On Summand Minimality of Generalized Zeckendorf Decompositions

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Introduction

Base d Representation and Zeckendorf's theorem

Theorem (Base d Representation)

Every natural number can be represented uniquely as a sum of powers of d with coefficients in $\{0, 1, \ldots, d-1\}$.

Theorem (Zeckendorf)

Every natural number can be represented uniquely as a sum of non-consecutive Fibonacci numbers (with $F_1 = 1$ and $F_2 = 2$).

Definition: PLRS

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A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence of the following form:

$$H_n = c_1 H_{n-1} + \cdots + c_t H_{n-t}$$

where all $c_i \geq 0$ and $c_1, c_t \geq 1$. We use **ideal initial conditions** $H_{-(n-1)} = 0, \ldots, H_{-1} = 0, H_0 = 1$.

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Definition

We call $(c_1, \ldots, c_t)$ the **signature of the sequence**.

Representation Notation

Definition

For a sequence $\{H_i\}$ with signature $(c_1, \ldots, c_t)$, suppose we have:

$$n = b_k H_k + b_{k-1} H_{k-1} + \cdots + b_0 H_0$$

Then we call $[b_k, b_{k-1}, \ldots, b_0, \infty_{(t-1)}]$ a **representation** of n over $\{H_i\}$.

Example: Representation Notation

Representations of 18

$\mathbf{F_7}$	$\mathbf{F_6}$	$\mathbf{F_5}$	$\mathbf{F_4}$	$\mathbf{F_3}$	$\mathbf{F_2}$	$\mathbf{F_1}$	$\mathbf{F_0}$	$\mathbf{F_{-1}}$
21	13	8	5	3	2	1	1	0
	1	0	1	0	0	0	0	∞
	1	0	0	1	1	0	0	∞

Allowable Blocks

Definition

Given a PLRS with signature $(c_1, \ldots, c_t)$, we say that $[b_1, \ldots, b_k]$ is an **allowable block** if $k \leq t$ and $b_i = c_i$ for $i < k$ and $0 \leq b_k < c_k$. In other words, allowable blocks are *lexicographically less than* the signature.

Examples:

$\sigma = (\mathbf{1}, \mathbf{1})$	$\sigma = (\mathbf{2}, \mathbf{2}, \mathbf{1})$	$\sigma = (\mathbf{3}, \mathbf{0}, \mathbf{2})$
$[0]$	$[0]$	$[0]$
$[1, 0]$	$[1]$	$[1]$
	$[2, 0]$	$[2]$
	$[2, 1]$	$[3, 0, 0]$
	$[2, 2, 0]$	$[3, 0, 1]$

Generalized Zeckendorf Decompositions

Theorem (Miller et al., Hamlin)

Given a positive linear recurrence sequence, every natural number has a unique representation composed of allowable blocks. This representation is called the ***Generalized Zeckendorf Decomposition*** *(GZD).*

Examples

Signature: $\sigma = (d)$
Allowable blocks: $\{[0], [1], \ldots, [d-1]\}$

1000	100	10	1
7	3	0	9

This is the base *d* representation.

Signature: $\sigma = (1, 1)$
Allowable blocks: $\{[0], [1, 0]\}$

5	3	2	1	1	0
1	0	0	1	0	∞

This is the Zeckendorf representation.

Example

Signature: $\sigma = (2, 2, 1)$
Allowable blocks: $\{[0], [1], [2, 0], [2, 1], [2, 2, 0]\}$

385	136	48	17	6	2	1	0	0
2	0	1	2	2	0	0	∞	∞

Summand minimality

Number of Summands

Signature: $\sigma = (1, 1)$
Representations of 6:

# of summands	5	3	2	1	1	0
2	1	0	0	1	0	∞
2		2	0	0	0	∞
3		1	1	1	0	∞
3			3	0	0	∞
4			2	2	0	∞
5			1	0	4	∞
6					6	∞

The GZD has the fewest number of summands.

Number of Summands

Signature: $\sigma = (3, 0, 2)$
Representations of 116:

# of summands	93	29	9	3	1	0	0
4		4	0	0	0	∞	∞
6	1	0	2	1	2	∞	∞

There exists a representation with fewer summands than the GZD.

Summand minimality

Definition

A positive linear recurrence sequence, H, is called **summand minimal** if for all $n \in \mathbb{N}$, the GZD of n has the fewest number of summands among all representations for n using H.

Question

When is a positive linear recurrence summand minimal?

Summand minimality

Definition

A positive linear recurrence sequence, H, is called **summand minimal** if for all $n \in \mathbb{N}$, the GZD of n has the fewest number of summands among all representations for n using H.

Question

When is a positive linear recurrence summand minimal?

Theorem (CHHMPT '16)

A positive linear recurrence sequence is summand minimal if and only if its signature is weakly decreasing, i.e. $\sigma = (c_1, c_2, \ldots, c_t)$ *with* $c_1 \geq c_2 \geq \cdots \geq c_t$*.*

Sketch of techniques

From one representation to another: borrowing

- Signature $\sigma = (3, 2)$
- Strings $[-1, 3, 2]$ and $[1, -3, -2]$ represent 0.
- Representations of 39

H_4	H_3	H_2	H_1	H_0	H_{-1}
139	39	11	3	1	0
	1	0	0	0	∞
	-1	3	2		
	0	3	2	0	∞
			-1	3	2
		3	1	3	∞

From one representation to another: carrying

- Signature $\sigma = (2, 1, 3)$
- Representations of 42

H_4	H_3	H_2	H_1	H_0	H_{-1}	H_{-2}
41	15	5	2	1	0	0
	1	3	4	4	∞	∞
	1	-2	-1	-3		
	2	1	3	1	∞	∞
1	-2	-1	-3			
1	0	0	0	1	∞	∞

From any representation to the GZD

Theorem (CHHMPT '16)

There exists an algorithm which, given any $n \in \mathbb{N}$ and any representation for n, terminates in the GZD for n through a finite sequence of borrows and carries.

Example of the algorithm

- Signature $\sigma = (3, 2, 2)$
- Representations of 44

H_5	H_4	H_3	H_2	H_1	H_0	H_{-1}	H_{-2}
537	148	41	11	3	1	0	0
			4	0	0	∞	∞
			-1	3	2	2	
			3	3	2	∞	∞
		1	-3	-2	-2		
		1	0	1	0	∞	∞

Weakly decreasing signature implies summand minimality

Corollary

The definition of the algorithm immediately implies that positive linear recurrences sequences with weakly decreasing signature are summand minimal.

Idea: after every borrow we are immediately able to carry which causes the number of summands to weakly decrease.

Summand minimality implies weakly decreasing signature

- What about non-weakly decreasing signature?
- When $c_1 > 1$, we can construct an explicit non-GZD representation with fewer summands than the GZD. Idea is to construct a representation such that during the algorithm we need at least two borrows before we can carry (so that the number of summands increases as we move towards the GZD).
- When $c_1 = 1$, using the theory of irreducible polynomials, we are able to non-constructively prove the existence of a non-GZD representation with fewer summands than the GZD.

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Acknowledgments

- Full paper available on arXiv:
  https://arxiv.org/abs/1608.08764
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