Lower Order Terms for the Variance of Gaussian Primes Across Sectors

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Gaussian Primes

An odd prime p is said to be a ***sum of squares*** if

$$p = a^2 + b^2.$$

Reciprocity Law (Fermat): An odd prime p is a sum of squares if and only if $p = 1$ *mod 4.*

Examples:

- 3 is not a sum of squares
- $5 = 2^2 + 1^2$ is a sum of squares
- $13 = 2^2 + 3^2$ is a sum of squares

Pierre de Fermat
(1601-1665)

Motivation – Gaussian Primes

Equivalently, an odd prime p is a sum of squares if and only if it ***splits*** in the **Gaussian Integers.**

Gaussian Integers:

$$Z[i] = \{a + bi: a, b \in Z\}$$

Carl Friedrich Gauss
(1777-1855)

Examples:

- 3 does not split
- $5 = (2 + i)(2 - i) = 2^2 + 1^2$ ***splits***
- $13 = (2 + 3i)(2 - 3i) = 2^2 + 3^2$ ***splits***

$3, 2 + i, 2 + 3i$, etc. are called ***Gaussian Primes.***

Gaussian Primes

Angle of a Gaussian prime: *For an odd prime* p *such that*

$$p = a^2 + b^2 = (a + bi)(a - bi)$$

let $e^{i\theta_p}$ *denote the argument of* $a + bi$ *so that*

$$\theta_p = \tan^{-1}(\frac{b}{a})$$

By convention we choose $0 \leq b \leq a$ *so that* $\frac{b}{a} \in [0,1]$*, i.e.* $\theta_p \in [0, \frac{\pi}{4}]$.

Examples:

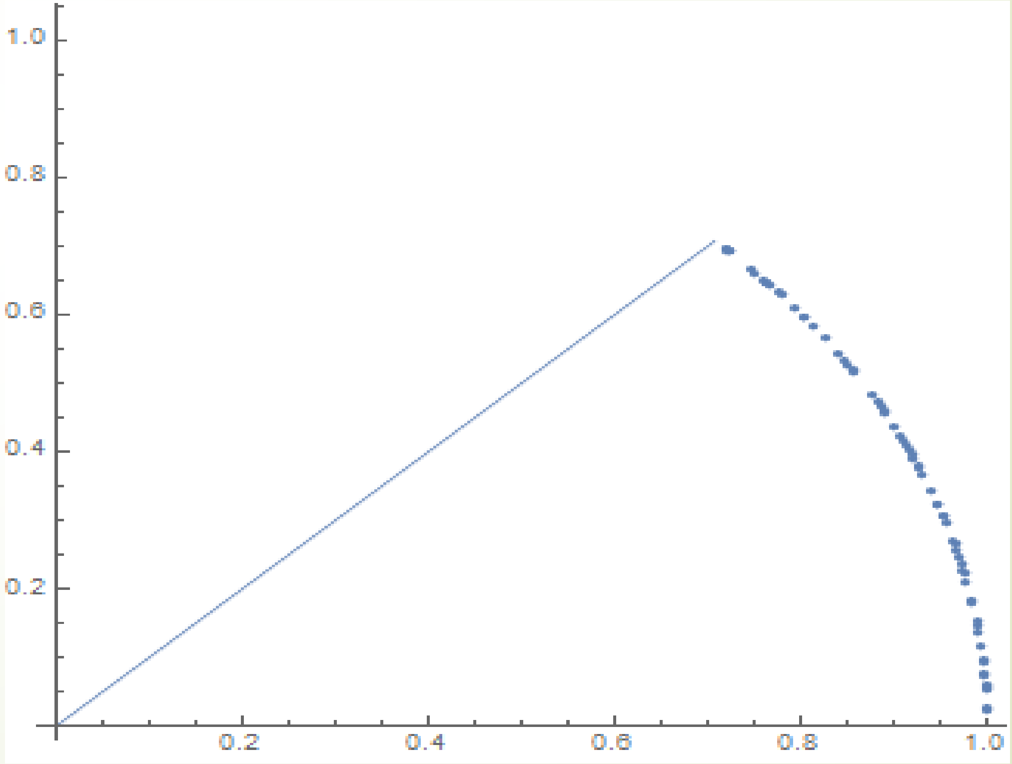
- $5 = 2^2 + 1^2 \Rightarrow (a, b) = (2,1) \Rightarrow \theta_5 = \tan^{-1}(\frac{1}{2})$
- $13 = 3^2 + 2^2 \Rightarrow (a, b) = (3,2) \Rightarrow \theta_{13} = \tan^{-1}(\frac{2}{3})$

Gaussian Primes

Are the Gaussian Primes "randomly" distributed?

Do the first X Gaussian prime angles have the same statistics as X random points in $[0, \frac{\pi}{4})$ *?*

"Random Points" – picked independently and uniformly in $\left[0, \frac{\pi}{4}\right)$



Angular distribution $(a + ib)/\sqrt{p}$ of the 67 primes 1000<p<2000, p=1 mod 4

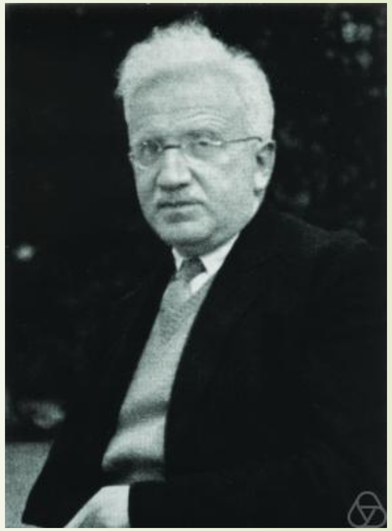
Gaussian Primes

Uniform Distribution

Let θ_n *denote the angle associated with the* n^{th} *prime* $p \equiv 1 \bmod 4$.

Hecke (1918): *The angles of Gaussian primes are* ***uniformly distributed*** *:* *for fixed* $0 \le \alpha \le \beta \le \frac{\pi}{4}$,

$$\lim_{N\to\infty} \# \frac{\{n \le N: \theta_n \in [\alpha,\beta]\}}{N} = \frac{\beta-\alpha}{\pi/4}.$$



Erich Hecke (1887-1947)

Variance of Gaussian Primes

Divide $[0, \pi/4]$ into K small arcs, and ask how many of the N prime angles fall into each arc. Let

$$N_{L,N}(\theta) = \#\left\{n \le N : \theta_n \in \left[\theta, \theta + \frac{\pi/4}{K}\right]\right\}.$$

The expected value is found to be

$$\langle N_{K,N}(\theta)\rangle = \frac{1}{\pi/4}\int_0^{\frac{\pi}{4}} N_{K,N}(\theta)\, d\theta \sim \frac{N}{K}.$$

Variance of Gaussian Primes

The limiting variance is then given by

$$Var(N_{K,N}) = \lim_{N\to\infty} \frac{1}{\pi/4} \int_0^{\frac{\pi}{4}} \left| N_{K,N}(\theta) - \frac{N}{K} \right|^2 d\theta.$$

In the region K $\gg N^{1+o(1)}$, i.e. very short intervals, we can calculate

$$Var(N_{K,N}) \sim \frac{N}{K}.$$

Theorem (Zeev Rudnick & EW): Under GRH,

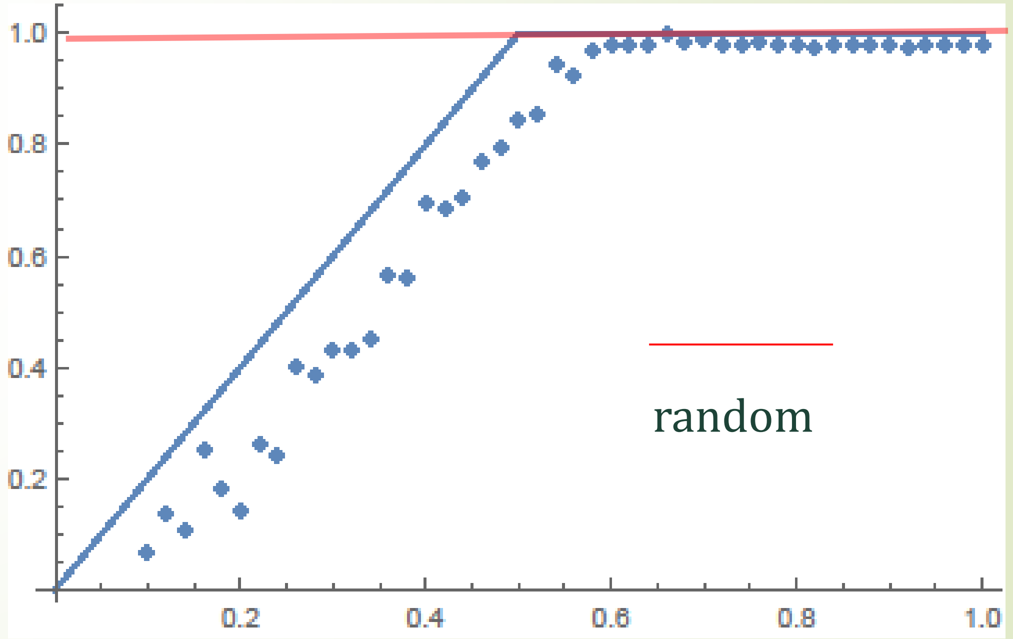
$$Var(N_{K,N}) \ll \frac{N}{K}(log\ K)^2.$$

For K $\ll N$ the assymptotics remain open.

Variance of Gaussian Primes

Conjecture: $Var(N_{K,N}) \sim \frac{N}{K} \min(1, 2\frac{\log K}{\log N})$

Compare: For N random points, $Var(N_{K,N}^{random}) \sim \frac{N}{K}$



Data: 35241 angles of the Gaussian primes $10^8 < p < 2 * 10^8$

$$\frac{Var(N_{K,N})}{N/K} \text{ vs. } \frac{\log K}{\log N}$$

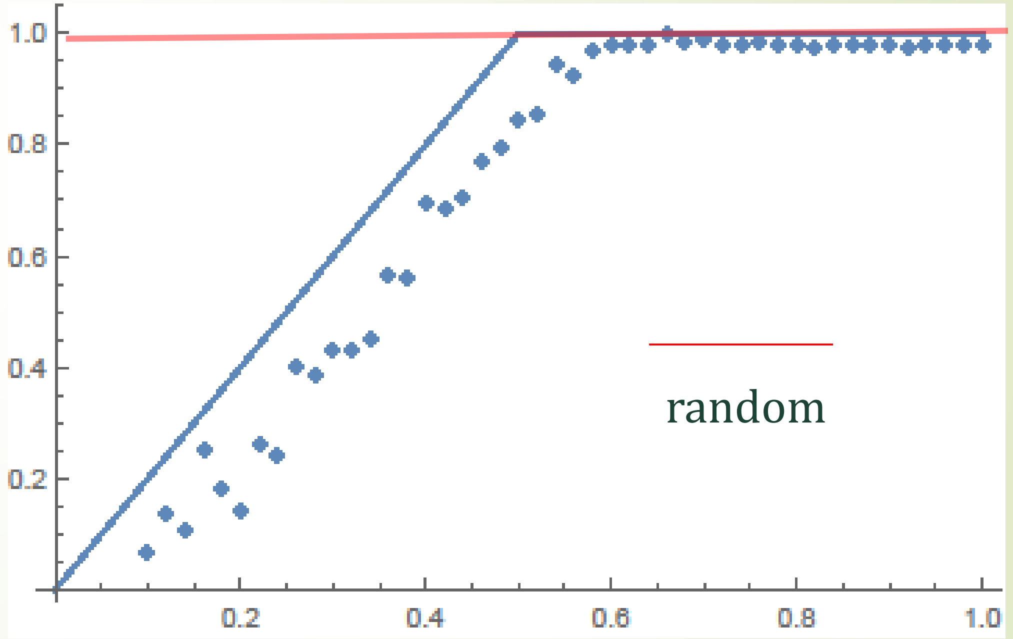
Motivations for Conjecture:

a) **Numerical Evidence**
b) **Random Matrix Model**: Express variance through zeros of a certain family of Hecke L-functions, then replace these zeros by eigenphases of a suitable ensemble of random matrices.
c) **Function Field Analogue**

Variance of Gaussian Primes

Question: Can we come up with a conjecture that fits the data even better?

Answer: Yes! Using the Ratios Conjecture



Data: 35241 angles of the Gaussian primes $10^8 < p < 2 * 10^8$

$$\frac{Var\left(N_{K,N}\right)}{N/K} \text{ vs. } \frac{\log K}{\log N}$$

Ratios Conjecture – A History

- Conrey-Farmer-Zinbauer (2007): conjecture an algorithm for computing the averages of products of L-functions over families

Ratios Conjecture – A History

- Conrey-Farmer-Zinbauer (2007): conjecture an algorithm for computing the averages of products of L-functions over families.
- Applications:
  - n-level correlations and n-level densities
  - Vanishing at the central point
  - Moments of L-functions

Ratios Recipe

- Use approximate functional equation to expand each term in the numerator (ignoring remainder term)

$$L_k(s) = \sum_{n<x} \frac{a_n(k)}{n^s} + \epsilon_k(s) \sum_{m<y} \frac{a_m(k)}{m^{1-s}} + remainder$$

- Expand the denominator by the generalized Mobius function

$$L_k(s) = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s}$$

- Replace each summand by its expected value when averaged over the family
- Replace each product of $\epsilon_k(s)$ factors by its expected value when averaged over the family

Hecke Größencharakters

In our case, we apply the conjecture to a family of L-functions attached to Hecke characters (**Größencharakters**). For a nonzero ideal $\alpha \in Z[i]$, define

$$\chi_k(\alpha) = e^{i4k\theta_{\boldsymbol{a}}}$$

which is well-defined on ideals. The associated **(Hecke) L-function** is defined as

$$L_k(s) = \sum_{\boldsymbol{a}\neq\boldsymbol{0}} \frac{\chi_k(\boldsymbol{a})}{N(\boldsymbol{a})^s} = \prod_{\boldsymbol{p}\ prime} (1 - \chi_k(\boldsymbol{p})N(\boldsymbol{p})^{-s})^{-1}$$

for Re(s)>1, where

$$N(\boldsymbol{a}) \coloneqq \#(Z[i]/\boldsymbol{a})$$

The Größencharakters can be used to detect elements in a given sector.

Ratios Conjecture

Ratios Conjecture Prediction: let

$$Y(\alpha,\beta,\gamma,\delta)=\frac{\zeta(1+2\alpha)\zeta(1+2\beta)\zeta(1+\gamma+\delta)\zeta(1+\alpha+\beta)}{\zeta(1+\alpha+\gamma)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)\zeta(1+\alpha+\delta)}\times\frac{L(1+2\gamma)L(1+2\delta)L(1+\gamma+\delta)L(1+\alpha+\beta)}{L(1+\alpha+\gamma)L(1+\beta+\gamma)L(1+\beta+\delta)L(1+\alpha+\delta)}$$

Then up to very good approximation,

$$\sum_{k\in K}\frac{L_k\left(\frac{1}{2}+\alpha\right)L_k\left(\frac{1}{2}+\beta\right)}{L_k\left(\frac{1}{2}+\gamma\right)L_k\left(\frac{1}{2}+\delta\right)}=Y(\alpha,\beta,\gamma,\delta)+\frac{1}{1-2\alpha}\left(\frac{\pi}{2K}\right)^{2\alpha}Y(-\alpha,\beta,\gamma,\delta)+\frac{1}{1-2\beta}\left(\frac{\pi}{2K}\right)^{2\beta}Y(\alpha,-\beta,\gamma,\delta)$$

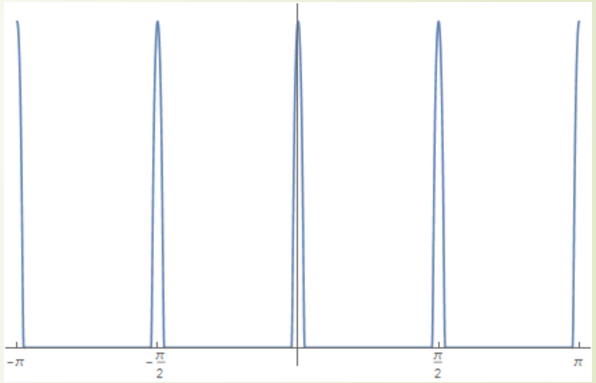
$$+\frac{1}{1-2\beta}\left(\frac{\pi}{2K}\right)^{2\beta}Y(\alpha,-\beta,\gamma,\delta)+\frac{1}{1-2(\alpha+\beta)}\left(\frac{\pi}{2K}\right)^{2(\alpha+\beta)}Y(-\alpha,-\beta,\gamma,\delta)$$

A smoothed number variance

Define

$$\psi_{K,X}(\theta) = \sum_{\alpha \subset Z[i]} \Phi\left(\frac{N(\alpha)}{X}\right) \Lambda(\alpha) F_K(\theta_\alpha - \theta)$$

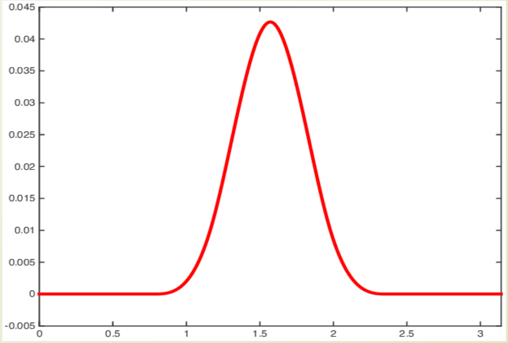
F_K, Φ= fixed window functions



- $F_K(\theta)$ is a smooth window function, localized at scale $1/K$, $\frac{\pi}{2}$-periodic.
- Φ is a smooth cut-off for the norm
- $\Lambda(\alpha)$ is the von-Mangoldt function, which is a weighted count of prime powers.

Rather than compute $\mathrm{Var}(N_{K,X})$ directly, we instead study

$$Var(\psi_{K,X}) = \frac{1}{\pi/2} \int_0^{\frac{\pi}{2}} \left|\psi_{K,X}(\theta) - \langle \psi_{K,X}(\theta) \rangle\right|^2 d\theta$$



A smoothed number variance

By making use of the explicit formula, we may write $Var(\psi_{K,X})$ in terms of the Hecke L-functions $L_k(s)$:

$$Var(\psi_{K,X}) = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k\neq 0} e^{4ki\theta} \frac{i}{2\pi K} \hat{f}\left(\frac{k}{K}\right) \int \frac{L_k{}'}{L_k}(s)\widetilde{\Phi}(s) X^s ds \right|^2 d\theta$$

- Opening up the bracket, and inserting the ratios conjecture, gives us a double integral.

A smoothed number variance

The integral then looks as follows:

$$Var(\psi_{K,X})$$
$$= -\frac{K^{\gamma-2}}{4\pi^2}\sum_{k\neq 0}\left|\hat{f}\left(\frac{k}{K}\right)\right|^2\left(\iint \frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha}Y(\alpha,\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}\widetilde{\Phi}\left(\frac{1}{2}+\alpha\right)\widetilde{\Phi}\left(\frac{1}{2}+\beta\right)X^\alpha X^\beta d\alpha d\beta\right.$$
$$+\left(\iint \frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha}\left(\frac{\pi}{2K}\right)^{2\beta}\frac{1}{1-2\beta}Y(\alpha,-\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}\widetilde{\Phi}\left(\frac{1}{2}+\alpha\right)\widetilde{\Phi}\left(\frac{1}{2}+\beta\right)X^\alpha X^\beta d\alpha d\beta\right.$$
$$+\iint \frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha}\left(\frac{\pi}{2K}\right)^{2\alpha}\frac{1}{1-2\alpha}Y(-\alpha,\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}\widetilde{\Phi}\left(\frac{1}{2}+\alpha\right)\widetilde{\Phi}\left(\frac{1}{2}+\beta\right)X^\alpha X^\beta d\alpha d\beta$$

where

$$Y(\alpha,\beta,\gamma,\delta) = \frac{\zeta(1+2\alpha)\zeta(1+2\beta)\zeta(1+\gamma+\delta)\zeta(1+\alpha+\beta)}{\zeta(1+\alpha+\gamma)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)\zeta(1+\alpha+\delta)}\frac{L(1+2\gamma)L(1+2\delta)L(1+\gamma+\delta)L(1+\alpha+\beta)}{L(1+\alpha+\gamma)L(1+\beta+\gamma)L(1+\beta+\delta)L(1+\alpha+\delta)}$$

A smoothed number variance

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$$= -\frac{K^{\gamma-2}}{4\pi^2}\sum_{k\neq 0}\left|\hat{f}\left(\frac{k}{K}\right)\right|^2\left(\iint \frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha}Y(\alpha,\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}\tilde{\Phi}\left(\frac{1}{2}+\alpha\right)\tilde{\Phi}\left(\frac{1}{2}+\beta\right)X^{\alpha}X^{\beta}d\alpha d\beta\right.$$

$$+\left(\iint \frac{\partial}{\partial\beta}\frac{\partial}{\partial\alpha}\left(\frac{\pi}{2K}\right)^{2\beta}\frac{1}{1-2\beta}Y(\alpha,-\beta,\gamma,\delta)\Big|_{\gamma=\alpha,\delta=\beta}\tilde{\Phi}\left(\frac{1}{2}+\alpha\right)\tilde{\Phi}\left(\frac{1}{2}+\beta\right)X^{\alpha}X^{\beta}d\alpha d\beta\right.$$

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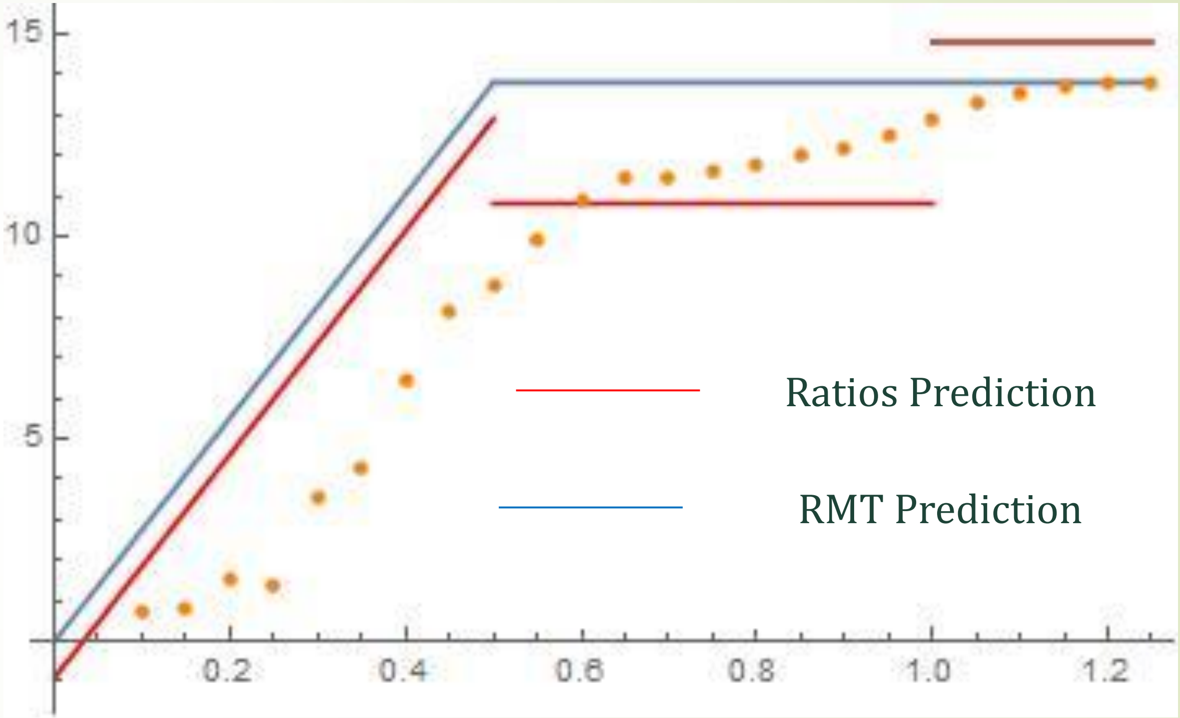
- The contour has single poles and double poles, coming from the singularity at $\zeta(1)$ within the ratios conjecture.
- The double poles contribute the main term; the single poles contribute the lower order terms.

New Conjecture

Let $X = K^\gamma$. Then

$$Var(\psi_{K,X}) = \begin{cases} \frac{X}{K}\left(2 \cdot \log K - \log\frac{\pi^2}{4}\right) & if \quad \gamma > 2 \\ \frac{X}{K}(\log X - 3) & if \quad 1 < \gamma < 2 \\ \frac{X}{K}(\log X + 1) & if \quad 1 < \gamma \end{cases}$$



Data: 78594 Gaussian primes and prime powers with norm $< 10^6$

$$\frac{Var(\Psi_{K,X})}{X/K} \text{ vs. } 1/\gamma$$

Computing the Variance

Function Fields (Theorem – EW & Rudnick):

$$\frac{Var(\psi_{\kappa,\nu})}{X/K} \sim \begin{cases} 2 \cdot \log_q K - 2, & \gamma > 2 \\ \log_q X - 1 + \eta(\log_q X), & 1 < \gamma < 2 \\ \log_q X + \eta(\log_q X), & \gamma < 1 \end{cases}$$

Number Fields (Ratios Conjecture):

$$\frac{Var(\psi_{K,X})}{X/K} \sim \begin{cases} 2 \cdot log\, K - log\frac{\pi^2}{4}, & \gamma > 2 \\ log\, X - 3, & 1 < \gamma < 2 \\ log\, X + 1, & \gamma < 1 \end{cases}$$

Thank you!