From Fibonacci Numbers to Central Limit Type Theorems

Yinghui Wang, MIT
Advisor: Steven J. Miller, Williams College

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

New Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

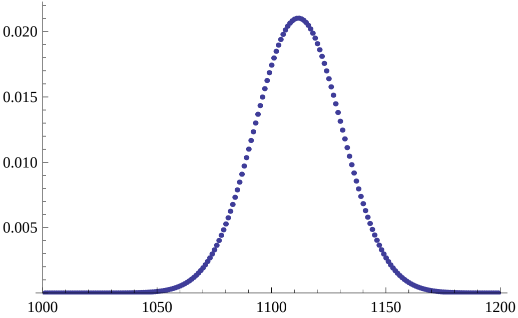

Figure: Number of summands in $[F_{2010}, F_{2011})$

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Cookie counting $\Rightarrow$ $p_{n,k} = \binom{n-k}{k-1}$.

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$y(x)$ is the root of $1 - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}$.

$y(1)$ is the root of $1 - c_1 y - c_2 y^2 - \cdots - c_L y^L$.

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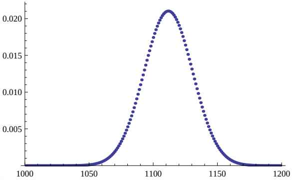
As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$ is Gaussian.



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Example: the Special Case of $L = 1$

$H_{n+1} = c_1 H_n$, $H_1 = 1$. $H_n = c_1^{n-1}$.

- Legal decomposition $\sum_{i=1}^{m} a_i H_i$:

  $a_i \in \{0, 1, \dots, c_1 - 1\}$ $(1 \le i < m)$, $a_m \in \{1, \dots, c_1 - 1\}$,
  equivalent to the c_1-base expansion.
- For $N \in [H_n, H_{n+1})$, $m = n$, i.e., the first term is $a_n H_n$.
- A_i: the corresponding random variable of a_i.
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Thank You!