).

5. Gaussian Behavior

In this section, we show that the distribution of the number of summands is Gaussian, which is equivalent to the following theorem according to the method of moments.

Theorem 5.1. *Let $\mu_n(m)$ be the m^{th} moment of $K_n - \mu_n$ and σ_n the standard deviation, then for any $u \geq 1$,*

$$\mu_n(2u-1) \to 0 \text{ and } \frac{\mu_n(2u)}{\sigma_n^{2u}} \to (2u-1)!!, \text{ as } u \to \infty. \tag{5.1}$$

Proof. Let $\tilde{\mu}_n = Cn + d$, where C and d are defined in (4.20), then $\mu_n = \tilde{\mu}_n + o(1)$ (see Remark 4.1 for the description of the $o(1)$ term). Define $\tilde{\mu}_n(m) = \sum_k p_{n,k}(k-\tilde{\mu}_n)^m/\Delta_n$. By some simple approximations (see Appendix I), we can see that $\mu_n(m) = \tilde{\mu}_n(m) + o(1)$.

Note that $\sigma_n^2 = \mu_n(2) = \tilde{\mu}_n(2) + o(1)$, therefore, (5.1) is equivalent to

$$\tilde{\mu}_n(2u-1) \to 0 \text{ and } \frac{\tilde{\mu}_n(2u)}{\tilde{\mu}_n(2)} \to (2u-1)!!, \text{ as } u \to \infty. \tag{5.2}$$

We will calculate the moments $\tilde{\mu}_n(m)$'s by applying the method of differentiating identities to g with g defined by (4.3).

Define

$$\tilde{g}_0(x) = \sum_k p_{n,k} x^{k-\tilde{\mu}_n - 1} = \frac{g(x)}{x^{\tilde{\mu}_n+1}}, \ \tilde{g}_{j+1}(x) = (x\tilde{g}_j(x))', \ j \geq 1, \ \forall\ x > 0. \tag{5.3}$$

Setting $x = 1$, we get

$$\tilde{g}_0(1) = \sum_k p_{n,k} = \Delta_n = \tilde{\mu}_n(0)\Delta_n.$$

When $m = 1$, by Definition (5.3) we get

$$\tilde{g}_1(x) = (x\tilde{g}_0(x))' = \left(\sum_k p_{n,k} x^{k-\tilde{\mu}_n}\right)' = \sum_k p_{n,k}(k-\tilde{\mu}_n) x^{k-\tilde{\mu}_n-1}. \tag{5.4}$$

When $m = 2$, by (5.3) and (5.4), we get

$$\tilde{g}_2(x) = (x\tilde{g}_1(x))' = \sum_k p_{n,k}(k-\tilde{\mu}_n)^2 x^{k-\tilde{\mu}_n-1}.$$

Setting $x = 1$, we get

$$\tilde{g}_2(1) = \sum_k p_{n,k}(k-\tilde{\mu}_n)^2 = \tilde{\mu}_n(2)\Delta_n.$$

By induction on m, we can prove the following proposition.

Proposition 5.2. *For any $m \geq 0$,we have*

$$\tilde{g}_m(x) = \sum_k p_{n,k}(k-\tilde{\mu}_n)^m x^{k-\tilde{\mu}_n-1} \text{ and } \tilde{g}_m(1) = \tilde{\mu}_n(m)\Delta_n. \tag{5.5}$$

Proof. We have proved the statement for $m = 0, 1, 2$. If (5.5) holds for m, then by the recurrence relation (5.3), we have

$$\tilde{g}_{m+1}(x) \quad = \quad (x\tilde{g}_m(x))' = \left(\sum_k p_{n,k}(k-\tilde{\mu}_n)^m x^{k-\tilde{\mu}_n}\right)'$$

$$= \sum_k p_{n,k}(k-\tilde{\mu}_n)^{m+1} x^{k-\tilde{\mu}_n-1}.$$

Setting $x=1$ gives $\tilde{g}_{m+1}(1) = \tilde{\mu}_n(m+1)\Delta_n$. Thus the statement holds for $m+1$ and hence for any $m \geq 0$. $\square$

Denote

$$\tilde{g}_{0,i}(x) = \frac{q_i(x)\alpha_i^n(x)}{x^{\tilde{\mu}_n}}, \text{ and } \tilde{g}_{j+1,i}(x) = (x\tilde{g}_{j,i}(x))' \tag{5.6}$$

for $x \in I_\varepsilon$ if $1 < i \leq L$ and for $x \in I_\varepsilon \cup \{1\}$ if $i=1$.

By Definition (5.6) and using the same approach as in Lemma 4.4, we can prove that

$$\sum_{i=2}^{L} \tilde{g}_{j,i}(x) = o(1)\alpha_1^n(x), \ \forall\ x \in I_\varepsilon \tag{5.7}$$

(see Remark 4.1 for the description of the $o(1)$ term). Thus refering to (5.3), we have

$$\tilde{g}_j(x) = \sum_{i=1}^{L} \tilde{g}_{j,i}(x) = \tilde{g}_{j,1}(x) + o(1)\alpha_1^n(x), \ \forall\ x \in I_\varepsilon. \tag{5.8}$$

Taking the limit as x approaches 1 yields

$$\tilde{g}_j(1) = \tilde{g}_{j,1}(1) + o(1)\alpha_1^n(1), \ \forall\ x \in I_\varepsilon. \tag{5.9}$$

Denoting $\tilde{g}_{j,1}(x)$ by $F_j(x)$, then

$$F_0(x) = q_1(x)\alpha_1^n(x)x^{-\tilde{\mu}_n} \text{ and } F_{j+1}(x) = (xF_j(x))'. \tag{5.10}$$

Note that $q_1(x)$ and $\alpha_1(x)$ are ℓ-times differentiable for any $\ell \geq 1$(see Claim 4.3). Thus when $j=0$, we get

$$\begin{aligned}
& F_1(x) = (xF_0(x))' = \left(q_1(x)\alpha_1^n(x)x^{-\tilde{\mu}_n}\right)' \\
= \;& nxq_1(x)\alpha_1'(x)\alpha_1^{n-1}(x)x^{-\tilde{\mu}_n} - (\tilde{\mu}_n - 1)q_1(x)\alpha_1^n(x)x^{-\tilde{\mu}_n} \\
& +xq_1'(x)\alpha_1^n(x)x^{-\tilde{\mu}_n} \\
= \;& nxq_1(x)\alpha_1'(x)\alpha_1^{n-1}(x)x^{-\tilde{\mu}_n} - (Cn+d-1)q_1(x)\alpha_1^n(x)x^{-\tilde{\mu}_n} \\
& +xq_1'(x)\alpha_1^n(x)x^{-\tilde{\mu}_n} \\
= \;& \alpha_1^n(x)x^{-\tilde{\mu}_n}\left[\left(\frac{x\alpha_1'(x)}{\alpha_1(x)} - C\right)q_1(x)n + (1-d)q_1(x) + xq_1'(x)\right] \\
= \;& \alpha_1^n(x)x^{-\tilde{\mu}_n}\left[h(x)q_1(x)n + d'q_1(x) + xq_1'(x)\right],
\end{aligned} \tag{5.11}$$

where $h(x)$ and d' are defined as

$$h(x) = \frac{x\alpha_1'(x)}{\alpha_1(x)} - C \text{ and } d' = 1-d = -\frac{q_1'(1)}{q_1(1)} \tag{5.12}$$

(see (4.20) for the definition of d). By (4.20), we have

$$h(1) = 0. \tag{5.13}$$

Moreover, since $\alpha_1(x)$ is ℓ-times differentiable at 1 and $\alpha_1(1) \neq 0$ (see Proposition 4.2), we have

$$h(x) \text{ is } \ell\text{−times differentiable at 1 for any } \ell \geq 1. \tag{5.14}$$

From (5.10) and (5.11), we observe that $F_m(x)$ can be written as a product of $\alpha_1^n(x)x^{-\tilde{\mu}_n}$ and a sum of other functions of n and x. In fact, we have the following proposition.

Proposition 5.3. *For any $m \geq 0$,*

(a) *We have $F_m(x)$ is of the form*

$$F_m(x) = \alpha_1^n(x)x^{-\tilde{\mu}_n} \sum_{i=0}^{m} f_{i,m}(x)n^i, \tag{5.15}$$

where the $f_{i,m}$'s are functions of x and $\alpha_1(x)$ but independent of n.

(b) *The $f_{i,m}$'s are ℓ-times differentiable at $x \in I_\varepsilon$ for any $\ell \geq 1$.*

(c) *Define*

$$f_{i,m}(x) = 0 \text{ if } i > m \text{ or } i < 0 \text{ or } m < 0, \tag{5.16}$$

then for $m > 0$, we have the following recurrence relation:

$$f_{i,m}(x) = h(x)f_{i-1,m-1}(x) + d' f_{i,m-1}(x) + x f'_{i,m-1}(x). \tag{5.17}$$

Proof. We proceed by induction on m.

For $m = 0$ and 1, (a) holds because of (5.10) and (5.11). Further, (5.10) and (5.11) give the expressions of $f_{0,0}$, $f_{0,1}$ and $f_{1,1}$:

$$f_{0,0}(x) = q_1(x), f_{0,1}(x) = d'q_1(x) + xq_1'(x), f_{1,1}(x) = h(x)q_1(x). \tag{5.18}$$

By Claim 4.3 and (5.14), they are differentiable ℓ-times at $x \in I_\varepsilon$ for any $\ell \geq 1$. Hence (b) holds for $m = 0$ and 1. Finally, with (5.18), it is easy to verify that (c) holds for $m = 0$ and 1.

If the statement holds for m, by (5.3) we have

$$F_{m+1}(x) = \left[\alpha_1^n(x)x^{-\tilde{\mu}_n} \sum_{i=0}^{m} x f_{i,m}(x)n^i\right]' = \sum_{i=0}^{m} \left[\alpha_1^n(x)x^{-\tilde{\mu}_n} x f_{i,m}(x)n^i\right]'.$$

For convenience, we denote $h_i(x) = \alpha_1^n(x)x^{-\tilde{\mu}_n} x f_{i,m}(x)n^i$ for $0 \leq i \leq m$. Thus

$$F_{m+1}(x) = \sum_{i=0}^{m} h_i'(x). \tag{5.19}$$

For each $0 \leq i \leq m$, we have

$$\begin{aligned}
& h_i'(x) \\
= \;& n^i \left[\alpha_1'(x)\alpha_1^{n-1}(x)x^{-\tilde{\mu}_n} x f_{i,m} - (\tilde{\mu}_n - 1)\alpha_1^n(x)x^{-\tilde{\mu}_n} f_{i,m}(x)\right. \\
& \left. + \alpha_1^n(x)x^{-\tilde{\mu}_n} x f'_{i,m}(x)\right] \\
= \;& n^i \alpha_1^n(x)x^{-\tilde{\mu}_n} \left[n f_{i,m}(x)\left(\alpha_1'(x)\alpha_1^{-1}(x)x - C\right) + (1-d)f_{i,m}(x)\right. \\
& \left. + x f'_{i,m}(x)\right] \\
= \;& n^i \alpha_1^n(x)x^{-\tilde{\mu}_n} \left[n h(x) f_{i,m}(x) + d' f_{i,m}(x) + x f'_{i,m}(x)\right] \\
= \;& \alpha_1^n(x)x^{-\tilde{\mu}_n} \left[n^{i+1} h(x) f_{i,m}(x) + n^i \left(d' f_{i,m}(x) + x f'_{i,m}(x)\right)\right]
\end{aligned} \tag{5.20}$$

(see (5.12) for the definitions of $h(x)$ and d').

Plugging (5.20) into (5.19) yields

$$F_{m+1}(x)$$

$$
\begin{aligned}
&= \alpha_1^n(x)x^{-\tilde{\mu}_n}\sum_{i=0}^{m}\left[n^{i+1}h(x)f_{i,m}(x)+n^i\left(d'f_{i,m}(x)+xf'_{i,m}(x)\right)\right] \\
&= \alpha_1^n(x)x^{-\tilde{\mu}_n}\left[n^{m+1}h(x)f_{m,m}(x)+\sum_{i=1}^{m}n^i\left(h(x)f_{i-1,m}(x)+d'f_{i,m}(x)\right.\right. \\
&\quad \left.\left.+xf'_{i,m}(x)\right)+d'f_{0,m}(x)+xf'_{0,m}(x)\right]. \qquad (5.21)
\end{aligned}
$$

Hence (5.15) holds for $m+1$ as desired.

For (b) and (c), from (5.21) we get

$$f_{m+1,m+1}(x) = h(x)f_{m,m}(x), \tag{5.22}$$

$$f_{i,m+1}(x) = h(x)f_{i-1,m}(x) + d'f_{i,m}(x) + xf'_{i,m}(x),\ 1 \le i \le m \tag{5.23}$$

and

$$f_{0,m+1}(x) = d'f_{0,m}(x) + xf'_{0,m}(x). \tag{5.24}$$

By Definition (5.16), we can combine (5.22), (5.23) and (5.24) into one recurrence relation (5.17) (with m replaced by $m+1$). With this recurrence relation, (5.14) and the induction hypothesis of (b) for m, we see that (b) also holds for $m+1$. This completes the proof. □

Proposition 5.4. *We have*

$$\tilde{\mu}_n(m) = \frac{1}{q_1(1)}\sum_{i=0}^{m} f_{i,m}(1)n^i + o(1). \tag{5.25}$$

Proof. From (5.5), (5.8), (4.19), the definition $F_m(x) = \tilde{g}_{m,1}(x)$ and Proposition 5.3, we obtain

$$
\begin{aligned}
\tilde{\mu}_n(m) &= \frac{\tilde{g}_m(1)}{\Delta_n} = \frac{\tilde{g}_{m,1}(1)+o(1)\alpha_1^n(1)}{\Delta_n} = \frac{\tilde{F}_m(1)+o(1)\alpha_1^n(1)}{\Delta_n} \\
&= \frac{(\sum_{i=0}^m f_{i,m}(1)n^i + o(1))\alpha_1^n(1)}{(q_1(1)+o(1))\,\alpha_1^n(1)} = \frac{1}{q_1(1)}\sum_{i=0}^{m} f_{i,m}(1)n^i + o(1).
\end{aligned}
$$

□

From Proposition 5.4, we see that the main term of $\tilde{\mu}_n(m)$ only depends on $q_1(1)$ and the $f_{i,m}(1)$'s. Note that to prove (5.2), it suffices to find the main term of $\tilde{\mu}_n(m)$. Thus the problem reduces to finding the $f_{i,m}(1)$'s. We first calculate the variance, namely $\tilde{\mu}_n(2)$.

Proposition 5.5. *The variance of* $K - \tilde{\mu}_n$

$$\tilde{\mu}_n(2) = h'(1)n + q_1''(1) + o(1) \tag{5.26}$$

with $h'(1) \neq 0$.

Proof. If $m = 2$, by (5.22) and (5.13) we get $f_{2,2}(1) = h(1)f_{1,1}(1) = 0$. Applying (5.17) to $(i,m) = (1,2)$ and plugging in (5.18) yields

$$
\begin{aligned}
f_{1,2}(x) &= h(x)f_{0,1}(x) + d'f_{1,1}(x) + xf'_{1,1}(x) \\
&= h(x)f_{0,1}(x) + d'h(x)q_1(x) + xh(x)q_1'(x) + xh'(x)q_1(x).
\end{aligned}
$$

Setting $x = 1$ and using $h(1) = 0$ (see (5.13)) yields

$$f_{1,2}(1) = h(1)f_{0,1}(1) + d'h(1)q_1(1) + h(1)q_1'(1) + h'(1)q_1(1) = h'(1)q_1(1).$$

Using (5.24) and (5.17), we can find $f_{0,2}(x)$ as follows.

$$\begin{aligned} f_{0,2}(x) &= d' f_{0,1}(x) + x f'_{0,1}(x) \\ &= d'^2 q_1(x) + d' x q'_1(x) + d' x q_1(x) + x q'_1(x) + x^2 q''_1(x). \end{aligned}$$

Setting $x = 1$ and substituting d' by $-\frac{q'_1(1)}{q_1(1)}$ (see (5.12)) yields

$$f_{0,2}(1) = q''_1(1).$$

Combining the above results with Proposition 5.4 gives (5.26). Thus it remains to show that $h'(1) \neq 0$. We can derive a formula of $h'(x)$ in terms of $y_1(x)$ by Definition (5.12), (4.21) and (4.10), and then prove that $h'(1) \neq 0$ by contradiction (see Appendix H). $\square$

From Propositions 5.4 and 5.5, we see that (5.2) is equivalent to

$$f_{i,2u-1}(1) = 0, \ \ i \geq u, \tag{5.27}$$

$$f_{i,2u}(1) = 0, \ \ i > u, \tag{5.28}$$

and

$$f_{u,2u}(1) = (2u-1)!! q_1(1) \left(h'(1)\right)^u. \tag{5.29}$$

For convenience, we denote

$$t^{(\ell)}_{i,m} = f^{(\ell)}_{i,m}(1), \ \ \ell \geq 0.$$

Note that if $\ell = 0$, then the definition is just $t_{i,m} = f_{i,m}(1)$.

Proposition 5.6. *For any* $0 \leq m < 2i$ *and* $\ell \geq 0$*, we have*

$$t^{(\ell)}_{i,m-\ell} = f^{(\ell)}_{i,m-\ell}(1) = 0. \tag{5.30}$$

Proof. If $\ell > m$ or $i > m - \ell$, according to Definition (5.16), we have $f_{i,m-\ell}(x) = 0$. Thus $f^{(\ell)}_{i,m-\ell}(x) = 0$ and (5.30) follows. Therefore, it suffices to prove for $0 \leq \ell \leq m < 2i$ and $i \leq m - \ell$, i.e.,

$$0 \leq \ell \leq m - i < i. \tag{5.31}$$

We proceed by induction on m.

If $m = 0$, then there is no i that satisfies (5.31). Thus the statement holds.

If $m = 1$, the only choice for i and ℓ that satisfies (5.31) is $i = 1$ and $\ell = 0$.

By (5.18) and (5.13), we get $t^{(\ell)}_{i,m-\ell} = t_{1,1} = f_{1,1}(1) = h(1)q_1(1) = 0$. Thus the statement holds for $m = 1$.

Assume that the statement holds for any $m' < m$ $(m \geq 2)$.

For any (i, m, ℓ) that satisfies (5.31) and $1 \leq j \leq \ell$, we have

$$2(i-1) = 2i - 2 > m - 2 \geq m - 1 - j,$$

thus we can apply the induction hypothesis (5.30) to $(i-1, m-1-j, \ell-j)$, $(i, m-1, \ell)$ and $(i, m-1-\ell+j, j)$ with $1 \leq j \leq \ell$ and obtain

$$f^{(\ell-j)}_{i-1,m-1-\ell}(1) = f^{(\ell)}_{i,m-1-\ell}(1) = f^{(j)}_{i,m-1-\ell}(1) = 0. \tag{5.32}$$

Taking the ℓ^{th} derivative of both sides of (5.17), we get

$$f^{(\ell)}_{i,m-\ell}(x) = \left(h(x) f_{i-1,m-1-\ell}(x)\right)^{(\ell)} + d' f^{(\ell)}_{i,m-1-\ell}(x) + \left(x f'_{i,m-1-\ell}(x)\right)^{(\ell)}$$

$$= \quad h(x) f_{i-1,m-1-\ell}^{(\ell)}(x) + \sum_{j=1}^{\ell} \binom{\ell}{j} h^{(j)}(x) f_{i-1,m-1-\ell}^{(\ell-j)}(x)$$
$$+d' f_{i,m-1-\ell}^{(\ell)}(x) + x f_{i,m-1-\ell}^{(\ell+1)}(x) + \sum_{j=1}^{\ell} f_{i,m-1-\ell}^{(j)}(x).$$

Setting $x = 1$ and using (5.32) and (5.13) yields

$$f_{i,m-\ell}^{(\ell)}(1) = f_{i,m-1-\ell}^{(\ell+1)}(1), \text{ i.e., } t_{i,m-\ell}^{(\ell)} = t_{i,m-1-\ell}^{(\ell+1)}. \tag{5.33}$$

Applying (5.33) to $\ell = 0, 1, \ldots, m$, we get

$$t_{i,m}^{(0)} = t_{i,m-1}^{(1)} = t_{i,m-2}^{(2)} = \cdots = t_{i,0}^{(m)} = t_{i,-1}^{(m+1)} = 0,$$

where the last step follows from (5.16).

Thus the statement holds for m as well. This completes the proof. □

Corollary 5.7. *For any $u \geq 1$, we have (5.27) and (5.28), i.e.,*

$$t_{i,2u-1} = 0, \; i \geq u \text{ and } t_{i,2u} = 0, \; i > u. \tag{5.34}$$

Proof. Applying Proposition 5.6 with $(i, m, \ell) = (i, 2u - 1, 0)$ $(i \geq u)$ and $(i, m, \ell) = (i, 2u - 1, 0)$ $(i > u)$. □

Thus it remains to show (5.29).

Proposition 5.8. *For any $u \geq 1$ we have*

(a)$f_{u,u+v}(x)$ *with $0 \leq v \leq u$ is of the form*

$$f_{u,u+v}(x) = r_{u,v} q_1(x) x^v h^{u-v}(x) \left(h'(x)\right)^v + s_{u,v}(x) h^{u+1-v}(x), \tag{5.35}$$

where $r_{u,v}$ is a constant determined by u and v, $s_{u,v}(x)$ is a polynomial of the $h^{(\ell)}(x)$'s and the $q_1^{(\ell)}(x)$'s $(\ell \geq 0)$ with coefficients polynomials of x.

(b) $r_{u,0} = 1$ *and*

$$r_{u,v} = r_{u-1,v} + (u - v + 1) r_{u,v-1}, \; r_{u,u} = r_{u,u-1}, \; 1 \leq v < u. \tag{5.36}$$

(c)
$$r_{u,u} = (2u - 1)!!. \tag{5.37}$$

Proof. We proceed by induction on $u + v$.

By (5.18) and (5.22), we get

$$f_{u,u}(x) = q_1(x) h^u(x), \; u \geq 1.$$

Hence (a) holds for $v = 0$ and $r_{u,0} = 1$.

Since the only (u, v) with $u + v = 1$ and $0 \leq v \leq u$ is $(0, 1)$, (a) holds for $u + v = 1$.

Assume that (a) holds for $u + v \leq t$ $(t \geq 1)$. If $u + v = t + 1$, we have shown that the statement holds for $v = 0$. For $1 \leq v \leq u$, we have three cases: $v = 1$, $1 < v < u$ and $1 < v = u$.

When $1 \leq v < u$, applying (5.17) to $(i, m, \ell) = (u, u + v, 0)$ and using the induction hypothesis for $(u - 1, v), (u, v - 1)$, we get

$$f_{u,u+v}(x) = h(x) f_{u-1,u+v-1} + d' f_{u,u+v-1} + x f'_{u,u+v-1} \tag{5.38}$$
$$= \quad h(x) \left[r_{u-1,v} q_1(x) x^v h^{u-1-v}(x) \left(h'(x)\right)^v + s_{u-1,v}(x) h^{u-v}(x)\right]$$

$$
\begin{aligned}
&+d'\left[r_{u,v-1}q_1(x)x^{v-1}h^{u-v+1}(x)\left(h'(x)\right)^{v-1}+s_{u,v-1}(x)h^{u+2-v}(x)\right]\\
&+x\left[r_{u,v-1}q_1(x)x^{v-1}h^{u-v+1}(x)\left(h'(x)\right)^{v-1}+s_{u,v-1}(x)h^{u+2-v}(x)\right]'\\
=\;&r_{u-1,v}q_1(x)x^{v}h^{u-v}(x)\left(h'(x)\right)^{v}+[s_{u-1,v}(x)\\
&+d'r_{u,v-1}q_1(x)x^{v-1}\left(h'(x)\right)^{v-1}+d's_{u,v-1}(x)h(x)]h^{u+1-v}(x)\\
&+x\left[r_{u,v-1}q_1(x)x^{v-1}h^{u-v+1}(x)\left(h'(x)\right)^{v-1}\right.\\
&\left.+\,s_{u,v-1}(x)h^{u+2-v}(x)\right]'.
\end{aligned}
\tag{5.39}
$$

Denote W the last two lines of (5.39).

Case 1. $v=1$.

We have

$$
\begin{aligned}
W\;=\;&x\left[r_{u,v-1}q_1(x)h^{u-v+1}(x)+s_{u,v-1}(x)h^{u+2-v}(x)\right]'\\
=\;&x\left[r_{u,v-1}q_1'(x)h^{u-v+1}(x)+(u-v+1)r_{u,v-1}q_1(x)h'(x)h^{u-v}(x)\right.\\
&\left.+(u+2-v)s_{u,v-1}(x)h'(x)h^{u+1-v}(x)\right]\\
=\;&x\left[r_{u,v-1}q_1'(x)+(u+2-v)s_{u,v-1}(x)h'(x)\right]h^{u-v+1}(x)\\
&+x(u-v+1)r_{u,v-1}q_1(x)h'(x)h^{u-v}(x).
\end{aligned}
$$

Noting that $v=1$, thus the above equation can be written as

$$
\begin{aligned}
W\;=\;&x\left[r_{u,v-1}q_1'(x)+(u+2-v)s_{u,v-1}(x)h'(x)\right]h^{u-v+1}(x)\\
&+(u-v+1)r_{u,v-1}q_1(x)x^{v}h^{u-v}(x)\left(h'(x)\right)^{v}.
\end{aligned}
$$

Plugging this into (5.39) yields

$$
\begin{aligned}
&f_{u,u+v}(x)\\
=\;&r_{u-1,v}q_1(x)x^{v}h^{u-v}(x)\left(h'(x)\right)^{v}+[s_{u-1,v}(x)\\
&+d'r_{u,v-1}q_1(x)x^{v-1}\left(h'(x)\right)^{v-1}+d's_{u,v-1}(x)h(x)]h^{u+1-v}(x)\\
&+x\left[r_{u,v-1}q_1'(x)+(u+2-v)s_{u,v-1}(x)h'(x)\right]h^{u-v+1}(x)\\
&+(u-v+1)r_{u,v-1}q_1(x)x^{v}h^{u-v}(x)\left(h'(x)\right)^{v}\\
=\;&[r_{u-1,v}+(u-v+1)r_{u,v-1}]q_1(x)x^{v}h^{u-v}(x)\left(h'(x)\right)^{v}+[s_{u-1,v}(x)\\
&+d'r_{u,v-1}q_1(x)x^{v-1}\left(h'(x)\right)^{v-1}+d's_{u,v-1}(x)h(x)+xr_{u,v-1}q_1'(x)\\
&+x(u+2-v)s_{u,v-1}(x)h'(x)]h^{u-v+1}(x).
\end{aligned}
$$

Hence $f_{u,u+v}(x)$ is of the form (5.35) and (5.36) holds.

Case 2. $1<v<u$.

We have

$$
\begin{aligned}
W\;=\;&x\left[r_{u,v-1}q_1(x)x^{v-1}h^{u-v+1}(x)\left(h'(x)\right)^{v-1}+s_{u,v-1}(x)h^{u+2-v}(x)\right]'\\
=\;&(u-v+1)r_{u,v-1}q_1(x)x^{v}h^{u-v}(x)\left(h'(x)\right)^{v}\\
&+[r_{u,v-1}q_1'(x)x^{v}+(v-1)r_{u,v-1}q_1(x)x^{v-1}\left(h'(x)\right)^{v-1}
\end{aligned}
$$

$$\begin{aligned}
&+(v-1)r_{u,v-1}q_1(x)x^v\left(h'(x)\right)^{v-2}h''(x)\\
&+(u+2-v)xs_{u,v-1}(x)h'(x)]h^{u+1-v}(x)
\end{aligned}$$

Plugging this into (5.39) yields

$$\begin{aligned}
&f_{u,u+v}(x)\\
= \;&[r_{u-1,v}+(u-v+1)r_{u,v-1}]q_1(x)x^vh^{u-v}(x)\left(h'(x)\right)^v+[s_{u-1,v}(x)\\
&+d'r_{u,v-1}q_1(x)x^{v-1}\left(h'(x)\right)^{v-1}+d's_{u,v-1}(x)h(x)+r_{u,v-1}q_1'(x)x^v\\
&+(v-1)r_{u,v-1}q_1(x)x^{v-1}\left(h'(x)\right)^{v-2}\left(h'(x)+xh''(x)\right)\\
&+(u+2-v)xs_{u,v-1}(x)h'(x)]h^{u+1-v}(x).
\end{aligned}$$

Hence $f_{u,u+v}(x)$ is of the form (5.35) and (5.36) holds in this case too.

Case 3. $1<v=u$. *Thus* $u\geq 2$.

From the recurrence relation (5.17) and the initial condition (5.18), we see that each $f_{i,m}$ is a polynomial of the $h^{(\ell)}(x)$'s and the $q_1^{(\ell)}(x)$'s ($\ell\geq 0$) with coefficients polynomials of x. By (5.38) and the induction hypothesis (5.35) for $(u,v)=(u,u-1)$, we get

$$\begin{aligned}
&f_{u,u+v}(x)=f_{u,2u-1}(x)=h(x)f_{u-1,2u-1}+d'f_{u,2u-1}+xf'_{u,2u-1}\\
= \;&h(x)f_{u-1,2u-1}+r_{u,u-1}q_1(x)x^{u-1}h(x)\left(h'(x)\right)^{u-1}+s_{u,u-1}(x)h^2(x)\\
&+x[r_{u,u-1}q_1(x)x^{u-1}h(x)\left(h'(x)\right)^{u-1}+s_{u,u-1}(x)h^2(x)]'\\
= \;&[f_{u-1,2u-1}+r_{u,u-1}q_1(x)x^{u-1}\left(h'(x)\right)^{u-1}+s_{u,u-1}(x)h(x)]h(x)\\
&+x[r_{u,u-1}q_1'(x)x^{u-1}h(x)\left(h'(x)\right)^{u-1}\\
&+(u-1)r_{u,u-1}q_1(x)x^{u-2}h(x)\left(h'(x)\right)^{u-1}+r_{u,u-1}q_1(x)x^{u-1}\left(h'(x)\right)^{u}\\
&+(u-1)r_{u,u-1}q_1(x)x^{u-1}h(x)\left(h'(x)\right)^{u-2}h''(x)+s'_{u,u-1}(x)h^2(x)\\
&+2s_{u,u-1}(x)h'(x)h(x)]\\
= \;&r_{u,u-1}q_1(x)x^u\left(h'(x)\right)^u+[f_{u-1,2u-1}+r_{u,u-1}q_1(x)x^{u-1}\left(h'(x)\right)^{u-1}\\
&+s_{u,u-1}(x)h(x)+r_{u,u-1}q_1'(x)x^u\left(h'(x)\right)^{u-1}\\
&+(u-1)r_{u,u-1}q_1(x)x^{u-1}\left(h'(x)\right)^{u-2}\left(h'(x)+xh''(x)\right)\\
&+xs'_{u,u-1}(x)h(x)+2xs_{u,u-1}(x)h'(x)]h(x).
\end{aligned}$$

Hence $f_{u,u+v}(x)$ is of the form (5.35) and (5.36) holds in this case.

We use generating functions to prove (c).

Lemma 5.9. *Define*

$$T_v(x)=\sum_{u=v}^{\infty}r_{u,v}x^{u-v},\;\;v\geq 0. \tag{5.40}$$

Then we have

(a)

$$T_v(x)=\frac{T'_{v-1}(x)}{1-x},\;\;v\geq 1. \tag{5.41}$$

(b)

$$T_0(x) = \frac{1}{1-x} \text{ and } T_v(x) = \frac{(2v-1)!!}{(1-x)^{2v+1}}, \ v \geq 1. \tag{5.42}$$

Proof. (a) According to Definition (5.40),

$$\begin{aligned}(1-x)T_v(x) &= \sum_{u=v}^{\infty} r_{u,v}x^{u-v} - \sum_{u=v}^{\infty} r_{u,v}x^{u-v+1} \\ &= \sum_{u=v}^{\infty} r_{u,v}x^{u-v} - \sum_{u=v+1}^{\infty} r_{u-1,v}x^{u-v} \\ &= r_{v,v} + \sum_{u=v+1}^{\infty} (r_{u,v} - r_{u-1,v})x^{u-v}.\end{aligned}$$

By the recurrence relation (5.36), we get

$$r_{u,v} - r_{u-1,v} = (u-v+1)r_{u,v-1} \text{ for } u \geq v+1, \text{ and } r_{v-1,v} = r_{v,v}.$$

Thus

$$\begin{aligned}(1-x)T_v(x) &= r_{v,v} + \sum_{u=v+1}^{\infty} (u-v+1)r_{u,v-1}x^{u-v} \\ &= r_{v-1,v} + \sum_{u=v+1}^{\infty} (u-v+1)r_{u,v-1}x^{u-v} \\ &= \sum_{u=v}^{\infty}(u-v+1)r_{u,v-1}x^{u-v}. \end{aligned} \tag{5.43}$$

On the other hand, taking the derivative of both sides of Definition (5.40), we see that $T'_{v-1}(x)$ also equals (5.43). Therefore (5.41) holds.

(b) Since $r_{u,0} = 1$ (see Proposition 5.8(b)), we have

$$T_0(x) = \sum_{u=0}^{\infty} r_{u,0}x^u = \sum_{u=0}^{\infty} x^u = \frac{1}{1-x}.$$

Applying (a) to $v = 1$, we get

$$T_1(x) = \frac{T'_0(x)}{1-x} = \frac{1}{1-x}\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^3}.$$

Thus (5.42) holds for $v = 1$.

Assume that (5.42) holds for $v-1$ ($v \geq 2$). It follows from (a) and the induction hypothesis that

$$T_v(x) = \frac{T'_{v-1}(x)}{1-x} = \frac{1}{1-x}\left(\frac{(2v-3)!!}{(1-x)^{2v-1}}\right)' = \frac{(2v-1)!!}{(1-x)^{2v+1}}.$$

Hence (5.42) holds for v and therefore for any $v \geq 1$. $\square$

Let us return to the proof of (c). For any $u \geq 1$,

$$r_{u,u} = T_u(0) = (2u-1)!!$$

by Definition (5.40) and Lemma 5.9. □

Setting $v = u$ and $x = 1$ in Proposition 5.8(a) and using (5.13) and (5.37), we get

$$f_{u,2u}(1) = r_{u,u} q_1(1) \left(h'(1)\right)^u = (2u-1)!! q_1(1) \left(h'(1)\right)^u,$$

as desired. □

6. Lekkerkerker and Gaussian Behavior in Hannah's Problem

In this section, we will apply the generating function approach to study the distributions of the numbers of positive summands and negative summands in the far-difference representation (see Definition 1.5). We will prove that these two distributions are bivariate Gaussian with a computable, negative correlation.

6.1. **Generating Function of the Probability Density.**

Let $p_{n,k,l}$ ($n > 0$) be the number of far-difference representations of numbers in $(S_{n-1}+1, S_n]$ with k positive summands and l negative summands. Clearly, $p_{n,k,l} = 0$ if $k \leq 0$ or $l < 0$. For every far-difference representation $N = \sum_{j=1}^m a_j F_{i_j} \in [S_{n-1}+1, S_n]$, $N' := \sum_{j=2}^m a_j F_{i_j}$ is also a far-difference representation. Theorem 1.6 states that $i_1 = n$ and $a_1 = 1$, therefore $N' \in [S_{n-1}+1-F_n, S_n - F_n] = [-S_{n-3}, S_{n-4}]$. Thus $p_{n,k,l}$ is the number of far-difference representations of numbers in $[-S_{n-3}, S_{n-4}]$ with $k-1$ positive summands and l negative summands.

Let $n \geq 4$. We have two cases $(k-1, l) \neq (0,0)$ and $(k-1, l) = (0,0)$.

Case 1. $(k-1, l) \neq (0,0)$.

Then $N' = N - a_1 F_{i_1} \neq 0$. Let $N(J, k, l)$ be the number of far-difference representations of integers in interval J with k positive summands and l negative summands. Thus

$$\begin{aligned} p_{n,k,l} &= N((0, S_{n-4}], k-1, l) + N([-S_{n-3}, 0), k-1, l) \\ &= N((0, S_{n-4}], k-1, l) + N((0, S_{n-3}], l, k-1) \\ &= \sum_{i=1}^{n-4} p_{i,k-1,l} + \sum_{i=1}^{n-3} p_{i,l,k-1}. \end{aligned} \tag{6.1}$$

For $n \geq 5$, replacing n with $n-1$ yields

$$p_{n-1,k,l} = \sum_{i=1}^{n-5} p_{i,k-1,l} + \sum_{i=1}^{n-4} p_{i,l,k-1}. \tag{6.2}$$

Subtracting (6.2) from (6.1), we get

$$p_{n,k,l} = p_{n-1,k,l} + p_{n-4,k-1,l} + p_{n-3,l,k-1}, \quad n \geq 5. \tag{6.3}$$

Case 2. $(k-1, l) = (0,0)$.

Then $N' = N - a_1 F_{i_1}$ can be 0. Thus we have

$$p_{n,k,l} = \sum_{i=1}^{n-4} p_{i,k-1,l} + \sum_{i=1}^{n-3} p_{i,l,k-1} + 1 \tag{6.4}$$

and

$$p_{n-1,k,l} = \sum_{i=1}^{n-5} p_{i,k-1,l} + \sum_{i=1}^{n-4} p_{i,l,k-1} + 1. \tag{6.5}$$

Subtracting (6.5) from (6.4), we see that (6.3) still holds.

Let $n \geq 9$. Replacing (n, k, l) in (6.3) with $(n-3, l, k-1)$ gives

$$p_{n-3,l,k-1} = p_{n-4,l,k-1} + p_{n-7,l-1,k-1} + p_{n-6,k-1,l-1}, \; n \geq 8. \tag{6.6}$$

Rearranging the terms of (6.3), we obtain

$$p_{n-3,l,k-1} = p_{n,k,l} - p_{n-1,k,l} - p_{n-4,k-1,l}, \; n \geq 5. \tag{6.7}$$

Replacing (n, k, l) in (6.3) with $(n-1, k, l)$ and $(n-4, k, l-1)$ (since $n \geq 9$, $n-1 > n-4 \geq 5$, thus (6.7) applies to $n-1$ and $n-4$), we get

$$p_{n-4,l,k-1} = p_{n-1,k,l} - p_{n-2,k,l} - p_{n-5,k-1,l} \tag{6.8}$$

and

$$p_{n-7,l-1,k-1} = p_{n-4,k,l-1} - p_{n-5,k,l-1} - p_{n-8,k-1,l-1}. \tag{6.9}$$

Plugging (6.6), (6.8) and (6.9) into (6.3) yields

$$\begin{aligned} p_{n,k,l} &= 2p_{n-1,k,l} - p_{n-2,k,l} + p_{n-4,k-1,l} + p_{n-4,k,l-1} - p_{n-5,k-1,l} \\ &\quad - p_{n-5,k,l-1} + p_{n-6,k-1,l-1} - p_{n-8,k-1,l-1}, \; n \geq 9. \end{aligned} \tag{6.10}$$

Let the generating function be $\hat{\mathscr{G}}(x, y, z) = \sum_{n>0,k>0,l\geq 0} p_{n,k,l} x^k y^l z^n$ (analogous to $\mathscr{G}(x, y)$, see Section 3).

Multiplying both sides of (6.10) by $x^k y^l z^n$, we get

$$\begin{aligned} p_{n,k,l} x^k y^l z^n &= 2z p_{n-1,k,l} x^k y^l z^{n-1} - z^2 p_{n-2,k,l} x^k y^l z^{n-2} \\ &\quad + xz^4 p_{n-4,k-1,l} x^{k-1} y^l z^{n-4} + yz^4 p_{n-4,k,l-1} x^{k-1} y^l z^{n-4} \\ &\quad - xz^5 p_{n-5,k-1,l} x^{k-1} y^l z^{n-5} - yz^5 p_{n-5,k,l-1} x^k y^{l-1} z^{n-5} \\ &\quad + xyz^6 p_{n-6,k-1,l-1} x^{k-1} y^{l-1} z^{n-6} \\ &\quad - xyz^8 p_{n-8,k-1,l-1} x^{k-1} y^{l-1} z^{n-8}. \end{aligned}$$

Summing both sides over $n \geq 9$ and recalling that $p_{n,k,l} = 0$ if $k \geq 0$ or $l < 0$, we obtain

$$\begin{aligned} &\hat{\mathscr{G}}(x, y, z) \\ = \; &2z\hat{\mathscr{G}}(x, y, z) - 2 \sum_{1<n\leq 8} p_{n-1,k,l} x^k y^l z^n - z^2 \hat{\mathscr{G}}(x, y, z) \\ &+ \sum_{2<n\leq 8} p_{n-2,k,l} x^k y^l z^n + xz^4 \hat{\mathscr{G}}(x, y, z) - \sum_{4<n\leq 8} p_{n-4,k-1,l} x^k y^l z^n \\ &+ yz^4 \hat{\mathscr{G}}(x, y, z) - \sum_{4<n\leq 8} p_{n-4,k,l-1} x^k y^l z^n - xz^5 \hat{\mathscr{G}}(x, y, z) \end{aligned}$$

$$\begin{aligned}
&+\sum_{5<n\leq 8} p_{n-5,k-1,l}x^k y^l z^n - yz^5\hat{\mathscr{G}}(x,y,z) + \sum_{5<n\leq 8} p_{n-5,k,l-1}x^k y^l z^n \\
&+xyz^6\hat{\mathscr{G}}(x,y,z) - \sum_{6<n\leq 8} p_{n-6,k-1,l-1}x^k y^l z^n - xyz^8\hat{\mathscr{G}}(x,y,z) \\
= \;& \left(2z - z^2 + xz^4 + yz^4 - xz^5 - yz^5 + xyz^6 - xyz^8\right)\hat{\mathscr{G}}(x,y,z) \\
&-2\sum_{1<n\leq 8} p_{n-1,k,l}x^k y^l z^n + \sum_{2<n\leq 8} p_{n-2,k,l}x^k y^l z^n \\
&-\sum_{4<n\leq 8} p_{n-4,k-1,l}x^k y^l z^n - \sum_{4<n\leq 8} p_{n-4,k,l-1}x^k y^l z^n \\
&+\sum_{5<n\leq 8} p_{n-5,k-1,l}x^k y^l z^n + \sum_{5<n\leq 8} p_{n-5,k,l-1}x^k y^l z^n \\
&-\sum_{6<n\leq 8} p_{n-6,k-1,l-1}x^k y^l z^n. \qquad (6.11)
\end{aligned}$$

We calculated all $p_{n,k,l}$'s for $n \leq 8$ and found that the only terms in the right-hand side of (6.11) that are not canceled are xz, $-xz^2$, xyz^4 and $-xyz^5$, therefore

$$\begin{aligned}
\hat{\mathscr{G}}(x,y,z) &= \frac{x(z-z^2) + xy(z^4 - z^5)}{1 - (2z - z^2 + xz^4 + yz^4 - xz^5 - yz^5 + xyz^6 - xyz^8)} \\
&= \frac{xz + xyz^4}{1 - z - (x+y)z^4 - xyz^6 - xyz^7}. \qquad (6.12)
\end{aligned}$$

6.2. **Lekkerkerker's Theorem and Gaussian Behavior.**

To show that $\mathcal{K}_n$ and $\mathcal{L}_n$ are bivariate Gaussian, it suffices to prove the Gaussian behavior of $\mathcal{K}_n$, $\mathcal{L}_n$ and $a\mathcal{K}_n + b\mathcal{L}_n$ for any a, b with $ab \neq 0$. Note that the coefficient of z^n in $\hat{\mathscr{G}}(x,y,z)$ is $\sum_{k>0,l\geq 0} p_{n,k,l}x^k y^l$. Setting $y = 1$, $x = 1$ and $(x,y) = (w^a, w^b)$ with $ab \neq 0$ and applying differential identities will give the moments of $\mathcal{K}_n$, $\mathcal{L}_n$ and $a\mathcal{K}_n + b\mathcal{L}_n$, respectively.

Let $\hat{A}(z)$ be the denominator of $\hat{\mathscr{G}}(x,y,z)$, i.e.,

$$\hat{A}(z) = 1 - z - (x+y)z^4 + xyz^6 + xyz^7 \qquad (6.13)$$

Clearly, 0 is not a root of $\hat{A}(z)$. When $x = y = 1$, we have

$$\hat{A}(z) = 1 - z - 2z^4 - z^6 - z^7 = -(z^2 + z - 1)(z^2 + 1)(z^3 + 1). \qquad (6.14)$$

Thus $\hat{A}(z)$ has no multiple roots; moreover, except $\frac{\sqrt{5}-1}{2}$, any other root z of $\hat{A}(z)$ satisfies $|z| \leq 1$. Note that in both cases $x = 1$ and $y = 1$, the coefficients of $\hat{A}(z)$ are polynomials in one variable and hence continuous, thus the roots of $\hat{A}(z)$ are continuous (see Appendix A).

6.2.1. *Distribution of the Number of Positive Summands.*

To study the number of positive summands, we set $y = 1$ and let $\hat{A}_x(z)$ be the $\hat{A}(z)$ when $y = 1$, then

$$\hat{A}_x(z) = 1 - z - (x+1)z^4 - xz^6 - xz^7. \qquad (6.15)$$

Note that $\hat{A}_1(z)$ has no multiple roots (see 6.14), thus similarly to Proposition 4.1, we have the following proposition (see Appendix J for the proof).

Proposition 6.1. *There exists $\varepsilon \in (0,1)$ such that for any $x \in (1-\varepsilon, 1+\varepsilon)$, we have*
(a) *$\hat{A}_x(z)$ has exactly 7 roots but no multiple roots.*
(b) *There exists a root $z_1(x)$ such that $|z_1(x)| < 1$ and $|z_1(x)| < |z_i(x)|$, $1 < i \le 7$.*
(c) *Each root $z_i(x)$ ($1 \le i \le 7$) is continuous and ℓ-times differentiable for any $\ell \ge 1$, and*

$$z_i'(x) = -\frac{z_i^4(x) + z_i^6(x) + z_i^7(x)}{1 + 4(x+1)z_i^3(x) + 6xz_i^5(x) + 7xz_i^6(x)}, \tag{6.16}$$

(d)
$$\frac{1}{\hat{A}_x(z)} = -\frac{1}{x}\sum_{i=1}^{7} \frac{1}{(z - z_i(x))\prod_{j\neq i}(z_j(x) - z_i(x))}. \tag{6.17}$$

Assume $x \in (1-\varepsilon, 1+\varepsilon)$. Combining (6.12) and Proposition 6.1(d), we get

$$\hat{\mathscr{G}}(x,1,z) = -(z + z^4)\sum_{i=1}^{7} \frac{1}{(z - z_i(x))\prod_{j\neq i}(z_j(x) - z_i(x))}.$$

Denote $\hat{g}_+(x)$ the coefficient of z^n in $\hat{\mathscr{G}}(x,1,z)$, i.e.,

$$\hat{g}_+(x) = \sum_{k>0,l\ge 0} p_{n,k,l}x^k$$

(analogous to $g(x)$, see (4.3)), then

$$\begin{aligned}
\hat{g}_+(x) &= \langle z^{n-4}\rangle \sum_{i=1}^{7} \frac{1}{(1 - \frac{z}{z_i(x)})z_i(x)\prod_{j\neq i}(z_j(x) - z_i(x))} \\
&\quad + \langle z^{n-1}\rangle \sum_{i=1}^{7} \frac{1}{(1 - \frac{z}{z_i(x)})z_i(x)\prod_{j\neq i}(z_j(x) - z_i(x))} \\
&= \sum_{i=1}^{7} \frac{1}{z_i^{n-3}(x)\prod_{j\neq i}(z_j(x) - z_i(x))} \\
&\quad + \sum_{i=1}^{7} \frac{1}{z_i^{n}(x)\prod_{j\neq i}(z_j(x) - z_i(x))} \\
&= \sum_{i=1}^{7} \frac{1 + z_i^3(x)}{z_i^{n}(x)\prod_{j\neq i}(z_j(x) - z_i(x))}.
\end{aligned} \tag{6.18}$$

Let

$$\hat{q}_{i+}(x) = \frac{1 + z_i^3(x)}{x\prod_{j\neq i}(z_j(x) - z_i(x))} \tag{6.19}$$

(analogous to $q_i(x)$, see (4.11)), then

$$\hat{g}_+(x) = \sum_{i=1}^{7} x\hat{q}_{i+}(x)z_i^{-n}(x). \tag{6.20}$$

Since for any ℓ, $z_i(x)$ is ℓ-times differentiable, so is $\hat{q}_{i+}(x)$.

Define

$$\hat{\mathcal{A}}_x(z) = z^7 \hat{A}_x\left(\frac{1}{z}\right) = z^7 - z^6 - (x+1)z^3 - xz - x$$

(analogous to $\mathcal{A}(x)$, see (4.13)), then we have the roots of $\hat{\mathcal{A}}_x(z)$ are $\hat{\alpha}_i(x) := (z_i(x))^{-1}$ (analogous to $\alpha_i(x)$). According to Proposition 6.1(b),

$$|\hat{\alpha}_1(x)| > 1 \text{ and } |\hat{\alpha}_1(x)| > |\hat{\alpha}_i(x)|, \ 1 < i \leq 7. \tag{6.21}$$

Substituting $z_i(x)$ by $(\hat{\alpha}_i(x))^{-1}$ in (6.20), we get

$$\hat{g}_+(x) = \sum_{i=1}^{7} x\hat{q}_{i+}(x)\hat{\alpha}_i^n(x)$$

(analogous to (4.15)).

Let $\hat{\mu}_{n+}$ be the mean of $\mathcal{K}_n$, then analogously to Theorem 1.2 and (4.20), we have

$$\hat{\mu}_{n+} = \hat{C}n + \hat{d}_+ + o(1), \tag{6.22}$$

where

$$\hat{C} = \frac{\hat{\alpha}_1'(1)}{\hat{\alpha}_1(1)} \text{ and } \hat{d}_+ = \frac{\hat{q}_{1+}(1) + \hat{q}_{1+}'(1)}{\hat{q}_{1+}(1)}. \tag{6.23}$$

Similarly to (4.21), we have

$$\hat{C} = \frac{\hat{\alpha}_1'(1)}{\hat{\alpha}_1(1)} = -\frac{z_1'(1)}{z_1(1)}. \tag{6.24}$$

From (6.14), we see that $z_1(1) = \frac{\sqrt{5}-1}{2}$. Then we can evaluate $\hat{C}$ by (6.16). Denote $\Phi = \frac{\sqrt{5}-1}{2}$, then $\Phi^2 + \Phi = 1$. Setting $i = x = 1$ in (6.16), we get

$$\hat{C} = -\frac{z_1'(1)}{z_1(1)} = \frac{\Phi^3 + \Phi^5 + \Phi^6}{1 + 8\Phi^3 + 6\Phi^5 + 7\Phi^6} = \frac{\Phi^2}{10\Phi^2} = \frac{1}{10}. \tag{6.25}$$

From (6.25), we also see that

$$z_1'(1) = -\frac{\Phi}{10}. \tag{6.26}$$

Next we calculate $d+$. Recall from (6.19) that

$$\hat{q}_{1+}(x) = \frac{1 + z_1^3(x)}{x\prod_{j\neq 1}(z_j(x) - z_1(x))}. \tag{6.27}$$

Let

$$\hat{E}(x) = \prod_{j\neq 1}(z_j(x) - z_1(x)), \tag{6.28}$$

then

$$\begin{aligned} &\frac{\hat{q}_{1+}(x) + \hat{q}_{1+}'(x)}{\hat{q}_{1+}(x)} = 1 + \frac{\hat{q}_{1+}'(1)}{\hat{q}_{1+}(1)} \\ = \quad & 1 + \frac{[(1+z_1^3(x))'x\hat{E}(x) - (x\hat{E}(x))'(1+z_1^3(x))]/(x\hat{E}(x))^2}{(1+z_1^3(x))/(x\hat{E}(x))} \\ = \quad & 1 + \frac{(1+z_1^3(x))'}{1+z_1^3(x)} - \frac{(x\hat{E}(x))'}{x\hat{E}(x)} = 1 + \frac{3z_1^2(x)z_1'(x)}{1+z_1^3(x)} - \frac{\hat{E}(x) + x\hat{E}'(x)}{x\hat{E}(x)}. \end{aligned}$$

Setting $x = 1$ and using (6.26), we get

$$\hat{d}_+ = \frac{3z_1^2(1)z_1'(1)}{1+z_1^3(1)} - \frac{\hat{E}'(1)}{\hat{E}(1)} = \frac{3\sqrt{5}-9}{40} - \frac{\hat{E}'(1)}{\hat{E}(1)} \tag{6.29}$$

(see (6.23) for the definition of $\hat{d}_+$).

Thus it remains to evaluate $\hat{E}(1)$ and $\hat{E}'(1)$.

Setting $z = z' + z_1(x)$ in (6.37), we get

$$\hat{A}_+(z) = 1 - z' - z_1(x) - (x+1)(z'+z_1(x))^4 - x(z'+z_1(x))^6 - x(z'+z_1(x))^7. \tag{6.30}$$

On the other hand, similar to (E.2), we have

$$\hat{A}_+(z) = -xz' \prod_{j\neq 1}(z' + z_1(x) - z_j(x)). \tag{6.31}$$

Comparing the coefficients of z' in (6.30) and (6.31) gives

$$x\prod_{j\neq 1}(z_1(x) - z_j(x)) = 1 + 4(1+x)z_1^3(x) + 6xz_1^5(x) + 7xz_1^6(x).$$

Thus

$$\hat{E}(x) = \prod_{j\neq 1}(z_1(x) - z_j(x)) = \frac{1}{x} + 4\left(1+\frac{1}{x}\right)z_1^3(x) + 6z_1^5(x) + 7z_1^6(x).$$

Taking the derivative of both sides, we obtain

$$\begin{aligned}\hat{E}'(x) &= -\frac{1}{x^2} - \frac{4}{x^2}z_1^3(x) + 12\left(1+\frac{1}{x}\right)z_1^2(x)z_1'(x) + 30z_1^4(x)z_1'(x)\\ &\quad +42z_1^5(x)z_1'(x).\end{aligned}$$

Therefore

$$\frac{\hat{E}'(1)}{\hat{E}(1)} = \frac{-1-4\Phi^3-24\Phi^2\frac{\Phi}{10}-30\Phi^4\frac{\Phi}{10}-42\Phi^5\frac{\Phi}{10}}{1+8\Phi^3+6\Phi^5+7\Phi^6} = \frac{29\sqrt{5}-95}{10}. \tag{6.32}$$

Plugging (6.32) into (6.29) yields

$$\hat{d}_+ = \frac{3\sqrt{5}-9}{40} - \frac{29\sqrt{5}-95}{10} = \frac{371-113\sqrt{5}}{40} \approx 2.95810796. \tag{6.33}$$

Thus we proved the Lekkerkerker's Theorem for the number of positive terms.

Theorem 6.2. *The mean of the numbers of positive summands in the far-difference representations of integers in* $(S_{n-1}, S_n]$

$$\hat{\mu}_{n+} = \frac{1}{10}n + \frac{371-113\sqrt{5}}{40} + o(1). \tag{6.34}$$

Using the same approach in Section 5 (see Proposition 5.5), we obtain the variance of k :

$$\mu_{n+}(2) = \tilde{\mu}_{n+}(2) + o(1) = \hat{h}'(1)n + \hat{q}_{1+}''(1) + o(1),$$

where

$$\hat{h}(x) = \frac{x\hat{\alpha}_1'(x)}{\hat{\alpha}_1(x)} - \hat{C} = -\frac{xz_1'(x)}{z_1(x)} - \hat{C}. \tag{6.35}$$

Applying (6.16), we get

$$\begin{aligned}
\hat{h}'(x) &= \left[-\frac{xz_1'(x)}{z_1(x)}\right]' = \left[\frac{xz_1^3(x)+xz_1^5(x)+xz_1^6(x)}{1+4(x+1)z_1^3(x)+6xz_1^5(x)+7xz_1^6(x)}\right]' \\
&= [1+4(x+1)z_1^3(x)+6xz_1^5(x)+7xz_1^6(x)]^{-1}[z_1^3(x) \\
&\quad +3xz_1^2(x)z_1'(x)+z_1^5(x)+5xz_1^4(x)z_1'(x) \\
&\quad +z_1^6(x)+6xz_1^5(x)z_1'(x)] - [1+4(x+1)z_1^3(x)+6xz_1^5(x) \\
&\quad +7xz_1^6(x)]^{-2}[xz_1^3(x)+xz_1^5(x)+xz_1^6(x)][4z_1^3(x) \\
&\quad +12(x+1)z_1^2(x)z_1'(x)+6z_1^5(x)+30xz_1^4(x)z_1'(x)+7z_1^6(x) \\
&\quad +42xz_1^5(x)z_1'(x)].
\end{aligned}$$

Setting $x=1$ and using $z_1(1)=\Phi$, $z_1'(1)=\frac{\Phi}{10}$ (see (6.25)), we get

$$\hat{h}'(x) = \frac{29\sqrt{5}-25}{1000} \approx 0.0398459713.$$

Hence we obtain the following theorem on the variance of the number of positive terms.

Theorem 6.3. *The variance of the numbers of positive summands in the far-difference representations of integers in* $(S_{n-1}, S_n]$

$$\hat{\mu}_{n+}(2) = \frac{29\sqrt{5}-25}{1000}n + \hat{q}_{1+}''(1) + o(1) \tag{6.36}$$

(note that $\hat{q}_{1+}''(1)$ *is a constant).*

Remark 6.1. *We already have the formulas for* $z_1'(x)$, $\hat{E}(x)$ *(as function of* $z_1(x)$*) and* $\hat{E}'(x)$ *(as functions of* $z_1(x)$ *and* $z_1'(x)$*), so we can derive the formula for* $z_1''(x)$ *and then for* $\hat{E}''(x)$ *(as functions of* $z_1(x)$ *and* $z_1'(x)$*). Then we will have a formula for* $\hat{q}_{1+}''(x) = \Big([1+z_1^3(x)]/x\hat{E}(x)\Big)''$ *(see (6.27) and (6.28)). Since the values of* $z_1(1)$ *and* $z_1'(1)$ *are known (which are* Φ *and* $\frac{\Phi}{10}$*), we can calculate the value of* $\hat{q}_{1+}''(1)$ *as well.*

Since the coefficient of n in the formula (6.36) of $\hat{\mu}_{n+}(2)$ is nonzero, we can apply the same procedure in the proof of Theorem 5.1 to prove that the distribution of k is Gaussian.

Theorem 6.4. *The distribution of the number of positive summands in the far-diffrence representations of integers in* $(S_{n-1}, S_n]$ *is Gaussian as* $n\to\infty$.

6.2.2. *Distribution of the Number of Negative Summands.*

Set $y=1$ and let $\hat{A}_y(z)$ be the $\hat{A}(z)$ when $x=1$, i.e.,

$$\hat{A}_y(z) = 1 - z - (y+1)z^4 - yz^6 - yz^7. \tag{6.37}$$

Since $\hat{A}(z)$ is symmetric with respect to x and y (see (6.13)), $\hat{A}_x(z)$ and $\hat{A}_y(z)$ are symmetric. Thus we have a counterpart of Proposition 6.1 with x replaced by y.

Assume $y\in I_\varepsilon$. Combining (6.12) and Proposition 6.1(d) (for y), we get

$$\hat{\mathscr{G}}(1,y,z) = -(z+z^4)\sum_{i=1}^{7}\frac{1}{y(z-z_i(y))\prod_{j\neq i}(z_j(y)-z_i(y))}.$$

Denote $\hat{g}_-(x)$ the coefficient of z^n in $\hat{\mathscr{G}}(1, y, z)$. Similarly to (6.19) and (6.20), we get

$$\hat{g}_-(x) = \sum_{i=1}^{7} x\hat{q}_{i-}(x)z_i^{-n}(x),$$

where

$$\hat{q}_{i-}(y) = \frac{1 + z_i^3(y)}{y^2 \prod_{j\neq i}(z_j(y) - z_i(y))}, \tag{6.38}$$

Similarly, for any ℓ, $\hat{q}_{i-}(y)$ is ℓ-times differentiable. We also have a counterpart of (6.22) and (6.23), i.e.,

$$\hat{\mu}_{n-} = \hat{C}n + \hat{d}_- + o(1),$$

where

$$\hat{C} = \frac{\hat{\alpha}_1'(1)}{\hat{\alpha}_1(1)} = \frac{1}{10} \text{ and } \hat{d}_- = \frac{\hat{q}_{1-}(1) + \hat{q}_{1-}'(1)}{\hat{q}_{1-}(1)}.$$

Recall from (6.38) that $\hat{q}_{1-}(y) = (1 + z_1^3(y))/(y^2\hat{E}(y))$ (see (6.28) for the definition of $\hat{E}$), then we get

$$\frac{\hat{q}_{1-}(y) + \hat{q}_{1-}'(y)}{\hat{q}_{1-}(y)} = 1 + \frac{\hat{q}_{1-}'(y)}{\hat{q}_{1-}(y)} = 1 + \frac{3z_1^2(y)z_1'(y)}{1 + z_1^3(y)} - \frac{2y\hat{E}(y) + y^2\hat{E}'(y)}{y^2\hat{E}(y)}.$$

Setting $y = 1$ yields

$$\hat{d}_- = 1 + \frac{3z_1^2(1)z_1'(1)}{1 + z_1^3(1)} - \frac{2\hat{E}(1) + \hat{E}'(1)}{\hat{E}(1)} = \frac{3z_1^2(1)z_1'(1)}{1 + z_1^3(1)} - \frac{\hat{E}'(1)}{\hat{E}(1)} - 1. \tag{6.39}$$

Comparing (6.39) to (6.29), we see that $\hat{d}_- = \hat{d}_+ - 1$; in other words, there is one more positive term.

Thus by (6.33), we have $\hat{d}_- = (331 - 113\sqrt{5})/40 \approx 1.95810796$.

Theorem 6.5. *The mean of the numbers of negative summands in the far-difference representations of integers in* $(S_{n-1}, S_n]$

$$\hat{\mu}_{n-} = \frac{1}{10}n + \frac{331 - 113\sqrt{5}}{40} + o(1) = \hat{\mu}_{n+} - 1 + o(1). \tag{6.40}$$

For variance and Gaussian behavior, we also have similar results as in Theorem 6.3 and Theorem 6.4 for the number of negative terms.

Theorem 6.6. *The variance of the numbers of negative terms in the far-difference representations of integers in* $(S_{n-1}, S_n]$

$$\hat{\mu}_{n-}(2) = \frac{15 + 21\sqrt{5}}{1000}n + \hat{q}_{1-}''(1) + o(1), \tag{6.41}$$

where $\hat{q}_{1-}''(1)$ *is computable (see Remark 6.1).*

Theorem 6.7. *The distribution of the number of negative terms in the far-diffrence representations of integers in* $(S_{n-1}, S_n]$ *is Gaussian as* $n \to \infty$.

6.2.3. *Distribution of $a\mathcal{K}_n + b\mathcal{L}_n$.*

To study the distribution of $a\mathcal{K}_n + b\mathcal{L}_n$ with $ab \neq 0$, we set $(x, y) = (w^a, w^b)$, then

$$\hat{A}_w(z) = 1 - z - (w^a + w^b)z^4 - w^{a+b}z^6 - w^{a+b}z^7.$$

We have the following proposition similarly to Proposition 6.1 (see Appendix K for the proof).

Proposition 6.8. *There exists $\varepsilon \in (0,1)$ such that for any $w \in I_\varepsilon = (1-\varepsilon, 1+\varepsilon)$,*

(a) *$\hat{A}_w(z)$ has exactly 7 roots but no multiple roots.*

(b) *There exists a root $e_1(w)$ such that $|e_1(w)| < 1$ and $|e_1(w)| < |e_i(w)|$, $1 < i \leq 7$.*

(c) *Each root $e_i(w)$ $(1 \leq i \leq 7)$ is continuous and ℓ-times differentiable for any $\ell \geq 1$, and*

$$e_i'(w) = -\frac{\left(aw^{a-1} + bw^{b-1}\right)e_i^4(w) + (a+b)w^{a+b-1}[e_i^6(w) + e_i^7(w)]}{1 + 4(w^a + w^b)e_i^3(x) + 6w^{a+b}e_i^5(w) + 7w^{a+b}e_i^6(w)} \tag{6.42}$$

(d)
$$\frac{1}{\hat{A}_w(z)} = -\frac{1}{w^{a+b}}\sum_{i=1}^{7}\frac{1}{(z - e_i(w))\prod_{j\neq i}(e_j(w) - e_i(w))}. \tag{6.43}$$

Assume $w \in I_\varepsilon$. Combining (6.12) and Proposition 6.1(d), we get

$$\hat{\mathscr{G}}(w^a, w^b, z) = -(z + w^b z^4)\sum_{i=1}^{7}\frac{1}{w^b(z - e_i(w))\prod_{j\neq i}(e_j(w) - e_i(w))}.$$

Denote $\hat{g}(w)$ the coefficient of z^n in $\hat{\mathscr{G}}(w^a, w^b, z)$, i.e.,

$$\hat{g}(w) = \sum_{k>0, l\geq 0} p_{n,k,l} w^{ak+bl},$$

then

$$\begin{aligned}
\hat{g}(w) &= \langle z^{n-4}\rangle \sum_{i=1}^{7}\frac{w^b}{(1 - \frac{z}{e_i(w)})e_i(w)\prod_{j\neq i}(e_j(w) - e_i(w))} \\
&\quad + \langle z^{n-1}\rangle \sum_{i=1}^{7}\frac{1}{(1 - \frac{z}{e_i(w)})e_i(w)\prod_{j\neq i}(e_j(w) - e_i(w))} \\
&= \sum_{i=1}^{7}\frac{w^b}{e_i^{n-3}(w)\prod_{j\neq i}(e_j(w) - e_i(w))} \\
&\quad + \sum_{i=1}^{7}\frac{1}{e_i^{n}(w)\prod_{j\neq i}(e_j(w) - e_i(w))} \\
&= \sum_{i=1}^{7}\frac{1 + w^b e_i^3(w)}{e_i^{n}(w)\prod_{j\neq i}(e_j(w) - e_i(w))}.
\end{aligned}$$

Let

$$\hat{q}(w) = \frac{1 + w^b e_i^3(w)}{w\prod_{j\neq i}(e_j(w) - e_i(w))}.$$

Since $e_i(x)$ is ℓ-times differentiable for any ℓ, so is $\hat{q}(x)$.

Similarly to the proof of Theorem 5.1 and Theorem 6.4, to show the Gaussian behavior of $a\mathcal{K}_n + b\mathcal{L}_n$, it suffices to verify that $\hat{h}'_{a,b}(1) \neq 0$, where

$$\hat{h}_{a,b}(w) = -\frac{we'_1(w)}{e_1(w)} - \hat{C}_{a,b}$$

with $\hat{C}_{a,b} = -e'_1(1)/e_1(1)$ constant (analogous to (6.35) and (6.24)). To prove, we derive a formula for $\hat{h}'_{a,b}(w)$ in terms of $e_1(w)$ by using (6.42). Then by $e_1(1) = \Phi$ we get

$$\hat{h}'_{a,b}(1) = \frac{\sqrt{5}-1}{200}\left[10\left(a^2+b^2\right) - \frac{20-\sqrt{5}}{5}(a+b)^2\right] \tag{6.44}$$

Finally, we verify that it is nonzero (details can be found in Appendix L).

Combining the Gaussian behavior of $a\mathcal{K}_n + b\mathcal{L}_n$ ($ab \neq 0$) with Theorem 6.4 and Theorem 6.7 gives the following theorem.

Theorem 6.9. *For any real numbers a and b, the distribution of $a\mathcal{K}_n + b\mathcal{L}_n$ is Gaussian as $n \to \infty$, where $\mathcal{K}_n$ and $\mathcal{L}_n$ are the numbers of positive and negative summands in the far-difference representations of integers in $(S_{n-1}, S_n]$, respectively. In other words, $\mathcal{K}_n$ and $\mathcal{L}_n$ are bivariate Gaussian as $n \to \infty$.*

Moreover, (6.44) also gives a formula for the variance of $a\mathcal{K}_n + b\mathcal{L}_n$. We can get the mean of $a\mathcal{K}_n + b\mathcal{L}_n$ from Theorem 6.2 and Theorem 6.5 as well.

Theorem 6.10. *The mean of $a\mathcal{K}_n + b\mathcal{L}_n$ is*

$$\frac{a+b}{10}n + \frac{371-113\sqrt{5}}{40}a + \frac{331-113\sqrt{5}}{40}b + o(1). \tag{6.45}$$

The variance of $a\mathcal{K}_n + b\mathcal{L}_n$ is

$$\frac{\sqrt{5}-1}{200}\left[10\left(a^2+b^2\right) - \frac{20-\sqrt{5}}{5}(a+b)^2\right] n + q_{a,b} + o(1), \tag{6.46}$$

where $q_{a,b}$ is a constant dependent only on a and b.

In particular, if we set $a = b = 1$ and $(a,b) = (1,1)$ in (6.46), we get

$$\operatorname{var}(\mathcal{K}_n + \mathcal{L}_n) = \frac{\sqrt{5}-1}{200}\left[20 - \frac{4(20-\sqrt{5})}{5}\right] n + O(1) = \frac{2\sqrt{5}}{125}n + O(1) \tag{6.47}$$

and

$$\operatorname{var}(\mathcal{K}_n - \mathcal{L}_n) = \frac{\sqrt{5}-1}{200} \cdot 20n + O(1) = \frac{\sqrt{5}-1}{10}n + O(1). \tag{6.48}$$

Hence

$$\begin{aligned} \operatorname{cov}(\mathcal{K}_n, \mathcal{L}_n) &= \frac{\operatorname{var}(\mathcal{K}_n + \mathcal{L}_n) - \operatorname{var}(\mathcal{K}_n - \mathcal{L}_n)}{4} \\ &= \frac{25-21\sqrt{5}}{1000}n + O(1) \approx -0.0219574275n + O(1). \end{aligned}$$

With Theorem 6.3, Theorem 6.6 and (6.49), we can compute the correlation between $\mathcal{K}_n$ and $\mathcal{L}_n$:

$$\begin{aligned}
\operatorname{corr}(\mathcal{K}_n, \mathcal{L}_n) &= \frac{\operatorname{cov}(\mathcal{K}_n, \mathcal{L}_n)}{\sqrt{\operatorname{var}(\mathcal{K}_n)\operatorname{var}(\mathcal{L}_n)}} \\
&= \frac{\frac{25-21\sqrt{5}}{1000}n + O(1)}{\sqrt{\left(\frac{29\sqrt{5}-25}{1000}n + O(1)\right)\left(\frac{29\sqrt{5}-25}{1000}n + O(1)\right)}} \\
&= \frac{\frac{25-21\sqrt{5}}{1000}n + O(1)}{\frac{29\sqrt{5}-25}{1000}n + O(1)} = \frac{25-21\sqrt{5}}{29\sqrt{5}-25} + o(1) \\
&= \frac{10\sqrt{5}-121}{179} + o(1) \approx -0.551057655 + o(1).
\end{aligned}$$

Since $\operatorname{var}(\mathcal{K}_n)$ and $\operatorname{var}(\mathcal{L}_n)$ are of size n and have the same coefficients of n, we have

$$\begin{aligned}
& \operatorname{cov}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n) \\
=\ & E\left[(\mathcal{K}_n - E[\mathcal{K}_n] + (\mathcal{L}_n - E[\mathcal{L}_n]))\,(\mathcal{K}_n - E[\mathcal{K}_n] - (\mathcal{L}_n - E[\mathcal{L}_n]))\right] \\
=\ & E[(\mathcal{K}_n - E[\mathcal{K}_n])^2 - (l - E[\mathcal{L}_n])^2] = \operatorname{var}(\mathcal{K}_n) - \operatorname{var}(\mathcal{L}_n) \\
=\ & O(1).
\end{aligned}$$

Further, we have the values of $\operatorname{var}(\mathcal{K}_n + \mathcal{L}_n)$ and $\operatorname{var}(\mathcal{K}_n - \mathcal{L}_n)$ from (6.47) and (6.48), thus

$$\begin{aligned}
\operatorname{corr}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n) &= \frac{\operatorname{cov}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n)}{\sqrt{\operatorname{var}(\mathcal{K}_n + \mathcal{L}_n)\operatorname{var}(\mathcal{K}_n - \mathcal{L}_n)}} \\
&= \frac{O(1)}{\sqrt{\left(\frac{2\sqrt{5}}{125}n + O(1)\right)\left(\frac{\sqrt{5}-1}{10}n + O(1)\right)}} \\
&= o(1).
\end{aligned}$$

Since $\mathcal{K}_n$ and $\mathcal{L}_n$ are bivariate Gaussian, $\mathcal{K}_n + \mathcal{L}_n$ and $\mathcal{K}_n - \mathcal{L}_n$ are independent as $n \to \infty$.

7. Conclusion

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Appendix A. Continuity of the Roots of A Polynomial with Continuous Coefficients

Lemma A.1. *Any root z of $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with $a_n \neq 0$ satisfies $|z| \leq \max\{1, (|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|)/|a_n|\}$.*

Proof. If $|z| \leq 1$, then we are done; else, if $|z| > 1$, then we get

$$\begin{aligned}
|a_n z^n| &= |a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0| \\
&\leq |a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-2} + \cdots + |a_0| \\
&\leq |a_{n-1}||z|^{n-1} + |a_{n-2}||z|^{n-1} + \cdots + |a_0||z|^{n-1}.
\end{aligned}$$

Thus the lemma follows. □

Theorem A.2. *Let the $a_i(x)$'s be continuous functions of x defined on $\mathbb{R}$ and the $y_i(x)$'s the roots of $P_x(y) = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x)$. For $x_0 \in \mathbb{R}$ with $a_n(x_0) \neq 0$, we have*

(a) *If $y_i(x_0)$ is of multiplicity m, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$ with $|x - x_0| < \delta$, $P_x(y)$ has at least m roots $y_{i_1}(x), y_{i_2}(x), \ldots, y_{i_m}(x)$ such that $|y_{i_j}(x) - y_i(x_0)| < \varepsilon$, $1 \leq j \leq m$.*

(b) *The $y_i(x)$'s are continuous at x_0.*

Proof. First note that since $a_n(x)$ is continuous at x_0, there exists $\delta_0 > 0$ such that $a_n(x) \neq 0$ for $x \in \mathbb{R}$ with $|x - x_0| < \delta_0$. Thus $P_x(y)$ is a polynomial of degree n and has n roots for $x \in \mathbb{R}$ with $|x - x_0| < \delta_0$.

(a) We prove by contradiction. Assume the contrary, i.e., there exist a sequence $\{x_k\}_{k=1}^{\infty}$ with $|x_k - x_0| < \delta_0$ and $x_k \to x_0$ such that there are at most $m-1$ roots of $P_{x_i}(y)$ in the set $\{z \in \mathbb{C} : |z - y_i(x_0)| < \varepsilon\}$.

Since $x_k \to x_0$, $\{x_k\}_{k=1}^{\infty}$ is bounded and so is each $\{a_i(x_k)\}_{k=1}^{\infty}$ $(1 \leq i \leq n)$. By Lemma A.1, the roots of $P_{x_k}(x)$ are also bounded. Therefore there exists a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ such that $(y_1(x_{k_j}),\ y_2(x_{k_j}),\ \ldots,\ y_n(x_{k_j}))$ converges (with respect to j). Let the limit be $(\tilde{y}_1,\ \tilde{y}_2,\ \ldots,\ \tilde{y}_n)$, then there exists a sufficiently large M such that for any $j > M$ and $1 \leq i \leq n$, $|y_i(x_{k_j}) - \tilde{y}_i| < \varepsilon$.

Since

$$\begin{aligned}
P_{x_{k_j}}(y) &= a_n(x_{k_j})(y - y_1(x_{k_j}))(y - y_2(x_{k_j})) \cdots (y - y_n(x_{k_j})) \\
&\to a_n(x_0)(y - \tilde{y}_1)(y - \tilde{y}_2) \cdots (y - \tilde{y}_n)
\end{aligned}$$

and

$$\begin{aligned}
P_{x_{k_j}}(y) &= a_n(x_{k_j})y^n + a_{n-1}(x_{k_j})y^{n-1} + \cdots + a_0(x_{k_j}) \\
&\to a_n(x_0)y^n + a_{n-1}(x_0)y^{n-1} + \cdots + a_0(x_0) \\
&\to a_n(x_0)(y - y_1(x_0))(y - y_2(x_0)) \cdots (y - y_n(x_0)),
\end{aligned}$$

the y_i's and the $y_i(x_0)$'s are equal (with multiplicity). As a result, there are m $\tilde{y}_i$'s equal to $y_i(x_0)$, then the corresponding $y_i(x_{k_j})$'s are in the set $\{z \in \mathbb{C} : |z - y_i(x_0)| < \varepsilon\}$, contradiction.

(b) Let z_i $(i = 1, 2, \ldots, t)$ be the distinct roots of $P_{x_0}(y)$ and m_i be the multiplicity of z_i. By (a), for any $\varepsilon > 0$ and $i \in \{1, 2, \ldots, t\}$, there exists $\delta_i \in (0, \delta_0)$ such that $P_x(y)$ has at least m_i roots $y_{i_1}(x), y_{i_2}(x), \ldots, y_{i_{m_i}}(x)$ such that $|y_{i_j}(x) - z_i| < \frac{1}{3}\min_{k_1 < k_2}\{\varepsilon, |z_{k_1} - z_{k_2}|\}$,

$1 \leq j \leq m_i$, for $x \in \mathbb{R}$ with $|x - x_0| < \delta_i$. Therefore for any $x \in \mathbb{R}$ with $|x - x_0| < \min_{1 \leq i \leq t}\{\delta_i\}$ and $i \in \{1, 2, \ldots, t\}$, $P_x(y)$ has such m_i roots. Then for any i_j and $i'_{j'}$ with $i \neq i'$, we have

$$\begin{aligned} |y_{i_j}(x) - y_{i'_{j'}}(x)| &= |(y_{i_j}(x) - z_i) + (z_i - z_{i'}) + (z_{i'} - y_{i'_{j'}}(x))| \\ &\geq |(z_i - z_j)| - |(y_{i_j}(x) - z_i) - (z_{i'} - y_{i'_{j'}}(x))| \\ &> \min_{k_1 < k_2}\{\varepsilon, |z_{k_1} - z_{k_2}|\} - 2 \cdot \frac{1}{3}\min_{k_1 < k_2}\{\varepsilon, |z_{k_1} - z_{k_2}|\} \\ &= \frac{1}{3}\min_{k_1 < k_2}\{\varepsilon, |z_{k_1} - z_{k_2}|\} > 0. \end{aligned}$$

Hence $y_{i_j}(x) \neq y_{i'_{j'}}(x)$ for any $i \neq i'$. Since the sum of the m_i's is the degree of $P_{x_0}(y)$, which is the same as that of $P_x(y)$, the $y_{i_j}(x)$'s are all of the roots of $P_x(y)$. Since $|y_{i_j}(x) - z_i| < \frac{1}{3}\min_{k_1 < k_2}\{\varepsilon, |z_{k_1} - z_{k_2}|\} < \varepsilon$, the roots of $P_x(y)$ are continuous at x_0. □

Appendix B. No Multiple Roots for $x \in I_\epsilon$

Assume that $L \geq 2$. We first show that there exists $x > 0$ such that $A(y)$ has no multiple roots.

Lemma B.1. *For any $n \geq 1$ and positive real numbers $a_0 \leq a_1 \leq \cdots \leq a_n$ but not all equal, any root z of $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ satisfies $|z| < 1$.*

Proof. Let z be a root of $P(x)$, then z is also a root of $(1-x)P(x)$. Thus

$$a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n - a_n z^n = 0.$$

If $|z| \geq 1$, then we get

$$\begin{aligned} |a_n z^n| &= |a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n| \\ &\leq |a_0| + |(a_1 - a_0)z| + |(a_2 - a_1)z^2| + \cdots + |(a_n - a_{n-1})z^n| \\ &= a_0 + (a_1 - a_0)|z| + (a_2 - a_1)|z|^2 + \cdots + (a_n - a_{n-1})|z|^n \\ &\leq a_0 + (a_1 - a_0)|z|^n + (a_2 - a_1)|z|^n + \cdots + (a_n - a_{n-1})|z|^n \\ &= a_n|z|^n = |a_n z^n|. \end{aligned}$$

Hence all of the equalities are achieved, i.e., $|z| = 1$ and $(a_1 - a_0)z, (a_2 - a_1)z^2, \ldots, (a_n - a_{n-1})z^n$ are real and nonnegative since a_0 is real and positive.

Since the a_i's are not all equal, there exists i such that $a_{i+1} > a_i$. Since $(a_{i+1} - a_i)z^{i+1})z$ is real and nonnegative, so is z. Therefore, $P(z) = a_0 + a_1 z + \cdots + a_n z^n \geq a_0 > 0$, contradiction. □

Lemma B.2. *Let $f_0(x) = 1 - x - x^2 - \cdots - x^n$ with $n \geq 2$, then*

(a) *$f_0(x)$ has a unique positive real root r_0, $0 < r_0 < 1$ and r_0 is not a multiple root of $f_0(x)$.*

(b) *Any root $z \neq r_0$ of $f_0(x)$ satisfies $|z| > 1$.*

Proof. (a) Since $f_0(x)$ is decreasing on $(0, \infty)$ and $f(0) = 1 > 0 > f(1)$, $Q(x)$ has a unique positive real root r and $0 < r < 1$.

Since $f_0'(x) = -1 - 2x - \cdots - nx^{n-1}$ and $r > 0$, $f_0'(r) < 0$. Therefore r is not a multiple root of $f_0(x)$.

(b) Note that $f_0(0) \neq 0$, thus 0 is not a root of $f_0(x)$. Let

$$f(x) = x^n f_0\left(\frac{1}{x}\right) = x^n - x^{n-1} - \cdots - x - 1,$$

then it suffices to show that any root $z \neq r$ of $f(x)$ satisfies $|z| < 1$ where $r = 1/r_0$.

Since r is a root of $f(x)$, $f(x)$ can be factored as

$$\begin{aligned} f(x) &= (x-r)(d_0x^{n-1} + d_1x^{n-2} + \cdots + d_{n-2}x + d_{n-1}) \qquad \text{(B.1)} \\ &= x^n + \sum_{i=1}^{n-1}(d_i - rd_{i-1})x_{n-i} - rd_{n-1}, \end{aligned}$$

where $d_0 = 1$.

Comparing the coefficients of x_{n-i} of both sides, we get $d_i - rd_{i-1} = -1$, i.e.,

$$d_i = rd_{i-1} - 1, \ 1 \leq i \leq n-1. \qquad \text{(B.2)}$$

Using $d_0 = 1$ and applying (C.2) repeatedly, we get

$$d_i = r^i - r^{i-1} - r^{i-2} - \cdots - 1, \ 1 \leq i \leq n-1.$$

Since $f(r) = 0$, for $1 \leq i \leq n-1$,

$$d_i = r^i - r^{i-1} - r^{i-2} - \cdots - 1 = \frac{1}{r^{n-i}}(r^{n-i-1} + r^{n-i-2} + \cdots + 1) > 0,$$

and for $1 \leq i \leq n-2$,

$$\begin{aligned} d_i &> \frac{1}{r^{n-i}}(r^{n-i-1} + r^{n-i-2} + \cdots + r) \\ &= \frac{1}{r^{n-i-1}}(r^{n-i-2} + r^{n-i-3} + \cdots + 1) \\ &= d_{i+1}. \end{aligned}$$

Hence $d_1 > d_2 > \cdots > d_{n-1} > 0$.

Since $f_0(r) = 0$, we have

$$r^n = r^{n-1} + r^{n-2} + \cdots + 1 = \frac{r^n - 1}{r-1},$$

which yields

$$r^n(r-1) \leq (r^n - 1) < r^n.$$

Hence $r - 1 < 1$ and therefore $d_1 = r - 1 < 1 = d_0$.

Let $P(x) = d_0x^{n-1} + d_1x^{n-2} + \cdots + d_{n-2}x + d_{n-1}$, then $f(x) = (x-r)P(x)$ (see (C.1)). Applying Lemma B.1 to $P(x)$, we see that $|z| < 1$ for any root z of $P(x)$, i.e., any root z of $f(x)$ such that $z \neq r$. □

Lemma B.3. *Let* $Q(x) = A(1) = 1 - x - \cdots - x^{s_L - 1}$ *and* $R(x) = A'(1) = -\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1}(m+1)x^j$, *then* $R(x)$ *and* $Q(x)$ *are coprime (see (4.6) for the definition of* $A(y)$*).*

Proof. Let $n = s_L - 1 \geq c_1 + c_L - 1 \geq 1$. If $n = 1$, then $c_1 = c_L = 1$ and the other c_i's are zero. Thus $Q(x) = -x$ and $R(x) = -1 - Lx$ are coprime.

Assume that $n \geq 2$. We prove by contradiction. Assume that $R(x)$ and $Q(x)$ are not coprime. Let $D(x) = \sum_{i=0}^{l} a_i x^i$ be a greatest common divisor of $Q(x)$ and $Q'(x)$ with $l, a_l > 0$. Let $Q(x) = D(x)Q_1(x)$, where $Q_1(x) = \sum_{j=0}^{t} b_j x^j \in \mathbb{Z}[x]$. Noting that the leading coefficient and the constant term of $Q(x)$ are -1 and 1, respectively, we get $a_l = 1$, $b_t = -1$ and $a_0 = b_0 \in \{\pm 1\}$.

Let z_i's be the roots of $D(x)$, then they are also the roots of $Q(x)$ and $R(x)$. Applying Lemma B.1 to $R(x)$, we see that any root of $R(x)$ has norm smaller than 1. Hence we have $|z_i| < 1$ for all i. On the other hand, by Lemma B.2 to $Q(x)$, any root of $Q(x)$ except one (the unique positive root) has norm greater than 1. Therefore $D(x)$ only has one root z_1, which is the unique positive root of $Q(x)$. This implies that $D(x)$ is of degree 1. Since $Q(x)$ is of degree $n \geq 2$ and $Q(x) = D(x)Q_1(x)$, $Q_1(x)$ is of degree at least 1. Since any root other than $z_1(x)$ of $Q(x)$ is a root of $Q_1(x)$ and thus has norm greater than 1, the norm of the product of roots of $Q_1(x)$ should be greater than 1; however by Vieta's Formula, the norm of the product is $|b_t/b_l| = 1$, contradiction. $\square$

Lemma B.4. *There are finitely many $x > 0$ such that $A(y)$ has multiple roots. As a consequence, there exists $\epsilon \in (0,1)$ such that for any $x \in I_\epsilon$, $A(y)$ has no multiple roots.*

Proof. If $x > 0$, then $A(y)$ is of degree $s_L - 1$ in terms of y. We proved in Lemma B.3 that

$$A(1) = \mathscr{A}(x,1) \text{ and } A'(1) = \left.\frac{d}{dy}\mathscr{A}(x,y)\right|_{y=1}$$

are coprime, hence $\mathscr{A}(x,y)$ and $\frac{d}{dy}\mathscr{A}(x,y)$ are coprime (see (3.1) for the definition of $\mathscr{A}(x,y)$).

Now, we regard $\mathscr{A}(x,y)$ and $\frac{d}{dy}\mathscr{A}(x,y)$ as polynomials $A(y)$ and $A'(y)$ of y with coefficients polynomials of x. We use the Euclidean algorithm to compute the great common divisor of $A(y)$ and $A'(y)$. In each step, the quotient and remainder are (fractional) polynomials of x. If we get a fractional polynomial, there are finitely many x's such that the denominator is zero. We exclude these values from the current *admissible* set of x and continue (the admissible set was $\{x > 0\}$ at the beginning).

Since $A(y)$ and $A'(y)$ are coprime, finally we will get a constant polynomial in terms of y, which is a nonzero fractional polynomial of x. Otherwise, we will get a common divisor of $A(y)$ and $A'(y)$, which is a polynomial of y of degree at least 1 with coefficients fractional polynomials of x. Denote this common divisor by $U(x,y)/V(x)$ with polynomials $U(x,y)$ and $V(x)$ coprime. Then there exists a polynomial $W(x,y)$ such that $W(x,y)U(x,y)/V(x) = \mathrm{GCD}(A(y), A'(y))$. Since $A(y)$ and $A'(y)$ are coprime, we get $W(x,y)U(x,y)|V(x)$, contradiction.

We exclude from the current admissible set the roots of the numerator and the denominator of the fractional polynomial we obtain at the last step.

In the above procedure, at each time we exclude finitely many values from the current admissible set. Since there are at most s_L steps, we exclude finitely many values in total. For any x in the last admissible set, $A(y)$ has no multiple roots. Hence there are finitely many $x \in \mathbb{R}$ such that $A(y)$ has multiple roots. $\square$

Appendix C. No Multiple Roots for Non-increasing c_i's

Proposition C.1. *If the c_i's are non-increasing, i.e., $c_1 \geq c_2 \geq \cdots \geq c_n$, then $A(y)$ has no multiple roots when $x = 1$.*

Proof. We first show by contradiction that when $x = 1$, $A(y)$ is irreducible in $\mathbb{Q}[y]$. It suffices to prove that $A(y)$ is irreducible in $\mathbb{Z}[y]$ since $A(y) \in \mathbb{Z}[y]$. Suppose instead that $A(y)$ is reducible, then $A(y)$ can be written as $A(y) = g(y)\tilde{h}(y)$, where $\tilde{g} = 1 + \tilde{g}_1 y + \cdots + \tilde{g}_{n_1} y^{n_1}$ and $\tilde{h} = 1 + \tilde{h}_1 y + \cdots + \tilde{h}_{n_2} y^{n_2} \in \mathbb{Z}[y]$ with $\tilde{g}_{n_1}, \tilde{h}_{n_2} > 0$ and $n_1, n_2 \geq 1$.

Lemma C.2. *For any positive real numbers $a_1 \geq a_2 \geq \cdots \geq a_L \geq 1$, $\tilde{f}(y) = 1 - a_1 y - a_2 y^2 - \cdots - a_L y^L$ has a unique positive real root $\tilde{r}$, $0 < \tilde{r} < 1$, and any root $\tilde{z} \neq \tilde{r}$ of $\tilde{f}$ satisfies $|\tilde{z}| < 1$.*

Proof. Since $\tilde{f}$ is decreasing on $(0, \infty)$ and $f(0) = 1 > 0 \geq 1 - a_1 > f(1)$, $\tilde{f}$ has a unique positive real root $\tilde{r}$.

Let $f(y) = y^L - a_1 y^{L-1} - a_2 y^{L-2} - \cdots - a_L$. Note that $f(0), \tilde{f}(0) \neq 0$ and

$$f(y) = y^L \tilde{f}\left(\frac{1}{y}\right), \ \tilde{f}(y) = y^L f\left(\frac{1}{y}\right),$$

thus f has a unique positive real root r, and $r > 1$. To prove that any root $\tilde{z} \neq \tilde{r}$ of $\tilde{f}$ satisfies $|\tilde{z}| < 1$, it suffice to prove that any root $z \neq r$ of f satisfies $|z| < 1$.

Since r is a root of f, f can be factored as

$$\begin{aligned} f(y) &= (y - r)(d_0 y^{L-1} + d_1 y^{L-2} + \cdots + d_{L-2} y + d_{L-1}) \qquad \text{(C.1)} \\ &= y^L + \sum_{i=1}^{L-1} (d_i - r d_{i-1}) y_{L-i} - r d_{L-1}, \end{aligned}$$

where $d_0 = 1$.

Comparing the coefficients of y_{L-i} of both sides, we get $d_i - r d_{i-1} = -a_i$, i.e.,

$$d_i = r d_{i-1} - a_i, \ 1 \leq i \leq L - 1. \qquad \text{(C.2)}$$

Using $d_0 = 1$ and applying (C.2) repeatedly, we get

$$d_i = r^i - a_1 r^{i-1} - a_2 r^{i-2} - \cdots - a_i, \ 1 \leq i \leq L - 1.$$

Since $f(r) = 0$, for $1 \leq i \leq L - 1$,

$$\begin{aligned} d_i &= r^i - a_1 r^{i-1} - a_2 r^{i-2} - \cdots - a_i \\ &= \frac{1}{r^{L-i}} (a_{i+1} r^{L-i-1} + a_{i+2} r^{L-i-2} + \cdots + a_L) \\ &> 0, \end{aligned}$$

and for $1 \leq i \leq L - 2$,

$$\begin{aligned} d_i &> \frac{1}{r^{L-i}} (a_{i+1} r^{L-i-1} + a_{i+2} r^{L-i-2} + \cdots + a_{L-1} r) \\ &\geq \frac{1}{r^{L-i}} (a_{i+2} r^{L-i-1} + a_{i+3} r^{L-i-2} + \cdots + a_L r) \\ &= \frac{1}{r^{L-i-1}} (a_{i+2} r^{L-i-2} + a_{i+3} r^{L-i-3} + \cdots + a_L) \\ &= d_{i+1}. \end{aligned}$$

Hence $d_1 > d_2 > \cdots > d_{L-1} > 0$.

Further, since $f(r) = 0$, we have

$$\begin{aligned} r^L &= a_1 r^{L-1} + a_2 r^{L-2} + \cdots + a_L \leq a_1 r^{L-1} + a_1 r^{L-2} + \cdots + a_1 \\ &= a_1 \frac{r^L - 1}{r - 1}, \end{aligned}$$

which yields

$$r^L(r-1) \leq a_1(r^L-1) < a_1 r^L.$$

Hence $r-1 < a_1$ and therefore $d_1 = r - a_1 < 1 = d_0$.

Let $P(y) = d_0 y^{L-1} + d_1 y^{L-2} + \cdots + d_{L-2} y + d_{L-1}$, then $f(x) = (y-r)P(y)$ (see (C.1)). Applying Lemma B.1 to $P(y)$, we get $|z| < 1$ for any root z of P. Since $r > 1$, r cannot be a root of P. Therefore r is a not a multiple root of f. □

By Lemma C.2, $A(y)$ has a unique positive real root $\tilde{r}$ and $0 < \tilde{r} < 1$. Then $\tilde{g}(\tilde{r})\tilde{h}(\tilde{r}) = \tilde{A}(\tilde{r}) = 0$. Without loss of generality, assume that $\tilde{g}(\tilde{r}) = 0$. Since any root $\tilde{z}$ of $A(y)$ satisfies $|\tilde{z}| > 1$, so does any root of $\tilde{h}$. Let $\tilde{z}_i$ $(1 \leq i \leq n_2)$ be the roots of $\tilde{h}$, then $|\tilde{z}_i| > 1$. By Vieta's formula, $|\tilde{h}_{n_2}| = |\left(\prod_{i=1}^{n_2} \tilde{z}_i\right)^{-1}| < 1$. This contradicts the fact that $\tilde{h}_{n_2}$ is a positive integer. Hence $A(y)$ is irreducible in $\mathbb{Q}[y]$.

If $A(y)$ has multiple roots, then $A(y)$ and $A'(y)$ are not coprime in $\mathbb{Q}[y]$, i.e., there exists $d(y) \in \mathbb{Q}[y]$ such that $\deg d \geq 1$ and d divides A and A'. Thus $\deg d \leq \deg A' < \deg A$. Hence $A(y)$ is reducible in $\mathbb{Q}[y]$, contradiction. □

Appendix D. Differentiability of the Roots

Proof. For fixed positive x and a small increment $\Delta x > 0$, letting $z_i(x) = y_i(x + \Delta x)$ $(1 \leq i \leq L)$, we have

$$1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y_i^{m+1}(x) = 0, \tag{D.1}$$

and

$$1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (x+\Delta x)^j z_i^{m+1}(x) = 0. \tag{D.2}$$

Subtracting (D.2) from (D.1), we get

$$\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left((x+\Delta x)^j z_i^{m+1} - x^j y_i^{m+1}(x)\right) = 0.$$

The left-hand side can be written as

$$\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left(z_i^{m+1}(x)\left((x+\Delta x)^j - x^j\right) + x^j\left(z_i^{m+1}(x) - y_i^{m+1}(x)\right)\right),$$

thus

$$\begin{aligned} &\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j \left(z_i^{m+1}(x) - y_i^{m+1}(x)\right) \\ = \ & -\sum_{m=0}^{L-1} \sum_{j=s'_m}^{s'_{m+1}-1} z_i^{m+1}(x)\left((x+\Delta x)^j - x^j\right). \end{aligned} \tag{D.3}$$

Since

$$z_i^{m+1}(x) - y_i^{m+1}(x) = (z_i(x) - y_i(x)) \sum_{l=0}^{m} z_i^l(x) y_i^{m-l}(x)$$

and

$$(x+\Delta x)^j - x^j = \Delta x \sum_{t=0}^{j-1} (x+\Delta x)^t x^{j-1-t},$$

(D.3) can be written as

$$\begin{aligned}
&(z_i(x)-y_i(x)) \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j \sum_{l=0}^{m} z_i^l(x) y_i^{m-l}(x) \\
= \quad & -\Delta x \sum_{m=0}^{L-1} \sum_{j=s'_m}^{s'_{m+1}-1} z_i^{m+1}(x) \sum_{t=0}^{j-1} (x+\Delta x)^t x^{j-1-t}. \qquad \text{(D.4)}
\end{aligned}$$

The coefficient of $z_i(x) - y_i(x)$ on the left-hand side is

$$\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j \sum_{l=0}^{m} z_i^l(x) y_i^{m-l}(x), \qquad \text{(D.5)}$$

which is nonzero for all but finitely many $z_i(x)$ (to see this, regard (D.5) as a polynomial of $z_i(x)$) and hence nonzero for all but finite Δx (regard (D.2) as polynomial of Δx). Therefore, there exists $\epsilon' > 0$ such that for any $\Delta x \in (0, \epsilon')$, (D.5) is not zero. Thus we can write (D.4) as

$$\frac{z_i(x)-y_i(x)}{\Delta x} = -\frac{\sum_{m=0}^{L-1} \sum_{j=s'_m}^{s'_{m+1}-1} z_i^{m+1}(x) \sum_{t=0}^{j-1} (x+\Delta x)^t x^{j-1-t}}{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j \sum_{l=0}^{m} z_i^l(x) y_i^{m-l}(x)}. \qquad \text{(D.6)}$$

To prove the differentiability of $y_i(x)$, it is equivalent to show that the limit of the right-hand side of (D.6) exists. Recall that $y_i(x)$ is continuous, so it suffices to verify that the denominator of the limit of (D.6) as $\Delta x \to 0$ is nonzero.

The limit of the denominator is

$$\begin{aligned}
\mathscr{R}_i(x) \quad &:= \quad \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j \sum_{l=0}^{m} y_i^l(x) y_i^{m-l}(x) = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1) x^j y_i^m(x) \\
&= \quad -A'(y_i(x)),
\end{aligned}$$

which is not zero as $y_i(x)$ is not a multiple root of $A(y)$. Since $y_i(x)$ is continuous, $\mathscr{R}_i(x)$ is continuous. Thus there exists $\epsilon \in (0, \epsilon')$ such that for any $x' \in (x-\epsilon, x+\epsilon)$,

$$\mathscr{R}_i(x') \neq 0. \qquad \text{(D.7)}$$

Hence for $x' \in (x-\epsilon, x+\epsilon)$, we can take the limits of both sides of (D.6) thus prove the differentiability of $y_i(x)$ and get

$$y'_i(x) = -\frac{\sum_{m=0}^{L-1} \sum_{j=s'_m}^{s'_{m+1}-1} j y_i^{m+1}(x) x^{j-1}}{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1) x^j y_i^m(x)}. \qquad \text{(D.8)}$$

We prove by induction on ℓ that $y_i^{(\ell)}(x)$ exists and is of the form

$$y_i^{(\ell)}(x) = \frac{\mathscr{P}_\ell(y_i(x))}{\mathscr{Q}^{2\ell-1}(y_i(x))}, \qquad \text{(D.9)}$$

where $\mathscr{P}_\ell$ and $\mathscr{Q}$ are polynomials with coefficients polynomials of x, and

$$\mathscr{Q}(y_i(x)) = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1)x^j y_i^m(x) = \mathscr{R}_i(x).$$

Note that $\mathscr{Q}(y_i(x)) = \mathscr{R}_i(x) \neq 0$ by (D.7).

When $\ell = 1$, from (4.10) we get

$$y_i'(x) = \frac{1}{\mathscr{Q}(y_i(x))} \sum_{m=0}^{L-1} \sum_{j=s_m'}^{s_{m+1}'-1} j y_i^{m+1}(x) x^{j-1}.$$

Thus $y_i'(x)$ is of the form (D.9).

Assume that (D.9) holds for $\ell (\geq 1)$. Since y_i is differentiable and $\mathscr{Q}(y_i(x)) \neq 0$, $(\mathscr{Q}(y_i(x)))^{-\ell}$ is differentiable. Thus we can take the derivative of both sides of (D.9) and get

$$y_i^{(\ell+1)}(x) = \frac{1}{\mathscr{Q}^{2\ell-1}(y_i(x))} \frac{d\mathscr{P}_\ell(y_i(x))}{dx} - \frac{(2\ell-1)y_i'(x)\mathscr{P}_\ell(y_i(x))}{\mathscr{Q}^{2\ell}(y_i(x))}.$$

Using the induction hypothesis (D.9) for $\ell = 1$, we obtain

$$y_i^{(\ell+1)}(x) = \frac{1}{\mathscr{Q}^{2\ell+1}(y_i(x))} \left[\mathscr{Q}^2(y_i(x)) \frac{d\mathscr{P}_\ell(y_i(x))}{dx} - (2\ell-1)\mathscr{P}_1(y_i(x)) \right].$$

Thus $y_i^{(\ell+1)}(x)$ is of the form (D.9) as well. This completes the proof. □

Appendix E. Differentiability of the $\alpha_i(x)$'s and the $q_i(x)$'s

Proof. For any $\ell \geq 1$, by Proposition 4.2, $y_i(x)$ is ℓ-times differentiable at $x \in I_\varepsilon$ and $y_1(x)$ is ℓ-times differentiable at 1. Further, $y_i(x) \neq 0$ for any i and $x > 0$ as $A(0) = 1 \neq 0$ (see (4.6) for the definition of $A(y)$), thus $\alpha_i(x) = (y_i(x))^{-1}$ is ℓ-times differentiable at $x \in I_\varepsilon$ and $\alpha_1(x) = (y_1(x))^{-1}$ is ℓ-times differentiable at 1.

By Definition (4.11), the denominator and the numerator of $q_i(x)$ are

$$\sum_{j=s_{L-1}+1}^{s_L} x^j \prod_{j \neq i} (y_j(x) - y_i(x)), \ \sum_{m=1}^{L} b_m(x) y_i^m(x),$$

which are ℓ-times differentiable at $x \in I_\varepsilon$ since each $y_j(x)$ is ℓ-times differentiable at $x \in I_\varepsilon$. (Recall from Definitions (3.11) and (4.2) that the $b_m(x)$'s are polynomials of x.) Further, since the denominator is nonzero when $x \in I_\varepsilon$, $q_i(x)$ is ℓ-times differentiable at $x \in I_\varepsilon$.

Let

$$E_i(x) = \prod_{j \neq i} (y_j(x) - y_i(x)). \tag{E.1}$$

Then the denominator of $q_1(x)$ is $x^{s_L} y_1(x) E_1(x)$, which is nonzero when $x = 1$. Since $\sum_{j=s_{L-1}+1}^{s_L} x^j$ and $y_1(x)$ are ℓ-times differentiable at 1, it suffices to show that $E_1(x)$ is ℓ-times differentiable at 1. Letting $y = y' + y_1(x)$ in (4.6), we get

$$A(y) = 1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j (y' + y_1(x))^{m+1}.$$

On the other hand, we have

$$\begin{aligned} A(y) &= -\sum_{j=s_{L-1}+1}^{s_L} x^j \prod_{j=1}^{L} (y - y_j(x)) \\ &= -\sum_{j=s_{L-1}+1}^{s_L} x^j \prod_{j=1}^{L} (y' + y_1(x) - y_j(x)) \\ &= -\sum_{j=s_{L-1}+1}^{s_L} x^j y' \prod_{j\neq 1} (y' + y_1(x) - y_j(x)) . \end{aligned} \tag{E.2}$$

Comparing the coefficients of y in (E.1) and (E.2) yields

$$-\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j (m+1) y_1^m(x) = -\sum_{j=s_{L-1}+1}^{s_L} x^j (-1)^{L-1} E_1(x).$$

Hence

$$E_1(x) = \frac{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j (m+1) y_1^m(x)}{\sum_{j=s_{L-1}+1}^{s_L} x^j (-1)^{L-1}}. \tag{E.3}$$

Since $y_1(x)$ is ℓ-times differentiable at 1 and the denominator in (E.3) is nonzero as $x > 0$, $E_1(x)$ is ℓ-times differentiable at 1. □

Appendix F. Main Term of $g^{(\ell)}(x)$ (Proof of Claim 4.4)

Proof. Suppose n is sufficiently large. We first express $q_i(x)$ in terms of the $\alpha_i(x)$'s. By Definition 4.11, (E.1) and $\alpha_i(x) = (y_i(x))$, we get

$$q_i(x) = \frac{\sum_{m=1}^{L} b_m(x) y_i^m(x)}{\sum_{j=s_{L-1}+1}^{s_L} x^j E_i(x)} = \sum_{m=1}^{L} \frac{b_m(x)}{\sum_{j=s_{L-1}+1}^{s_L} x^j E_i(x) \alpha_i^m(x)},$$

where

$$\begin{aligned} E_i(x) &= \prod_{j\neq i} (y_j(x) - y_i(x)) = \prod_{j\neq i} \left[\frac{1}{\alpha_j(x)} - \frac{1}{\alpha_i(x)} \right] \\ &= \frac{\prod_{j\neq i}(\alpha_i(x) - \alpha_j(x))}{\alpha_i^{L-1}(x) \prod_{j\neq i} \alpha_j(x)} = \frac{\prod_{j\neq i}(\alpha_i(x) - \alpha_j(x))}{\alpha_i^{L-2}(x) \prod_{j=1}^{L} \alpha_j(x)} \\ &= \frac{(-1)^{L-1} \prod_{j\neq i}(\alpha_j(x) - \alpha_i(x))}{\alpha_i^{L-2}(x) (-1)^L \sum_{j=s_{L-1}}^{s_L-1} x^j} \\ &= -\frac{\prod_{j\neq i}(\alpha_j(x) - \alpha_i(x))}{\alpha_i^{L-2}(x) \sum_{j=s_{L-1}}^{s_L-1} x^j} \end{aligned} \tag{F.1}$$

by Vieta's Formula. Thus

$$q_i(x) = -\sum_{m=1}^{L} \frac{b_m(x)}{x\alpha_i^{L-2+m}(x)} \prod_{j\neq i} [\alpha_j(x) - \alpha_i(x)]^{(-1)},$$

and

$$\sum_{i=2}^{L} xq_i(x)\alpha_i^n(x) = -\sum_{m=1}^{L} b_m(x)\frac{\alpha_i^{n-L+2-m}(x)}{\prod_{j\neq i}(\alpha_j(x)-\alpha_i(x))}.$$

Since L is fixed, it suffices to show that

$$\left[\sum_{i=2}^{L}\frac{\alpha_i^n(x)}{\prod_{j\neq i}(\alpha_j(x)-\alpha_i(x))}\right]^{(\ell)} = o(1)\alpha_1^n(x),$$

Let

$$\mathcal{P}(x) = \sum_{i=2}^{L}\frac{\alpha_i^n(x)}{\prod_{j\neq i}(\alpha_j(x)-\alpha_i(x))}.$$

Then $\mathcal{P}$ is a symmetric function of $\alpha_2(x),\dots,\alpha_L(x)$. For $1 < i_0 < j_0$, we have

$$\begin{aligned}
&(\alpha_{i_0}(x)-\alpha_{j_0}(x))\mathcal{P}(x)\\
=\ & \sum_{i\neq 1,i_0,j_0}\frac{\alpha_i^n(x)(\alpha_{i_0}(x)-\alpha_{j_0}(x))}{\prod_{j\neq i}(\alpha_j(x)-\alpha_i(x))} - \frac{\alpha_{i_0}^n(x)}{\prod_{j\neq i_0,j_0}(\alpha_j(x)-\alpha_{i_0}(x))}\\
&+\frac{\alpha_{j_0}^n(x)}{\prod_{j\neq i_0,j_0}(\alpha_j(x)-\alpha_{j_0}(x))},
\end{aligned}$$

which equals zero if $\alpha_{i_0}(x)=\alpha_{j_0}(x)$. Hence the polynomial

$$\prod_{1\leq i<j\leq L}(\alpha_j(x)-\alpha_i(x))\mathcal{P}(x) \tag{F.2}$$

of $\alpha_1(x),\dots,\alpha_L(x)$ is divided by $\alpha_{i_0}(x)-\alpha_{j_0}(x)$ for any $1<i_0<j_0$. Therefore

$$\prod_{j\neq 1}(\alpha_j(x)-\alpha_1(x))\mathcal{P}(x) \tag{F.3}$$

is a polynomial of $\alpha_1(x),\dots,\alpha_L(x)$.

Since (F.2) is homogeneous of order $n-(L-1)+\frac{1}{2}(L-1)L$, the polynomial in (F.3) is homogeneous of order $n-(L-1)+\frac{1}{2}(L-1)L-\frac{1}{2}(L-2)(L-1)=n$. Furthermore, note that (F.2) is a sum of $O(1)$ terms with each summand a product of $\alpha_i^n(x)$ $(i>1)$ and a polynomial of $\alpha_1(x),\dots,\alpha_L(x)$ independent of n, thus we can divide the summands into $O(1)$ pairs with each pair of the form $\tilde{\mathcal{P}}(x)(\alpha_{i_0}^l(x)-\alpha_{j_0}^l(x))$ where $\tilde{\mathcal{P}}(x)$ is a polynomial of $\alpha_1(x),\dots,\alpha_L(x)$ independent of n and $l\leq n$. Dividing each pair by $\alpha_{i_0}^l(x)-\alpha_{j_0}^l(x)$, we get

$$\frac{\tilde{\mathcal{P}}(x)(\alpha_{i_0}^l(x)-\alpha_{j_0}^l(x))}{\alpha_{i_0}(x)-\alpha_{j_0}(x)} = \tilde{\mathcal{P}}(x)\sum_{t=0}^{l}\alpha_{i_0}^t(x)\alpha_{j_0}^{l-t}(x),$$

which is a sum of $O(n)$ terms with each summand a product of at most n element (with multiplicity) from $\{\alpha_i(x)\}_{i>1}$ and a polynomial of $\alpha_1(x),\dots,\alpha_L(x)$ independent of n, hence dividing (F.2) by $\alpha_{i_0}(x)-\alpha_{j_0}(x)$ yields a sum of $O(n)$ terms with each summand a product of at most n element (with multiplicity) from $\{\alpha_i(x)\}_{i>1}$ and a polynomial of $\alpha_1(x),\dots,\alpha_L(x)$ independent of n.

Repeating the proceeding procedure, namely dividing (F.2) by $\alpha_{i_0}(x)-\alpha_{j_0}(x)$ for all $1<i_0<j_0$, finally, we get a sum of $O(n^{N_0})$ terms with each term a product of at most n element

(with multiplicity) from $\{\alpha_i(x)\}_{i>1}$ and a polynomial of $\alpha_1(x),\ldots,\alpha_L(x)$ independent of n, where N_0 is determined by L and independent of n, namely

$$\mathcal{P}(x) = \frac{\sum_i \mathcal{P}_{i,0}(x)\prod_{j=2}^{L}\alpha_j^{i_j}(x)}{\prod_{j\neq 1}(\alpha_j(x)-\alpha_1(x))}, \tag{F.4}$$

where $\sum_{j=2}^{L} i_j \leq n$ and the $\mathcal{P}_i(x)$'s are polynomials of $\alpha_1(x),\ldots,\alpha_L(x)$ independent of n. Since the denominator of $\mathcal{P}(x)$ is continuous, nonzeron and well-defined at $x=1$, the claim in the case $\ell=0$ follows by Proposition 4.1.

Let

$$\mathcal{E}_i(x) = \prod_{j\neq i}(\alpha_j(x)-\alpha_i(x)). \tag{F.5}$$

Plugging Definition (F.5) with $i=1$ into (F.4), we get

$$\mathcal{P}(x) = \frac{1}{\mathcal{E}_1(x)}\sum_i \mathcal{P}_{i,0}(x)\prod_{j=2}^{L}\alpha_j^{i_j}(x).$$

Thus

$$\mathcal{P}'(x) = \left[\frac{1}{\mathcal{E}_1(x)}\right]'\sum_i \mathcal{P}_{i,0}(x)\prod_{j=2}^{L}\alpha_j^{i_j}(x) + \frac{1}{\mathcal{E}_1(x)}\left[\sum_i \mathcal{P}_{i,0}(x)\prod_{j=2}^{L}\alpha_j^{i_j}(x)\right]'. \tag{F.6}$$

By (F.1), we get

$$\mathcal{E}_i(x) = -\alpha_i^{L-2}(x)\sum_{j=s_{L-1}}^{s_L-1} x^j E_i(x).$$

Plugging in (E.3) with the index 1 replaced by i yields

$$\mathcal{E}_i(x) = \frac{(-1)^L\alpha_i^{L-2}(x)}{x}\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1}(m+1)x^j y_i^m(x).$$

Since $\alpha_i(x)$ and $y_i(x)$ are ℓ'-times differentiable at $x\in I_\varepsilon$ for all i and at $x=1$ for $i=1$ for all ℓ', so is $\mathcal{E}_i(x)$. Note from (4.10) that

$$\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1}(m+1)x^j y_i^m(x) = -\frac{1}{y_i'(x)}\sum_{m=0}^{L-1}\sum_{j=s'_m}^{s'_{m+1}-1} j y_i^{m+1}(x)x^{j-1},$$

thus

$$\begin{aligned}
\mathcal{E}_i(x) &= \frac{(-1)^{L-1}\alpha_i^{L-2}(x)}{xy_i'(x)}\sum_{m=0}^{L-1}\sum_{j=s'_m}^{s'_{m+1}-1} j y_i^{m+1}(x)x^{j-1}\\
&= \frac{(-1)^{L}\alpha_i^{L}(x)}{x\alpha_i'(x)}\sum_{m=0}^{L-1}\sum_{j=s'_m}^{s'_{m+1}-1} j \alpha_i^{-m-1}(x)x^{j-1}\\
&= \frac{(-1)^{L}}{x\alpha_i'(x)}\sum_{m=0}^{L-1}\sum_{j=s'_m}^{s'_{m+1}-1} j \alpha_i^{L-m-1}(x)x^{j-1}.
\end{aligned} \tag{F.7}$$

Therefore

$$\alpha_i'(x) = \frac{(-1)^L}{x\mathcal{E}_i(x)} \sum_{m=0}^{L-1} \sum_{j=s_m'}^{s_{m+1}'-1} j\alpha_i^{L-m-1}(x)x^{j-1}. \tag{F.8}$$

Note that $\left[\sum_i \mathcal{P}_{i,0}(x) \prod_{j=2}^L \alpha_j^{i_j}(x)\right]'$ is a sum of $O(n^{N_1'})$ terms with each summand a product of $\alpha_t'(x) \prod_{j=2}^L \alpha_j^{i_j}(x)$ and a polynomial of $\alpha_1(x), \dots, \alpha_L(x)$ independent of n, where N_1' is also independent of n, $t > 1$ and $\sum_{j=2}^L i_j \le n$. By (F.8), each summand is of the form

$$\frac{(-1)^L}{x\mathcal{E}_t(x)} \sum_{m=0}^{L-1} \sum_{j'=s_m'}^{s_{m+1}'-1} j'\alpha_t^{L-m-1}(x)x^{j'-1} \prod_{j=2}^L \alpha_j^{i_j}(x).$$

Since $\mathcal{P}(x)$ is symmetric with respect to $\alpha_2(x), \alpha_3(x), \dots, \alpha_L(x)$, so is $\sum_i \mathcal{P}_{i,0}(x) \prod_{j=2}^L \alpha_j^{i_j}(x)$ and its derivative. Thus, by the same approach as in the case $\ell = 0$, we can prove that

$$\left[\sum_i \mathcal{P}_{i,0}(x) \prod_{j=2}^L \alpha_j^{i_j}(x)\right]' = \frac{1}{x\mathcal{E}_1(x)} \sum_{i'} \hat{\mathcal{P}}_{i',1}(x) \prod_{j=2}^L \alpha_j^{i_j'}(x),$$

where there are at most $O(n^{N_1''})$ summands and $\sum_{j=2}^L i_j' \le n + M_1'$ with N_1'' and M_1' independent of n and the $\mathcal{P}_{i',1}(x)$'s are polynomials of $\alpha_1(x), \dots, \alpha_L(x)$ and x that are also independent of n.

Using this result and (F.6), we obtain

$$\mathcal{P}'(x) = \frac{\sum_{i'} \mathcal{P}_{i',1}(x) \prod_{j=2}^L \alpha_j^{i_j'}(x)}{x\mathcal{E}_1^2(x)},$$

where there are at most $O(n^{N_1})$ summands and $\sum_{j=2}^L i_j' \le n + M_1$ with N_1 and M_1 independent of n and the $\mathcal{P}_{i',1}(x)$'s are polynomials of $\alpha_1(x), \dots, \alpha_L(x), \mathcal{E}_1(x), \mathcal{E}_1'(x)$ and x that are also independent of n. Since the denominator of $\mathcal{P}'(x)$, namely $x\mathcal{E}_1^2(x)$ is continuous, well-defined and nonzero at $x = 1$, the claim in the case $\ell = 1$ then follows by Proposition 4.1.

By induction and the same approach, we can show that for each ℓ, we have

$$\mathcal{P}^{(\ell)}(x) = \frac{\sum_i \mathcal{P}_{i,\ell}(x) \prod_{j=2}^L \alpha_j^{i_j}(x)}{x^{2^{\ell-1}} \mathcal{E}_1^{2^\ell}(x)},$$

where there are at most $O(n^{N_\ell})$ summands and $\sum_{j=2}^L i_j \le n + M_\ell$ with N_ℓ and M_ℓ independent of n and the $\mathcal{P}_{i,1}(x)$'s are polynomials of $\alpha_1(x), \dots, \alpha_L(x), \mathcal{E}_1^{(l)}(x)$ $(1 \le l \le \ell)$ and x that are also independent of n. Since the denominator of $\mathcal{P}^{(\ell)}(x)$, namely $x^{2^{\ell-1}} \mathcal{E}_1^{2^\ell}(x)$ is continuous, well-defined and nonzero at $x = 1$, the claim then follows by Proposition 4.1. □

Appendix G. Upper and Lower Bound for C

If $L = 1$, then $C = \frac{1}{2}(s_0 + s_1 - 1) = \frac{c_1 - 1}{2}$.

If $L \ge 2$, for each $m \in \{0, 1, \dots, L-1\}$, we have

$$\frac{\frac{1}{2}(s_m + s_{m+1} - 1)}{m+1} \le \frac{mc_1 + (m+1)c_1 - 1}{2(m+1)} = c_1 - \frac{c_1 + 1}{2(m+1)}$$

$$\leq \quad c_1 - \frac{c_1+1}{2L} = \frac{(2L-1)c_1-1}{2L} < c_1. \tag{G.1}$$

Note that when $L=1$, $\frac{(2L-1)c_1-1}{2L} = \frac{c_1-1}{2}$, hence (G.1) holds in this case as well. Thus we get an upper bound for C:

$$C \leq \frac{(2L-1)c_1-1}{2L} < c_1.$$

If $m=0$, then

$$\frac{\frac{1}{2}(s_m+s_{m+1}-1)}{m+1} = \frac{c_1+m-1+c_1+m-1}{2(m+1)} = \frac{c_1-1}{2}.$$

If $m \geq 1$ and $c_1 \geq 2$, then

$$\frac{s_m+s_{m+1}-1}{2(m+1)} \geq \frac{c_1+m-1+c_1+m-1}{2(m+1)} = \frac{c_1-2}{m+1}+1 \geq \frac{c_1-2}{L}+1.$$

Thus

$$C \geq \min\{\frac{c_1-1}{2},\ \frac{c_1-2}{L}+1\}. \tag{G.2}$$

Note that when $c_1=1$, the right-hand side of (G.2) is 0, and when $L=1$, the right-hand side of (G.2) is $\min\{\frac{1}{2}(c_1-1),\ c_1-1\} = \frac{1}{2}(c_1-1)$. Thus (G.2) gives a lower bound for C for all L.

Appendix H. Proof of $h'(1) \neq 0$

Proof. When $L=1$, we have $c_1 > 1$ (see the assumption of Theorem 1.2) and $\alpha_1(x) = 1+x+x^2+\cdots+x^{c_1-1}$. Thus

$$\alpha_1'(x) = 1+2x+3x^2+\cdots+(c_1-1)x^{c_1-2}$$

and

$$\alpha_1''(x) = \begin{cases} 2\cdot 1+3\cdot 2x+\cdots+(c_1-1)(c_1-2)x^{c_1-3},\ c_1>2 \\ 0,\ c_1=2. \end{cases}$$

Setting $x=1$ gives

$$\alpha_1(1)=c_1,\ \alpha_1'(1)=\frac{c_1(c_1-1)}{2},\ \alpha_1''(1)=\frac{c_1(c_1-1)(c_1-2)}{3}.$$

By Definition (5.12), we get

$$h'(x) = \left(\frac{x\alpha_1'(x)}{\alpha_1(x)} - C\right)' = \frac{\alpha_1(x)\left(\alpha_1'(x)+x\alpha_1''(x)\right) - x\left(\alpha_1'(x)\right)^2}{\alpha_1^2(x)}.$$

Setting $x=1$ yields

$$\alpha_1^2(1)h'(1) = \alpha_1(1)\left(\alpha_1'(1)+\alpha_1''(1)\right) - \left(\alpha_1'(1)\right)^2 = \frac{c_1^2(c_1-1)(c_1+1)}{12} \neq 0.$$

We prove by contradiction for $L \geq 2$. Assume $h'(1)=0$. From (4.21), we get

$$h(x) = \frac{x\alpha_1'(x)}{\alpha_1(x)} - C = -\frac{xy_1'(x)}{y_1(x)} - C.$$

Thus

$$h'(x) = \left(-\frac{xy_1'(x)}{y_1(x)}\right)'.$$

Plugging in (4.10) yields

$$h'(x) = \left(\frac{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} jx^j y_1^m(x)}{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1)x^j y_1^m(x)} \right)'.$$

Since $h'(1) = 0$, we get

$$\begin{aligned} &\left(\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} jx^j y_1^m(x) \right)' \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1)x^j y_1^m(x) \\ = \;& \left(\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1)x^j y_1^m(x) \right)' \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} jx^j y_1^m(x), \text{ when } x = 1, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} jy_1^m(1)}{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} (m+1)y_1^m(1)} \qquad \text{(H.1)} \\ = \;& \frac{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left(j^2 x^{j-1} y_1^m(x) + mjx^j y_1^{m-1}(x) y_1'(x) \right)}{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left((m+1)jx^{j-1} y_1^m(x) + m(m+1)x^j y_1^{m-1}(x) y_1'(x) \right)} \\ = \;& \frac{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left(j^2 y_1^m(1) + mjy_1^{m-1}(1) y_1'(1) \right)}{\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left((m+1)jy_1^m(1) + m(m+1)y_1^{m-1}(1) y_1'(1) \right)}, \; x = 1. \end{aligned}$$

From (4.22), we see that (H.1) is exactly $-(y_1'(1))/(y_1(1))$, thus

$$\begin{aligned} & y_1'(1) \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left((m+1)jy_1^m(1) + m(m+1)y_1^{m-1}(1) y_1'(1) \right) \\ + \;& y_1(1) \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} \left(j^2 y_1^m(1) + mjy_1^{m-1}(1) y_1'(1) \right) = 0. \end{aligned}$$

Rearranging the terms, we get

$$\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} y_1^{m-1}(1)[j^2 y_1^2(1) + (2m+1)jy_1(1)y_1'(1) + m(m+1)\left(y_1'(1)\right)^2] = 0.$$

Adding $\sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} y_1^{m-1}(1)[jy_1(1)y_1'(1) + (m+1)\left(y_1'(1)\right)^2]$ to both sides yields

$$\begin{aligned} & \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} y_1^{m-1}(1)[j^2 y_1^2(1) + (2m+2)jy_1(1)y_1'(1) \qquad \text{(H.2)} \\ & +(m+1)^2 \left(y_1'(1)\right)^2] \\ = \;& \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} y_1^{m-1}(1)[jy_1(1)y_1'(1) + (m+1)\left(y_1'(1)\right)^2] \\ = \;& y_1'(1) \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} [jy_1^m(1) + (m+1)y_1^{m-1}(1)y_1'(1)] \end{aligned}$$

$$= y_1'(1)\left[\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} j y_1^m(1) + \frac{y_1'(1)}{y_1(1)}\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1}(m+1)y_1^m(1)\right]$$
$$= 0$$

by (4.22).

On the other hand, we can rewrite (H.2) as

$$\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} y_1^{m-1}(1)[jy_1(1)+(m+1)y_1'(1)]^2.$$

Since $y_1(1) > 0$, each $jy_1(1) + (m+1)y_1'(1)$ should be 0. Therefore

$$\frac{j}{m+1} = -\frac{y_1'(1)}{y_1(1)},\ \forall\ m \in [0, L-1],\ \forall\ j \in [s_m, s_{m+1}-1].$$

Letting $m = 0,\ j = 0$ and $m = 1,\ j = s_1$ (since $L \geq 2$, m can be 1), we get

$$\frac{0}{1} = -\frac{y_1'(1)}{y_1(1)} = \frac{s_1}{2},$$

contradiction. Hence $h'(1) \neq 0$. □

Appendix I. Proof of $\mu_n(m) = \tilde{\mu}_n(m) + o(1)$

Since

$$\tilde{\mu}_n(m) = \sum_k \frac{p_{n,k}(k-\tilde{\mu}_n)^m}{\Delta_n} = \sum_k \mathrm{Prob}(n,k)(k-\tilde{\mu}_n)^m.$$

and

$$\mu_n(m) = \sum_k \frac{p_{n,k}(k-\mu_n)^m}{\Delta_n} = \sum_k \mathrm{Prob}(n,k)(k-\mu_n)^m,$$

we have

$$\begin{aligned}
&|\mu_n(m) - \tilde{\mu}_n(m)| \\
=& \left|\sum_k \mathrm{Prob}(n,k)(k-\mu_n)^m - \sum_k \mathrm{Prob}(n,k)(k-\tilde{\mu}_n - o(1))^m\right| \\
=& \left|o(1)\sum_k \mathrm{Prob}(n,k)\sum_{i=0}^{m}(k-\tilde{\mu}_n)^{m-i}o^i(1)\right| \\
\leq& \left|o(1)\sum_k \sum_{i=0}^{m}(k+\tilde{\mu}_n)^{m-i}o^i(1)\right| \leq \left|o(1)n\sum_{i=0}^{m}(n+\tilde{\mu}_n)^{m-i}o^i(1)\right| \\
\leq& \left|o(1)n(n+Cn+n)^m\sum_{i=0}^{m}o^i(1)\right| \leq o(1),
\end{aligned}$$

for finite m and sufficiently large n (see Remark 4.1 for the description of the $o(1)$ term and note that $C > 0$). Hence $\mu_n(m) = \tilde{\mu}_n(m) + o(1)$.

Appendix J. Proof of Proposition 6.1

Proof. Since the roots of $\hat{A}_x(z)$ are continuous and (a), (b) hold for $x = 1$, they also hold for a sufficiently small neighborhood I_ε of 1.

For (c), since $z_i(x)$ is a root of $\hat{A}_x(z)$, we have

$$0 = 1 - z_i(x) - (x+1)z_i(x)^4 - xz_i(x)^6 - xz_i(x)^7. \tag{J.1}$$

Let Δx be a small increment, we have

$$\begin{aligned} 0 &= 1 - z_i(x+\Delta x) - (x+\Delta x+1)z_i^4(x+\Delta x) \\ &\quad -(x+\Delta x)z_i^6(x+\Delta x) - (x+\Delta x)z_i^7(x+\Delta x). \end{aligned} \tag{J.2}$$

Subtracting (J.2) from (J.1) yields

$$\begin{aligned} 0 &= z_i(x+\Delta x) - z_i(x) + (x+1)[z_i^4(x+\Delta x) - z_i^4(x)] \\ &\quad +\Delta x \cdot z_i^4(x+\Delta x) + x[z_i^6(x+\Delta x) - z_i^6(x)] + \Delta x \cdot z_i^6(x+\Delta x) \\ &\quad +x[z_i^7(x+\Delta x) - z_i^7(x)] + \Delta x \cdot z_i^7(x+\Delta x) \\ &= [z_i(x+\Delta x) - z_i(x)]\Bigg[1 + (x+1)\sum_{j=0}^{3} z_i^j(x+\Delta x)z_i^{3-j}(x) \\ &\quad +x\sum_{j=0}^{5} z_i^j(x+\Delta x)z_i^{5-j}(x) + x\sum_{j=0}^{6} z_i^j(x+\Delta x)z_i^{6-j}(x)\Bigg] \\ &\quad +\Delta x[z_i^4(x+\Delta x) + z_i^6(x+\Delta x) + z_i^7(x+\Delta x)]. \end{aligned} \tag{J.3}$$

Since $z_i(x)$ is continuous, the coefficient of $z_i(x+\Delta x) - z_i(x)$ converges as $\Delta x \to 0$ and its limit is

$$1 + 4(x+1)z_i^3(x) + 6xz_i^5(x) + 7xz_i^6(x),$$

which is exactly $-\hat{A}_x'(z)$ (with respect to z) at $z_i(x)$ and therefore nonzero since $\hat{A}_x(z)$ has no multiple roots. The coefficient of Δx in (J.3) also converges as $\Delta x \to 0$ and its limit is $z_i^4(x) + z_i^6(x) + z_i^7(x)$. Thus we have

$$z_i'(x) = \frac{z_i(x+\Delta x) - z_i(x)}{\Delta x} \to -\frac{z_i^4(x) + z_i^6(x) + z_i^7(x)}{1 + 4(x+1)z_i^3(x) + 6xz_i^5(x) + 7xz_i^6(x)}, \tag{J.4}$$

as $\Delta x \to 0$.

Since the denominator of $z_i'(x)$ is not zero, by the same approach in Proposition 4.2, we can show that $z_i(x)$ is ℓ-times differentiable for any $\ell \geq 1$.

Finally, with (a), Part (d) can be shown in the exactly same way as in Proposition 4.1(b). □

Appendix K. Proof of Proposition 6.8

Proof. Since the roots of $\hat{A}_w(z)$ are continuous and (a), (b) hold for $x = 1$, they also hold for a sufficiently small neighborhood I_ε of 1.

For (c), since $e_i(w)$ is a root of $\hat{A}_w(z)$, we have

$$0 = 1 - e_i(w) - (w^a + w^b)e_i^4(w) - w^{a+b}e_i^6(w) - w^{a+b}e_i^7(w). \tag{K.1}$$

For a small increment Δw, we have

$$\begin{aligned}0 = 1 &- e_i(w+\Delta w) - [(w+\Delta w)^a + (w+\Delta w)^b]e_i^4(w+\Delta w)\\ &- (w+\Delta w)^{a+b}e_i^6(w+\Delta w) - (w+\Delta w)^{a+b}e_i^7(w+\Delta w).\end{aligned} \tag{K.2}$$

Subtracting (K.2) from (K.1) yields

$$\begin{aligned}0 &= e_i(w+\Delta w) - e_i(w) + (w^a + w^b)[e_i^4(w+\Delta w) - e_i^4(w)]\\ &\quad + [(w+\Delta w)^a + (w+\Delta w)^b - w^a - w^b]e_i^4(w+\Delta w)\\ &\quad + w^{a+b}[e_i^6(w+\Delta w) - e_i^6(w)] + [(w+\Delta w)^{a+b} + w^{a+b}]e_i^6(w+\Delta w)\\ &\quad + w^{a+b}[e_i^7(w+\Delta w) - e_i^7(w)] + [(w+\Delta w)^{a+b} - w^{a+b}]e_i^7(w+\Delta w)\\ &= [e_i(w+\Delta w) - e_i(w)]\left[1 + (w^a+w^b)\sum_{j=0}^{3} e_i^j(w+\Delta w)e_i^{3-j}(w)\right.\\ &\quad \left. + w^{a+b}\sum_{j=0}^{5} e_i^j(w+\Delta w)e_i^{5-j}(w) + w^{a+b}\sum_{j=0}^{6} e_i^j(w+\Delta w)e_i^{6-j}(w)\right]\\ &\quad + \Delta w\left[\left(\frac{(w+\Delta w)^a - w^a}{\Delta w} + \frac{(w+\Delta w)^b - w^b}{\Delta w}\right)e_i^4(w+\Delta w)\right.\\ &\quad \left. + \frac{(w+\Delta w)^{a+b} - w^{a+b}}{\Delta w}\left(e_i^6(w+\Delta w) + e_i^7(w+\Delta w)\right)\right].\end{aligned} \tag{K.3}$$

Since $e_i(w)$ is continuous, the coefficient of $[e_i(w+\Delta w) - e_i(w)]$ converges as $\Delta w \to 0$ and its limit is

$$1 + 4(w^a + w^b)e_i^3(x) + 6w^{a+b}e_i^5(w) + 7w^{a+b}e_i^6(w),$$

which is exactly $-\hat{A}_w'(z)$ (with respect to z) at $e_i(w)$ and therefore nonzero since $\hat{A}_w(z)$ has no multiple roots. Since w^a, w^b and w^{a+b} are differentiable at $w = 1$, the coefficient of Δw in (K.3) also converges as $\Delta w \to 0$ and its limit is

$$\left(aw^{a-1} + bw^{b-1}\right)e_i^4(w) + (a+b)w^{a+b-1}[e_i^6(w) + e_i^7(w)].$$

Thus $e_i'(w)$ exists and

$$\begin{aligned}e_i'(w) &= \frac{e_i(w+\Delta w) - e_i(w)}{\Delta w}\\ \to\quad &-\frac{\left(aw^{a-1} + bw^{b-1}\right)e_i^4(w) + (a+b)w^{a+b-1}[e_i^6(w) + e_i^7(w)]}{1 + 4(w^a + w^b)e_i^3(x) + 6w^{a+b}e_i^5(w) + 7w^{a+b}e_i^6(w)}\end{aligned} \tag{K.4}$$

as $\Delta w \to 0$. Since the denominator of $e_i'(w)$ is not zero, by the same approach in Proposition 4.2, we can show that $e_i(w)$ is ℓ-times differentiable for any $\ell \geq 1$.

Finally, with (a), Part (d) can be shown in the exactly same way as in Proposition 4.1(b). □

Appendix L. Proof of $h_{a,b}'(1) \neq 0$

Proof. By (6.42), we have

$$\frac{we_1'(w)}{e_1(w)} = -\frac{\left(aw^a + bw^b\right)e_1^3(w) + (a+b)w^{a+b}[e_1^5(w) + e_1^6(w)]}{1 + 4(w^a + w^b)e_1^3(w) + 6w^{a+b}e_1^5(w) + 7w^{a+b}e_1^6(w)}. \tag{L.1}$$

Thus

$$\begin{aligned}
&\hat{h}'_{a,b}(w) \\
= &\left[\frac{\left(aw^a + bw^b\right) e_1^3(w) + (a+b)w^{a+b}(e_1^5(w) + e_1^6(w))}{1 + 4(w^a + w^b)e_1^3(w) + 6w^{a+b}e_1^5(w) + 7w^{a+b}e_1^6(w)}\right]' \\
= &\Big[\left[\left(aw^a + bw^b\right) e_1^3(w) + (a+b)w^{a+b}(e_1^5(w) + e_1^6(w))\right]' \\
&\cdot \left[1 + 4(w^a + w^b)e_1^3(w) + 6w^{a+b}e_1^5(w) + 7w^{a+b}e_1^6(w)\right] \\
&-[\left(aw^a + bw^b\right) e_1^3(w) + (a+b)w^{a+b}\left(e_1^5(w) + e_1^6(w)\right)] \\
&\cdot \left[1 + 4(w^a + w^b)e_1^3(w) + w^{a+b}\left(6e_1^5(w) + 7e_1^6(w)\right)\right]'\Big] \\
&\cdot \left[1 + 4(w^a + w^b)e_1^3(w) + w^{a+b}\left(6e_1^5(w) + 7e_1^6(w)\right)\right]^{-2}. \qquad \text{(L.2)}
\end{aligned}$$

Setting $w = 1$ in (L.1) and using $e_1(1) = \Phi$, we get

$$\frac{e_1'(1)}{e_1(1)} = -\frac{(a+b)(\Phi^3 + \Phi^5 + \Phi^6)}{1 + 8\Phi^3 + 6\Phi^5 + 7\Phi^6} = -\frac{a+b}{10}.$$

Thus

$$e_1'(1) = -\frac{a+b}{10}\Phi. \qquad \text{(L.3)}$$

Plugging $e_1(1) = \Phi$ and (L.3) into (L.2) with $w = 1$ yields

$$\begin{aligned}
\hat{h}'_{a,b}(1) &= \left[\Phi^5\left[10\left(a^2 + b^2\right) + (a+b)^2\left(-3 + 10\Phi - 5\Phi^2 - 6\Phi^3\right)\right]\right. \\
&\quad \left.-\Phi^5(a+b)^2\left(1.6 + 3\Phi^2 + 2.8\Phi^3\right)\right]/(100\Phi^4) \\
&= \frac{\sqrt{5}-1}{200}\left[10\left(a^2+b^2\right) - \frac{20-\sqrt{5}}{5}(a+b)^2\right] \qquad \text{(L.4)}
\end{aligned}$$

Since $\frac{20-\sqrt{5}}{5} < 4$ and $a^2 + b^2 > 0$, we have

$$\frac{20-\sqrt{5}}{5}\left(a^2+b^2\right) < 4(a+b)^2 \leq 8\left(a^2+b^2\right) < 10\left(a^2+b^2\right).$$

Hence $\hat{h}'_{a,b}(1) \neq 0$. $\square$