# Ranks of One-Parameter Families of Elliptic Curves Over $\mathbb{Q}(t)$ and Thoughts on the Excess Rank Question.

Steven Miller

Princeton University

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http://www.princeton.edu/$\sim$sjmiller/math/talks/talks.html

# Outline of the Talk

- Review of $1$ and $2$-Level Densities.

- Constructing $1$-Parameter Families with rank over $\mathbb{Q}(t)$.

- $n$-Level Density and Bounding Average Rank.

- Numerically Calculating (Approximating) Ranks.

- Potential Lower Order Density Corrections.

- Calculations of Excess Rank in Families.

# Elliptic Curves

Consider $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $a_i \in \mathbb{Q}$ and its $L$-function

$$L(s,E) = \prod_{p|\Delta}\left(1 - a_p p^{-s}\right)^{-1} \prod_{p \ \dagger\Delta}\left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1}$$

By GRH: All non-trivial zeros on the critical line, can talk about spacings between zeros.

Rational solutions form a group: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$, $T$ is the torsion points, $r$ is the geometric rank.

Birch and Swinnerton-Dyer Conjecture: Geometric rank equals the analytic rank, the order of vanishing of $L(s,E)$ at $s = \frac{1}{2}$.

One-parameter families: $a_i = a_i(t) \in \mathbb{Z}[t]$.

# $n$-Level Density: I

Let $f(x) = \prod_i f_i(x_i)$, $f_i$ even Schwartz functions whose Fourier Transforms are compactly supported.

$$D_{n,E}(f) \;=\; \sum_{\substack{j_1,\dots,j_n \\ distinct}} f_1\left(L_E \gamma_E^{(j_1)}\right) \cdots f_n\left(L_E \gamma_E^{(j_n)}\right)$$

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

To any geometric family, Katz-Sarnak predict the $n$-level density depends only on a symmetry group attached to the family.

# $n$-Level Density: II

Let $f(x) = \prod_i f_i(x_i)$, $f_i$ even Schwartz functions whose Fourier Transforms are compactly supported.

$$D_{n,E}(f) \;=\; \sum_{\substack{j_1,\dots,j_n \\ distinct}} f_1\Big(L_E\gamma_E^{(j_1)}\Big)\cdots f_n\Big(L_E\gamma_E^{(j_n)}\Big)$$

Hope: for $f$ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} D_{n,E}(f) \;=\; \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{\substack{j_1,\dots,j_n \\ j_i\neq\pm j_k}}\prod_i f_i\left(\frac{\log N_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to\; \int\cdots\int f(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx$$

$$=\; \int\cdots\int \widehat{f}(u)\widehat{W_{n,\mathcal{G}(\mathcal{F})}}(u)du.$$

# Explicit Formula

Relates sums of test functions over zeros to sums over primes of $a_E(p)$ and $a_E^2(p)$.

$$\begin{aligned}
\sum_{\gamma_E^{(j)}} G\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) \;&=\; \widehat{G}(0) + G(0) \\
&\quad - 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p}\widehat{G}\left(\frac{\log p}{\log N_E}\right) a_E(p) \\
&\quad - 2\sum_p \frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{G}\left(\frac{2\log p}{\log N_E}\right) a_E^2(p) \\
&\quad + O\left(\frac{\log\log N_E}{\log N_E}\right).
\end{aligned}$$

**Modified Explicit Formula:**

$$\begin{aligned}
\sum_{\gamma_E^{(j)}} G\left(\frac{\log X}{2\pi}\gamma_E^{(j)}\right) \;&=\; \frac{\log N_E}{\log X}\widehat{G}(0) + G(0) \\
&\quad - 2\sum_p \frac{\log p}{\log X}\frac{1}{p}\widehat{G}\left(\frac{\log p}{\log X}\right) a_E(p) \\
&\quad - 2\sum_p \frac{\log p}{\log X}\frac{1}{p^2}\widehat{G}\left(\frac{2\log p}{\log X}\right) a_E^2(p) \\
&\quad + O\left(\frac{\log\log X}{\log X}\right).
\end{aligned}$$

# New Results

**Rational Surfaces Density Theorem:** $1$-*parameter family, rank* $r$ *over* $\mathbb{Q}(t)$ *and a rational surface. Assume*

- GRH;
- $j(t)$ non-constant;
- ABC (or Sq-Free Sieve) if $\Delta(t)$ has an irred. factor of degree $\geq 4$.

*Let* $m =$ *deg* $C(t)$*,* $f_i$ *even Schwartz with support* $\sigma_i$*,* $\sigma_1 + \sigma_2 < \frac{1}{3m}$*. Possibly after passing to a subsequence, we observe two pieces. The first equals the expected contribution from* $r$ *zeros at the critical point (agreeing with what B-SD suggests). The second is*

$$\begin{aligned} D_{1,\mathcal{F}}^{(r)}(f_1) &= \widehat{f_1}(0) + \frac{1}{2}f_1(0) \\ D_{2,\mathcal{F}}^{(r)}(f) &= \prod_{i=1}^{2}\left[\widehat{f_i}(0) + \frac{1}{2}f_i(0)\right] + 2\int_{-\infty}^{\infty}|u|\widehat{f_1}(u)\widehat{f_2}(u)du \\ &\quad -2\widehat{f_1 f_2}(0) - f_1(0)f_2(0) + (f_1 f_2)(0)N(\mathcal{F},-1) \end{aligned}$$

*where* $N(\mathcal{F},-1)$ *is the percent of curves with odd sign.*

$1$ and $2$-level densities confirm Katz-Sarnak predictions for small support.

# Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free: all even.
2. $y^2 = x^3 \pm 4(4t+2)x$, $4t+2$ Sq-Free: $+$ yields all odd, $-$ yields all even.
3. $y^2 = x^3 + tx^2 - (t+3)x + 1$, $t^2+3t+9$ Sq-Free: all odd.

First two rank $0$ over $\mathbb{Q}(t)$; third is rank $1$. **Only** assume GRH for first two; **add** B-SD to interpret third.

Family of Rank $6$ over $\mathbb{Q}(t)$ (modulo reasonable conjs):

$$\begin{aligned} y^2 \;=\;& x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ & + (2ct - D)(t^2 + 2t - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8916100448256000000 \\ B &= -811365140824616222208 \\ C &= 26497490347321493520384 \\ D &= -343107594345448813363200 \\ a &= 16660111104 \\ b &= -1603174809600 \\ c &= 2149908480000 \end{aligned}$$

# Mestre's Construction

Mestre ([Mes1], [Mes2]): families of rank $11$ and $12$ over $\mathbb{Q}(t)$; within such a family, Nagao [Na1] finds a curve with rank $\geq 21$.

Consider $6$-tuple of integers $a_i$.

Let $q(x) = \prod_{i=1}^{6}(x - a_i)$, $p(t, x) = q(x - t)q(x + t)$.

$\exists g(t, x)$ of degree $6$ in $x$ and $r(t, x)$ of degree at most $5$ in $x$ such that $p(t, x) = g^2(t, x) - r(t, x)$.

Consider $y^2 = r(t, x)$. If $r(t, x)$ is of degree $3$ or $4$ in $x$, we obtain an elliptic curve, with points $P_{\pm i}(t) = \Big(\pm t + a_i, g(\pm t + a_i)\Big)$.

If $r(t, x)$ has degree $4$, change variables to make the coefficient of $x^4$ a perfect square (see [Mor], [Na1]). Two $6$-tuples that work (see [Na1]) are $(-17, -16, 10, 11, 14, 17)$ and $(399, 380, 352, 47, 4, 0)$.

Get families with rank up to $14$ over $\mathbb{Q}(t)$.

Shioda [Sh] gives explicit constructions for rank $2, 4, 6, 7$ and $8$ over $\mathbb{Q}(t)$ and generators of the Mordell-Weil groups.

# Rosen-Silverman Theorem

$$A_{\mathcal{E}}(p) \;=\; \frac{1}{p}\sum_{t=0}^{p-1} a_t(p) \;=\; \frac{1}{p}A_{1,\mathcal{F}}(p).$$

The elliptic surface $y^2 = x^3 + A(t)x + B(t)$, $t \in \mathbb{Z}$, is rational if and only if one of the following holds

1. $0 < \max\{3\text{deg } A(t), 2\text{deg } B(t)\} < 12$.
2. $3\text{deg } A(t) = 2\text{deg } B(t) = 12$, $\text{ord}_{t=0} t^{12}\Delta(t^{-1}) = 0$

**Thm [R-S]:** *Let $\mathcal{E} : y^2 = x^3 + A(t)x + B(t)$, and assume Tate's conjecture (known for rational surfaces) for the surface. Then*

$$\lim_{X\to\infty} \frac{1}{X}\sum_{p\le X} -A_{\mathcal{E}}(p)\log p = \text{rank } \mathcal{E}(\mathbb{Q}(t)).$$

We construct $\mathcal{E}$ with $A_{\mathcal{E}}(p) = -r + O(\frac{1}{p})$. The PNT implies $r$ is the rank over $\mathbb{Q}(t)$.

# Quadratic Legendre Sums

**LEMMA: Quadratic Legendre Sums:** *Assume $a$ and $b$ are not both zero mod $p$ and $p > 2$. Then*

$$\sum_{t=0}^{p-1} \left(\frac{at^2+bt+c}{p}\right) = \begin{cases} (p-1)\left(\frac{a}{p}\right) \text{ if } p \mid b^2-4ac \\ -\left(\frac{a}{p}\right) \text{ otherwise} \end{cases}$$

Moral: We can handle linear and quadratic Legendre sums; for cubic best is Hasse, giving $O(\sqrt{p})$.

# Rank $6$ Rational Surfaces over $\mathbb{Q}(t)$

$$\begin{aligned} y^2 = f_t(x) &= x^3t^2 + 2g(x)t - h(x) \\ g(x) &= x^3 + ax^2 + bx + c,\ c \neq 0 \\ h(x) &= (A-1)x^3 + Bx^2 + Cx + D \\ D_t(x) &= g(x)^2 + x^3h(x). \end{aligned}$$

$D_t(x)$ is one-fourth the discriminant of the quadratic (in $t$) polynomial $x^3t^2 + 2g(x)t - h(x)$.

The number of distinct, non-zero roots of $D_t(x)$ will control the rank.

$$a_t(p) = -\sum_{x(p)} \left(\frac{f_t(x)}{p}\right) = -\sum_{x(p)} \left(\frac{x^3t^2 + 2g(x)t - h(x)}{p}\right).$$

Study $-A_{1,\mathcal{F}}(p) = \sum_{x=0}^{p-1} \sum_{t=0}^{p-1} \left(\frac{f_t(x)}{p}\right)$.

$x = 0$, the $t$-sum vanishes if $c \neq 0$, as it is just $\sum_{t=0}^{p-1} \left(\frac{2ct-D}{p}\right)$.

# Rank $6$: II

Studying $\displaystyle\sum_{t(p)} \sum_{x \not\equiv 0(p)} \left(\frac{x^3t^2 + 2g(x)t - h(x)}{p}\right)$.

For $x \not\equiv 0$, have

$$\sum_{t=0}^{p-1} \left(\frac{x^3t^2 + 2g(x)t - h(x)}{p}\right) = \begin{cases} (p-1)\left(\frac{x^3}{p}\right) & \text{if } p \mid D_t(x) \\ -1\left(\frac{x^3}{p}\right) & \text{otherwise} \end{cases}$$

Find coefficients $a, b, c, A, B, C, D$ with $D_t(x)$ six distinct, non-zero perfect square roots.

Gives $6(p-1) - 1(-6) + 0 = 6p$.

# Rank $6$: III

For $1 \leq i \leq 6$, let $r_i = \rho_i^2$.

$$\begin{aligned}
D_t(x) &= g(x)^2 + x^3 h(x) \\
&= (x^3 + ax^2 + bx + c)^2 + x^3\big((A-1)x^3 + Bx^2 + Cx + D\big) \\
&= Ax^6 + (B+2a)x^5 + (C + a^2 + 2b)x^4 + (D + 2ab + 2c)x^3 \\
&\quad + (2ac + b^2)x^2 + (2bc)x + c^2 \\
&= A\Big(x^6 + \frac{B+2a}{A}x^5 + \frac{C+a^2+2b}{A}x^4 + \frac{D+2ab+2c}{A}x^3 \\
&\quad + \frac{2ac+b^2}{A}x^2 + \frac{2bc}{A}x + \frac{c^2}{A}\Big) \\
&= A(x^6 + R_5x^5 + R_4x^4 + R_3x^3 + R_2x^2 + R_1x + R_0) \\
&= A(x-r_1)(x-r_2)(x-r_3)(x-r_4)(x-r_5)(x-r_6) \\
&= A\Phi(x).
\end{aligned}$$

# Determining Constants $a, \ldots, D$

Can matching the $x^5, x^4, x^3$ terms ($B$, $C$, $D$).

We must simultaneously solve

$$\begin{aligned} 2ac + b^2 &= R_2 A \\ 2bc &= R_1 A \\ c^2 &= R_0 A. \end{aligned} \tag{0.0.1}$$

Send $A \to Aw^2$, rescaling $b$ and $c$ by $w$.

Have $2ac + b^2 w = R_2 Aw$. Taking $w = 2c$ simplifies to $a + b^2 = R_2 A$.

Take $A = 4R_0$. Then $c^2 = 4R_0^2$, or $c = 2R_0$.

$2bc = R_1 A$ yields $b = R_1$.

We now send $A \to Aw^2 = (4R_0) \cdot (2c)^2 = 64R_0^3$.

# Determining Constants $a, \ldots, D$: II

$$\begin{aligned}
c^2 &= 64R_0^4 &\to\ c &= 8R_0^2\\
2bc &= 64R_0^3R_1 &\to\ b &= 4R_0R_1\\
2ac+b^2 &= 64R_0^3R_2 &\to\ a &= 4R_0R_2 - R_1^2.
\end{aligned}$$

Take $r_i = \rho_i^2 = i^2$. Then

$$\begin{aligned}
A\Phi(x) &= A(x-1)(x-4)(x-9)(x-16)(x-25)(x-36)\\
&= A\Big(x^6 - 91x^5 + 3003x^4 - 44473x^3\\
&\quad +296296x^2 - 773136x + 518400\Big)\\
&= A\Big(x^6 + \frac{B+2a}{A}x^5 + \frac{C+a^2+2b}{A}x^4 + \frac{D+2ab+2c}{A}x^3\\
&\quad + \frac{2ac+b^2}{A}x^2 + \frac{2bc}{A}x + \frac{c^2}{A}\Big)\\
&= A(x^6 + R_5x^5 + R_4x^4 + R_3x^3 + R_2x^2 + R_1x + R_0).
\end{aligned}$$

We have

$$R_0 = 518400,\ R_1 = -773136,\ R_2 = 296296.$$

Hence

$$\begin{aligned}
A &= 64R_0^3 &= 8916100448256000000\\
c &= 8R_0^2 &= 2149908480000\\
b &= 4R_0R_1 &= -1603174809600\\
a &= 4R_0R_2 - R_1^2 &= 16660111104
\end{aligned}$$

# Determining Constants $a, \ldots, D$: III

$$\begin{aligned} B &= R_5A - 2a &= -8113651408246162222208 \\ C &= R_4A - a^2 - 2b &= 26497490347321493520384 \\ D &= R_3A - 2ab - 2c &= -34310759434544881336320 0 \end{aligned}$$

We convert $y^2 = f_t(x)$ to $y^2 = F_t(x)$ in Weierstrass normal form.

$y \to \frac{y}{t^2+2t-A+1}$, $x \to \frac{x}{t^2+2t-A+1}$, multiply through by $(t^2 + 2t - A + 1)^2$.

$$\begin{aligned} f_t(x) &= (t^2 + 2t - A + 1)x^3 + (2at - B)x^2 \\ &\quad + (2bt - C)x + (2ct - D) \end{aligned}$$

$$\begin{aligned} F_t(x) &= x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ &\quad +(2ct - D)(t^2 + 2t - A + 1)^2. \end{aligned}$$

Except for finitely many primes, $F_t$ will give same sums (modulo lower order terms).

Is a rational surface; Rosen-Silverman Theorem is applicable.

# Future Work

By using cubics, quartics, and higher degrees in $t$, can we force sums to be large?

Will not be possible (in general) to evaluate most $\sum_{t(p)}$ for fixed $x$.

Might hope to get a large contribution from $x$ where the discriminant (in $t$) vanishes, and hope everything else "cancels".

# $n$-Level Density and Excess Rank Bounds

For $n = 1$ and $2$, consider the test functions

$$\begin{aligned} \widehat{f}_i(u) &= \frac{1}{2}\Big(\frac{1}{2}\sigma_n - \frac{1}{2}|u|\Big),\ |u| \le \sigma \\ f_i(x) &= \frac{\sin^2(2\pi\frac{1}{2}\sigma_n x)}{(2\pi x)^2}. \end{aligned}$$

Expect $\sigma_2 = \frac{\sigma_1}{2}$; only able to prove for $\sigma_2 = \frac{\sigma_1}{4}$.

Note $f_i(0) = \frac{\sigma_n^2}{4}$, $\widehat{f}_i(0) = f_i(0)\frac{1}{\sigma_n}$.

Assume B-SD, Equidistribution of Sign

# Notation

Family with rank $r$, $D_{1,\mathcal{F}}(f) = \widehat{f}(0) + \frac{1}{2}f(0) + rf(0)$.

By even (odd) we mean a curve whose rank $r_E$ has $r_E - r$ even (odd).

$P_0$: probability even curve has rank $\geq r + 2a_0$.

$P_1$: probability odd curve has rank $\geq r + 1 + 2b_0$.

$$D_{1,\mathcal{F}}(f) = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{\gamma_E} f\left(\frac{\log N_E}{2\pi}\gamma_E\right),$$

$\gamma_E$ is the imaginary part of the zeros.

# Average Rank: $1$-Level Bounds

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E f(0) \;\le\; \widehat{f_1}(0) + \frac{1}{2}f_1(0) + r f_1(0)$$

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E \;\le\; \frac{1}{\sigma_1} + \frac{1}{2} + r.$$

- All Curves: $r = 0$, $\sigma = \frac{4}{7}$, giving $2.25$ (Brumer, Heath-Brown: [Br], [BHB3], [BHB5])
- $1$-Parameter Families: $\left(\deg(N(t)) + r + \frac{1}{2}\right)\cdot(1 + o(1))$ (Silverman [Si3]).

Hope $1$-Level Density true for $\sigma \to \infty$.

Would yield average rank is $r + \frac{1}{2}$.

# Excess Rank: $1$-Level Bounds

Assume half even, half odd.

Even curves: $1 - P_0$ have rank $\leq r + 2a_0 - 2$; replace ranks with $r$. $P_0$ have rank $\geq r+2a_0$; replace with $r+2a_0$.

Odd curves: $1 - P_1$ contributing $r + 1$. $P_1$ contributing $r + 1 + 2b_0$.

$$
\begin{aligned}
\frac{1}{\sigma_1} + \frac{1}{2} + r \;&\geq\; \frac{1}{2}\Big[(1-P_0)r + P_0(r+2a_0)\Big] \\
&\quad + \frac{1}{2}\Big[(1-P_1)(r+1) + P_1(r+1+2b_0)\Big] \\
\frac{1}{\sigma_1} \;&\geq\; a_0P_0 + b_0P_1.
\end{aligned}
$$

**$1$-Level Density Bounds for Excess Rank**

$$
\begin{aligned}
P_0 \;&\leq\; \frac{1}{a_0\sigma_1} \\
P_1 \;&\leq\; \frac{1}{b_0\sigma_1} \\
\text{Prob}\{\text{rank} \geq r + 2a_0\} \;&\leq\; \frac{1}{a_0\sigma_1}.
\end{aligned}
$$

# $2$-Level Bounds:

$$\begin{aligned}
D_{2,\mathcal{F}}(f) &= D^*_{2,\mathcal{F}}(f) - 2D_{1,\mathcal{F}}(f_1f_2) + f_1(0)f_2(0)N(\mathcal{F},-1) \\
D^*_{2,\mathcal{F}}(f) &= \prod_{i=1}^{2}\left[\widehat{f}_i(0) + \frac{1}{2}f_i(0)\right] + 2\int |u|\widehat{f}_1(u)\widehat{f}_2(u)du \\
&\quad + r\widehat{f}_1(0)f_2(0) + rf_1(0)\widehat{f}_2(0) + (r^2+r)f_1(0)f_2(0) \\
D_{1,\mathcal{F}}(f) &= \widehat{f}(0) + \frac{1}{2}f(0) + rf(0).
\end{aligned}$$

$D^*_{2,\mathcal{F}}(f)$ is over all zeros. Gives

$$\begin{aligned}
\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}} r_E^2 &\leq \frac{1}{\sigma_2^2} + \frac{1}{\sigma_2} + \frac{1}{4} + \frac{1}{3} + \frac{2r}{\sigma_2} + r^2 + r \\
&= \frac{1}{\sigma_2^2} + \frac{2r+1}{\sigma_2} + \frac{1}{12} + r^2 + r + \frac{1}{2}.
\end{aligned}$$

# Excess Rank: $2$-Level Bounds: I

Similar proof yields

**Theorem: First $2$-Level Density Bounds**

$$P_0 \ \leq \ \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r+\frac{1}{2}}{\sigma_2}}{a_0(a_0+r)}$$

$$P_1 \ \leq \ \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r+\frac{1}{2}}{\sigma_2}}{b_0(b_0+r+1)}.$$

For $\sigma_2 = \frac{\sigma_1}{4}$, $r = 0$, $a_1 = 1$: ***worse*** than $1$-level density.

For fixed $\sigma_2 = \frac{\sigma_1}{4}$ and $r$, as we increase $a_0$ we eventually do get a better bound.

Proportional to $\frac{1}{(a_0\sigma_1)^2}$ instead of $\frac{1}{a_0\sigma_1}$.

# Excess Rank: $2$-Level Bounds: II

Use $D_{2,\mathcal{F}}(f)$ instead of $D^*_{2,\mathcal{F}}(f)$.

$r_E$ = number of zeros of curve $E$. Sum over $j_1 \neq j_2$.

$r_E$ even, get $r_E(r_E-2)$ (each zero matched with $r_E-2$ others).

$r_E$ odd: $(r_E-1)(r_E-2)+(r_E-1) = r_E(r_E-2)+1$.

**Theorem: Second $2$-Level Density Bounds**

$$P_0 \ \leq \ \frac{\frac{1}{2\sigma_2^2}+\frac{1}{24}+\frac{r}{\sigma_2}-\frac{1}{6\sigma_2}}{a_0(a_0+r-1)}$$
$$P_1 \ \leq \ \frac{\frac{1}{2\sigma_2^2}+\frac{1}{24}+\frac{r}{\sigma_2}-\frac{1}{6\sigma_2}}{b_0(b_0+r)},$$

where $a_0 \neq 1$ if $r=0$.

$\sigma_2 = \frac{\sigma_1}{4}$ and $r=0$, better for $a_0 > \frac{\sigma_1^2+8\sigma_1+192}{24\sigma_1}$.
$r=1$, better for $a_0 > \frac{\sigma_1^2+80\sigma_1+192}{24\sigma_1}$.

Decay is proportional to $\frac{1}{(a_0\sigma_1)^2}$.
Note the numerator is never negative; at least $\frac{1}{18}$.

# Excess Rank: $2$-Level Bounds: IIIa

$r_E \;=\; r + z_E.$

$\sum_{j_1}\sum_{j_2} f_1(L\gamma_{E_{j_1}}) f_2(L\gamma_{E_{j_2}})$. Let $j_1$ be one of the $r$ family zeros, varying $j_2$ gives $f_1(0)D_{1,E}(f_2)$. Interchanging $j_1$ and $j_2$ we get a contribution of $D_{1,E}(f_1)f_2(0)$ for each of the $r$ family.

Only double counting when $j_1$ and $j_2$ are both a family zero. Subtract off $r^2 f_1(0)f_2(0)$.

For the other $z_E$ zeros: already taken into account contribution from $j_1$ one of the $z_E$ zeros and $j_2$ one of the $r$ family zeros (and vice-versa).

Thus, for a given curve, a lower bound of the contribution from all pairs $(j_1, j_2)$ is

$$r f_1(0)D_{1,E}(f_2) + rD_{1,E}(f_1)f_2(0) - r^2 f_1(0)f_2(0) + z_E^2.$$

# Excess Rank: $2$-Level Bounds: IIIb

Summing over all $E \in \mathcal{F}$ and simplifying gives

$$\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} z_E^2 \ \leq \ \frac{1}{\sigma_2^2} + \frac{1}{\sigma_2} + \frac{1}{12} + \frac{1}{2}.$$

Similar calculation gives

**Theorem: Third $2$-Level Density Bounds**

$$P_0 \ \leq \ \frac{\frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_2} + \frac{1}{24}}{a_0^2}$$

$$P_1 \ \leq \ \frac{\frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_2} + \frac{1}{24}}{b_0 + b_0^2}$$

$\sigma_2 = \frac{\sigma_1}{4}$: beats $1$-level for $a_0 > \frac{\sigma_1^2 + 48\sigma_1 + 192}{24\sigma_1}$.
$r \neq 0$: beats first $2$-level once $a_0 > \frac{\sigma_1^2 + 48\sigma_1 + 192}{96\sigma_1}$.

$r \geq 1$: beats second $2$-level once $a_0 > \frac{3(r-1)}{3r-2}\frac{\sigma_1^2 + 48\sigma_1 + 192}{96\sigma_1}$.

# Heath-Brown & Brumer

Family of all elliptic curves $E_{a,b}$:

$$\mathcal{F}_T \;=\; \{y^2 = x^3 + ax + b; |a| \le T^{\frac{1}{3}}, |b| \le T^{\frac{1}{2}}.$$

From 1-Level Expansion, get

$$r(E_{a,b}) \;\le\; 2+\frac{\log T}{\log X}-2\sum_{p\le X} a_P(E_{a,b}) h\left(\frac{\log p}{\log X}\right)+O\left(\frac{1}{\log X}\right).$$

If $r(E_{a,b}) \ge r \ge 3 + 2\frac{\log T}{\log X}$, then $|U(E_{a,b},X)| \ge \frac{\log T}{2}$.

Led to

$$\#\{E_{a,b} \in \mathcal{F}_T : r(E_{a,b}) \ge r\}\cdot\left(\frac{\log T}{2}\right)^{2k} \;\le\; \sum_{E_{a,b}\in\mathcal{F}} |U(E_{a,b},X)|^{2k}.$$

Find $X = T^{\frac{1}{10k}}$, $k = \left[\frac{r-3}{20}\right]$. Yields

$$\begin{aligned}\text{Prob}\left(\text{rank}(E_{a,b}) \ge r\right) &\ll (11r)^{-\frac{r}{20}}\\ \text{rank}(E_{a,b}) &\le 17\frac{\log T}{\log\log T}.\end{aligned}$$

# Numerically Approximating Ranks: Preliminaries

Cusp form $f$, level $N$, weight $2$:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \quad \epsilon = \pm 1.$$

To each $E$ corresponds an $f$, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

# Algorithm for $L^r(s, E)$: I

$$
\begin{aligned}
\Lambda(E, s) &= \int_0^\infty f(iy/\sqrt{N}) y^{s-1} dy \\
&= \int_0^1 f(iy/\sqrt{N}) y^{s-1} dy + \int_1^\infty f(iy/\sqrt{N}) y^{s-1} dy \\
&= \int_1^\infty f(iy/\sqrt{N}) (y^{s-1} + \epsilon y^{1-s}) dy.
\end{aligned}
$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E, s) = \int_1^\infty f(iy/\sqrt{N}) (\log y)^k (y^{s-1} + \epsilon(-1)^k y^{1-s}) dy.$$

At $s = 1$,

$$\Lambda^{(k)}(E, 1) = (1 + \epsilon(-1)^k) \int_1^\infty f(iy/\sqrt{N}) (\log\ y)^k dy.$$

Trivially zero for half of $k$; let $r$ be analytic rank.

# Algorithm for $L^r(s, E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy \\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

# Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-r}}{n^r\cdot n!}x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t;\\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12};\\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3};\\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160};\\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\,e^{-2\pi ny/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

# Potential Lower Order Density Corrections

$$D_{1,\mathcal{F}}(f) = \widehat{f}(0) + f(0) - 2\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p}\widehat{f}\Big(\frac{\log p}{\log N_E}\Big)a_t(p)$$
$$-2\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{f}\Big(2\frac{\log p}{\log N_E}\Big)a_t(p)^2.$$

$j(t)$ non-const $\Rightarrow$ $A_{2,\mathcal{F}}(p) = \sum_{t(p)} a_t(p)^2 = p^2 + O(p^{\frac{3}{2}})$.

The main term of $A_{2,\mathcal{F}}(p)$ contributes $-\frac{1}{2}f(0)$ in the limit. For a great many families, we can do better than Michel's $O(p^{\frac{3}{2}})$.

Have errors of size $\frac{1}{\log N}$ and $\frac{\log\log N}{\log N}$ from the Explicit Formula.

The conductor dependence in the Gamma factors of the Explicit Formula is easily managed. The real difficulty is handling the primes which divide the discriminant and the sums of $a_E^m(p)$, $m \geq 3$.

For convenience, rescale each curve's zeros by the average log-conductor.

# Observed $A_{2,\mathcal{F}}(p)$ Corrections

| Family: $y^2 =$ | $A_{1,\mathcal{F}}(p)$ | $A_{2,\mathcal{F}}(p)$ |
|---|---|---|
| All Curves | $0$ | $p^3 - p^2$ |
| $x^3 + 2^4(-3)^3(9t+1)^2$ | $0$ | $\begin{cases} 2p^2-2p & p\equiv 2(3) \\ 0 & p\equiv 1(3) \end{cases}$ |
| $x^3 \pm 4(4t+2)x$ | $0$ | $\begin{cases} 2p^2-2p & p\equiv 1(4) \\ 0 & p\equiv 3(3) \end{cases}$ |
| $x^3 + (t+1)x^2 + tx$ | $0$ | $p^2 - 2p - 1$ |
| $x^3 + x^2 + 2t + 1$ | $0$ | $p^2 - 2p - \left(\frac{-3}{p}\right)$ |
| $x^3 + tx^2 + 1$ | $-p$ | $p^2 - c_1(p)p$ |
| $x^3 - t^2x + t^2$ | $-2p$ | $p^2 - p - c_2(p)p - c_3(p)$ |
| $x^3 - t^2x + t^4$ | $-2p$ | $p^2 - p - c_2(p)p - c_3(p)$ |

$$\begin{aligned} c_1(p) &= \begin{cases} 3p+1 & p \equiv 2(3) \\ 3ph_{3,p}(2) + 1 - p\sum_{x(p)}\left(\frac{4x^3+1}{p}\right) & p \equiv 1(3) \end{cases} \\ c_2(p) &= \left[\sum_{x(p)}\left(\frac{x^3-x}{p}\right)\right]^2 \\ c_3(p) &= \left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right), \end{aligned}$$

and $h_{3,p}(2)$ is one if 2 is a cube mod $p$ and zero otherwise.

## Contributions from $A_{2,\mathcal{F}}(p)$ Corrections

Correction from $-m_{\mathcal{F}}p$, $m_{\mathcal{F}} > 0$. Have $\frac{N}{p}$ full sums and one partial sum. Substituting yields

$$\begin{aligned} C_2 &= \frac{2}{N}\sum_p \frac{\log p}{\log M}\widehat{f}\Big(2\frac{\log p}{\log M}\Big)\frac{1}{p^2}\frac{N}{p}m_{\mathcal{F}}p \\ &= \frac{2m_{\mathcal{F}}}{\log M}\sum_p \widehat{f}\Big(2\frac{\log p}{\log M}\Big)\frac{\log p}{p^2}. \end{aligned}$$

Correction of size $\frac{1}{\log N}$. Consider

$$\begin{aligned} \widehat{f}(u) &= \frac{1}{2}\left(\frac{1}{2}\sigma - \frac{1}{2}|u|\right),\ |u| \leq \sigma \\ f(x) &= \frac{\sin^2(2\pi\frac{1}{2}\sigma x)}{(2\pi x)^2}. \end{aligned}$$

Gives

$$\text{Ave Rank} \ \leq \ \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log M}\right)\frac{m_{\mathcal{F}}}{\log M}.$$

## Observed $A_{2,\mathcal{F}}(p)$ Corrections

Family of all elliptic curves, $\sigma = \frac{4}{7}$. Ignoring the correction term, AveRank $\leq 2.25$. Correction is $.05$.

## Extra Contributions: $m_{\mathcal{F}} = 1$

| $M$ | $\sigma = \frac{4}{7}$ | $\sigma = 1$ | $\sigma = 2$ | $\sigma = \infty$ |
|---|---|---|---|---|
| $10^6$ | $.077$ | $.056$ | $.032$ | $0$ |
| $10^{12}$ | $.051$ | $.032$ | $.017$ | $0$ |
| $10^{18}$ | $.036$ | $.022$ | $.011$ | $0$ |
| $10^{24}$ | $.025$ | $.015$ | $.007$ | $0$ |

Fermigier observes average rank $\approx (r + \frac{1}{2}) + .40$.

Cond $\approx 10^{12}$: $\sigma = 1$: AveRank $\leq (r + \frac{1}{2}) + 1 + .03$.

Cond $\approx 10^{12}$: $\sigma = 2$: AveRank $\leq (r + \frac{1}{2}) + \frac{1}{2} + .02$.

Hopeless to get support $\geq 2$. ILS obtain such large support for some of their families, but only because of great averaging formulas. No good analogue, and our conductors grow very quickly.

# All Curves:

Complete sums of $a_{a,b}^m(p)$ vanish for $m$ odd, and

$$\begin{aligned}
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} a_{a,b}^4(p) &= 3p - 3p^2 - 2p^3 + 2p^4 \\
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} a_{a,b}^6(p) &= 5p + 4p^2 - 9p^3 - 5p^4 + 5p^5 \\
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} a_{a,b}^8(p) &= 7p + 13p^2 + 8p^3 - 28p^4 - 14p^5 + 14p^6.
\end{aligned}$$

Explicit Formula gives sums of $\alpha_E^m(p) + \beta_E^m(p)$, $m \geq 3$.

For $p \nmid N_E$, $\alpha_E(p) + \beta_E(p) = a_E(p)$ and $\alpha_E(p)\beta_E(p) = p$. For $p \nmid N_E$,

$$\begin{aligned}
\alpha_E^3(p) + \beta_E^3(p) &= a_E^3(p) - 3pa_E(p) \\
\alpha_E^4(p) + \beta_E^4(p) &= a_E^4(p) - 4pa_E^2(p) + 2p^2 \\
\alpha_E^5(p) + \beta_E^5(p) &= a_E^5(p) - 5p(\alpha_E^3(p) - \beta_E^3(p)) + 10p^2 a_E(p) \\
\alpha_E^6(p) + \beta_E^6(p) &= a_E^6(p) - 6pa_E^4(p) + 9p^2 a_E^2(p) - 2p^3 \\
\alpha_E^7(p) + \beta_E^7(p) &= a_E^7(p) - 7p(\alpha_E^5(p) + \beta_E^5(p)) - 21p^2(\alpha_E^3(p) + \beta_E^3(p)) - 35p^3 a_E \\
\alpha_E^8(p) + \beta_E^8(p) &= a_E^8(p) - 8pa_E^6(p) + 20p^2 a_E^4(p) - 16p^3 a_E^2(p) + 2p^4.
\end{aligned}$$

Have $\frac{1}{N^5}\frac{N^2}{p}\frac{N^3}{p}$ complete sums, multiply by $\frac{1}{p^{2m}}$ (and other factors) and sum over the primes.

The main contribution to the potential density correction is from $m = 2$. Complete sum of $a_E^2(p)$ gives $p^3 - p^2$, with the $p^2$ term leading to a sum of $\frac{1}{p^2}$.

# All Curves (cont)

$$\begin{aligned}
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} a_{a,b}^4(p) &= 3p - 3p^2 - 2p^3 + 2p^4 \\
\alpha_E^4(p) + \beta_E^4(p) &= a_E^4(p) - 4pa_E^2(p) + 2p^2
\end{aligned}$$

$$\begin{aligned}
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} a_{a,b}^6(p) &= 5p + 4p^2 - 9p^3 - 5p^4 + 5p^5 \\
\alpha_E^6(p) + \beta_E^6(p) &= a_E^6(p) - 6pa_E^4(p) + 9p^2a_E^2(p) - 2p^3
\end{aligned}$$

$$\begin{aligned}
\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} a_{a,b}^8(p) &= 7p + 13p^2 + 8p^3 - 28p^4 - 14p^5 + 14p^6. \\
\alpha_E^8(p) + \beta_E^8(p) &= a_E^8(p) - 8pa_E^6(p) + 20p^2a_E^4(p) - 16p^3a_E^2(p) + 2p^4.
\end{aligned}$$

When $m = 4$, potential term of size $\frac{1}{p^2}$, as $\frac{1}{p^{2\cdot 2}}$ exactly balances $p^{2+2}$. Note, the test function will be evaluated at $\frac{3\log p}{\log M}$ and not $\frac{2\log p}{\log M}$.

The $p^4$ terms (which yield the main term) exactly cancel. Thus, the $m = 4$ term contributes a sum of size $\frac{1}{p^3}$ and not of size $\frac{1}{p^2}$.

For $m \geq 6$, the contributions will be of size $\frac{1}{p^3}$ or less.

Hence, subject to proving the expansion formulas, for the family of all elliptic curves (no sieving, rescaling by the average log-conductor), the $m \geq 1$ terms contribute a potential lower order correction, where we sum over the primes terms of size $\frac{1}{p^2}$.

# Excess Rank Calculations
# Families with $y^2 = f_t(x)$; $D(t)$ SqFree

| Family | $t$ Range | Num $t$ | $r$ | $r$ | $r+1$ | $r+2$ | $r+3$ |
|---|---|---|---|---|---|---|---|
| $+4(4t+2)$ | $[2,2002]$ | 1622 | 0 | 95.44 | | 4.56 | |
| $-4(4t+2)$ | $[2,2002]$ | 1622 | 0 | 70.53 | | 29.35 | |
| $9t+1$ | $[2,247]$ | 169 | 0 | 71.01 | | 28.99 | |
| $t^2+9t+1$ | $[2,272]$ | 169 | 1 | 71.60 | | 27.81 | |
| $t(t-1)$ | $[2,2002]$ | 643 | 0 | 40.44 | 48.68 | 10.26 | 0.62 |
| $(6t+1)x^2$ | $[2,101]$ | 93 | 1 | 34.41 | 47.31 | 17.20 | 1.08 |
| $(6t+1)x$ | $[2,77]$ | 66 | 2 | 30.30 | 50.00 | 16.67 | 3.03 |

1. $x^3 + 4(4t+2)x$, $4t+2$ Sq-Free, odd.
2. $x^3 - 4(4t+2)x$, $4t+2$ Sq-Free, even.
3. $x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free, even.
4. $x^3 + tx^2 - (t+3)x + 1$, $t^2+3t+9$ Sq-Free, odd.
5. $x^3 + (t+1)x^2 + tx$, $t(t-1)$ Sq-Free, rank $0$.
6. $x^3 + (6t+1)x^2 + 1$, $4(6t+1)^3 + 27$ Sq-Free, rank $1$.
7. $x^3 - (6t+1)^2x + (6t+1)^2$, $(6t+1)[4(6t+1)^2 - 27]$ Sq-Free, rank $2$.

# Excess Rank Calculations
# Families with $y^2 = f_t(x)$; All $D(t)$

| Family | $t$ Range | Num $t$ | $r$ | $r$ | $r+1$ | $r+2$ | $r+3$ |
|---|---|---|---|---|---|---|---|
| $+4(4t+2)$ | $[2,2002]$ | 2001 | 0 | 6.45 | 85.76 | 3.95 | 3.85 |
| $-4(4t+2)$ | $[2,2002]$ | 2001 | 0 | 63.52 | 9.90 | 25.99 | .50 |
| $9t+1$ | $[2,247]$ | 247 | 0 | 55.28 | 23.98 | 20.73 | |
| $t^2+9t+1$ | $[2,272]$ | 271 | 1 | 73.80 | | 25.83 | |
| $t(t-1)$ | $[2,2002]$ | 2001 | 0 | 42.03 | 48.43 | 9.25 | 0.30 |
| $(6t+1)x^2$ | $[2,101]$ | 100 | 1 | 32.00 | 50.00 | 17.00 | 1.00 |
| $(6t+1)x$ | $[2,77]$ | 76 | 2 | 32.89 | 50.00 | 14.47 | 2.63 |

1. $x^3 + 4(4t+2)x$, $4t+2$ Sq-Free, odd.
2. $x^3 - 4(4t+2)x$, $4t+2$ Sq-Free, even.
3. $x^3 + 2^4(-3)^3(9t+1)^2$, $9t+1$ Sq-Free, even.
4. $x^3 + tx^2 - (t+3)x + 1$, $t^2+3t+9$ Sq-Free, odd.
5. $x^3 + (t+1)x^2 + tx$, $t(t-1)$ Sq-Free, rank $0$.
6. $x^3 + (6t+1)x^2 + 1$, $4(6t+1)^3 + 27$ Sq-Free, rank $1$.
7. $x^3 - (6t+1)^2x + (6t+1)^2$, $(6t+1)[4(6t+1)^2 - 27]$ Sq-Free, rank $2$.

# Appendix I: Standard Conjectures

**Generalized Riemann Hypothesis (for Elliptic Curves)** *Let $L(s, E)$ be the (normalized) $L$-function of the elliptic curve $E$. Then the non-trivial zeros of $L(s, E)$ satisfy Re$(s) = \frac{1}{2}$.*

**Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2]** *Let $E$ be an elliptic curve of geometric rank $r$ over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$, $T$ is the subset of torsion points). Then the analytic rank (the order of vanishing of the $L$-function at the critical point) is also $r$.*

**Tate's Conjecture for Elliptic Surfaces [Ta]** *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the $L$-series attached to $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to **C** and satisfies $-$ord$_{s=2}L_2(\mathcal{E}, s) =$ rank $NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line Re$(s) = 2$.*

Most of the one-parameter families we investigate are rational surfaces, where Tate's conjecture is known. See, for example, [RSi].

# Appendix II: Equidistribution of Signs

**ABC Conjecture** *Fix* $\epsilon > 0$*. For co-prime positive integers* $a$*,* $b$ *and* $c$ *with* $c = a + b$ *and* $N(a, b, c) = \prod_{p|abc} p$*,* $c \ll_\epsilon N(a, b, c)^{1+\epsilon}$*.*

The full strength of ABC is never needed; rather, we need a consequence of ABC, the Square-Free Sieve (see [Gr]):

**Square-Free Sieve Conjecture** *Fix an irreducible polynomial* $f(t)$ *of degree at least* $4$*. As* $N \to \infty$*, the number of* $t \in [N, 2N]$ *with* $f(t)$ *divisible by* $p^2$ *for some* $p > \log N$ *is* $o(N)$*.*

For irreducible polynomials of degree at most $3$, the above is known, complete with a better error than $o(N)$ ([Ho], chapter $4$).

**Restricted Sign Conjecture (for the Family** $\mathcal{F}$**)** *Consider a one-parameter family* $\mathcal{F}$ *of elliptic curves. As* $N \to \infty$*, the signs of the curves* $E_t$ *are equidistributed for* $t \in [N, 2N]$*.*

The Restricted Sign conjecture often fails. First, there are families with constant $j(E_t)$ where all curves have the same sign. Helfgott [He] has recently related the Restricted Sign conjecture to the Square-Free Sieve conjecture and standard conjectures on sums of Moebius:

**Polynomial Moebius** *Let* $f(t)$ *be a non-constant polynomial such that no fixed square divides* $f(t)$ *for all* $t$*. Then* $\sum_{t=N}^{2N} \mu(f(t)) = o(N)$*.*

The Polynomial Moebius conjecture is known for linear $f(t)$.

Helfgott shows the Square-Free Sieve and Polynomial Moebius imply the Restricted Sign conjecture for many families. More precisely, let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$.

**Theorem: Equidistribution of Sign in a Family [He]:** *Let* $\mathcal{F}$ *be a one-parameter family with* $a_i(t) \in \mathbb{Z}[t]$*. If* $j(E_t)$ *and* $M(t)$ *are non-constant, then the signs of* $E_t$*,* $t \in [N, 2N]$*, are equidistributed as* $N \to \infty$*. Further, if we restrict to good* $t$*,* $t \in [N, 2N]$ *such that* $D(t)$ *is good (usually square-free), the signs are still equidistributed in the limit.*

# Appendix III: Quadratic Legendre Sums

**Lemma:** *For* $p > 2$*,*

$$S(n) = \sum_{x=0}^{p-1} \left(\frac{n_1 + x}{p}\right)\left(\frac{n_2 + x}{p}\right) = \begin{cases} p - 1 & \text{if } p \mid n_1 - n_2 \\ -1 & \text{otherwise} \end{cases}$$

Proof: Shifting $x$ by $-n_2$, we need only prove the lemma when $n_2 = 0$. Assume $(n, p) = 1$ as otherwise the result is trivial. For $(a, p) = 1$ we have:

$$\begin{aligned} S(n) &= \sum_{x=0}^{p-1} \left(\frac{n + x}{p}\right)\left(\frac{x}{p}\right) \\ &= \sum_{x=0}^{p-1} \left(\frac{n + a^{-1}x}{p}\right)\left(\frac{a^{-1}x}{p}\right) \\ &= \sum_{x=0}^{p-1} \left(\frac{an + x}{p}\right)\left(\frac{x}{p}\right) = S(an) \end{aligned} \tag{0.0.2}$$

Hence

$$\begin{aligned} S(n) &= \frac{1}{p-1} \sum_{a=1}^{p-1} \sum_{x=0}^{p-1} \left(\frac{an + x}{p}\right)\left(\frac{x}{p}\right) \\ &= \frac{1}{p-1} \sum_{a=0}^{p-1} \sum_{x=0}^{p-1} \left(\frac{an + x}{p}\right)\left(\frac{x}{p}\right) - \frac{1}{p-1} \sum_{x=0}^{p-1} \left(\frac{x}{p}\right)^2 \\ &= \frac{1}{p-1} \sum_{x=0}^{p-1} \left(\frac{x}{p}\right) \sum_{a=0}^{p-1} \left(\frac{an + x}{p}\right) - 1 \\ &= 0 - 1 = -1 \end{aligned} \tag{0.0.3}$$

Where do we use $p > 2$? We used $\sum_{a=0}^{p-1} \left(\frac{an+x}{p}\right) = 0$ for $(n, p) = 1$. This is true for all odd primes (as there are $\frac{p-1}{2}$ quadratic residues, $\frac{p-1}{2}$ non-residues, and $0$); for $p = 2$, there is one quadratic residue, no non-residues, and $0$.

**Lemma: Quadratic Legendre Sums:** *Assume $a$ and $b$ are not both zero mod $p$ and $p > 2$. Then*

$$\sum_{t=0}^{p-1}\left(\frac{at^2+bt+c}{p}\right) = \begin{cases} (p-1)\left(\frac{a}{p}\right) \; if \; p \mid b^2-4ac \\ -\left(\frac{a}{p}\right) \; otherwise \end{cases}$$

Proof: Assume $a \not\equiv 0(p)$ as otherwise the proof is trivial. Let $\delta = 4^{-1}(b^2 - 4ac)$. Then

$$\begin{aligned}
\sum_{t=0}^{p-1}\left(\frac{at^2+bt+c}{p}\right) &= \sum_{t=0}^{p-1}\left(\frac{a^{-1}}{p}\right)\left(\frac{a^2t^2+bat+ac}{p}\right) \qquad (0.0.4)\\
&= \sum_{t=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{t^2+bt+ac}{p}\right)\\
&= \sum_{t=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{t^2+bt+4^{-1}b^2+ac-4^{-1}b^2}{p}\right)\\
&= \sum_{t=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{(t+2^{-1}b)^2-4^{-1}(b^2-4ac)}{p}\right)\\
&= \sum_{t=0}^{p-1}\left(\frac{a}{p}\right)\left(\frac{t^2-\delta}{p}\right)\\
&= \left(\frac{a}{p}\right)\sum_{t=0}^{p-1}\left(\frac{t^2-\delta}{p}\right)
\end{aligned}$$

If $\delta \equiv 0(p)$ we get $p-1$. If $\delta \equiv \eta^2, \eta \neq 0$, then by the previous Lemma

$$\sum_{t=0}^{p-1}\left(\frac{t^2-\delta}{p}\right) = \sum_{t=0}^{p-1}\left(\frac{t-\eta}{p}\right)\left(\frac{t+\eta}{p}\right) = -1.$$

We note that $\sum_{t=0}^{p-1}\left(\frac{t^2-\delta}{p}\right)$ is the same for all non-square $\delta$'s (let $g$ be a generator of the multiplicative group, $\delta = g^{2k+1}$, change variables by $t \to g^k t$). Denote this sum by $S$, the set of non-zero squares mod $p$ by $\mathcal{R}$, and the non-squares mod $p$ by $\mathcal{N}$. Since $\sum_{\delta=0}^{p-1}\left(\frac{t^2-\delta}{p}\right) = 0$ we have

$$
\begin{aligned}
\sum_{\delta=0}^{p-1}\sum_{t=0}^{p-1}\left(\frac{t^2-\delta}{p}\right) &= \sum_{t=0}^{p-1}\left(\frac{t^2}{p}\right)+\sum_{\delta\in\mathcal{R}}\sum_{t=0}^{p-1}\left(\frac{t^2-\delta}{p}\right)+\sum_{\delta\in\mathcal{N}}\sum_{t=0}^{p-1}\left(\frac{t^2-\delta}{p}\right) \\
&= (p-1)+\frac{p-1}{2}(-1)+\frac{p-1}{2}S=0.
\end{aligned}
$$

Hence $S=-1$, proving the lemma.

# Appendix IV: Contributions from Sub-Families

A strong possible explanation for the observed excess rank is the presence of sub-families of higher rank. These families' contributions will be dwarfed in the limit, but noticeable for small $N$.

Consider an elliptic curve $E : y^2 = x^3 + ax + b$ over $\mathbb{Q}$ and its family of twists $E_d : dy^2 = x^3 + ax + b$ by square-free $d$. Assuming the Restricted Sign conjecture for twists, Gouvéa and Mazur [GM] prove that there exist positive constants $C_0(\epsilon, E)$ and $C_1(\epsilon, E)$ such that, if $N \geq C_0(\epsilon, E)$, the number of square-free $d \leq N$ with the rank of $E_d$ even and at least 2 is at least $C_1(\epsilon, E)N^{\frac{1}{2}-\epsilon}$.

Similarly, Mai [Mai] considered $E_d : x^3 + y^3 = d$ for cube-free $d$. Assuming the Restricted Sign conjecture for cubic twists, he proved there exist positive constants $C_2(\epsilon, E)$ and $C_3(\epsilon, E)$ such that, if $N \geq C_2(\epsilon, E)$, the number of cube-free $|d| \leq N$ with the rank of $E_d$ even and at least 2 is at least $C_3(\epsilon, E)N^{\frac{2}{3}-\epsilon}$.

Stewart and Top [ST] generalize these results and remove the dependence on the Restricted Sign conjecture, though at the cost of weaker bounds. They prove:

**Theorem 0.0.1 (Cubic Twists).** *For the family $x^3 + y^3 = d$, there exists a universal constant $C_4$ such that, for cube-free $|d| \leq N$, if $N \geq 657$, at least $C_4 N^{\frac{1}{6}}$ curves have rank at least 3.*

**Theorem 0.0.2 (Quadratic Twists).** *Let $E$ be an elliptic curve over $\mathbb{Q}$ with $j(E) \neq 0, 1728$, and let $E_d$ be a quadratic twist. There exist constants $C_5(E)$ and $C_6(E)$ such that, for square-free $|d| \leq N$, if $N \geq C_5(E)$, then at least $C_6(E)N^{\frac{1}{7}} \cdot \log^{-2} N$ have rank even and at least 2.*

In all of the above results, there is very slow decay with respect to $N$. The cardinality in these families is a multiple of $N$. Thus, we have contributions ranging from $N^{\frac{1}{7}}/N$ to $N^{\frac{1}{2}}/N$ to $N^{\frac{2}{3}}/N$. Taking $N = 100$, $1000$ and $10000$ yields

| $N$ | $N^{-\frac{6}{7}}$ | $N^{-\frac{1}{2}}$ | $N^{-\frac{2}{3}}$ |
|---|---|---|---|
| 100 | .0193 | .1000 | .2154 |
| 1000 | .0027 | .0316 | .1000 |
| 10000 | .0004 | .0100 | .0464 |

In Fermigier's [Fe2] investigations, $N$ ranges from $250$ to $1000$. Thus, it is very likely, depending on the size of the constants (and the true values of the exponents) that there may be higher rank sub-families of cardinality $N^c$ lurking within our families, $0 < c < 1$. While not contributing in the limit, they will be very noticeable for small values of $N$.

To determine the analytic rank (the order of vanishing of $L(s, E)$ at the critical point $s = 1$) requires studying sums of the coefficients $a_E(n)$ for $n \leq \sqrt{N_E} \log N_E$. See, for example, [Cr]. As the conductors grow polynomially in our families, it already requires several hours to investigate families for $t$ up to $1000$. It thus seems unlikely that we will be able to attain large enough ranges of $t$ to get past the contributions of these possibly lower cardinality sub-families.

# Appendix V: Heath-Brown and Brumer Bounds

Some very quick notes about the details of the proof for the family of elliptic curves (the exponential decay).

We have the following expansion of $\left(\frac{x}{p}\right)$:

$$\left(\frac{x}{p}\right) = G_p^{-1} \sum_{c=1}^{p} \left(\frac{c}{p}\right) \mathbf{e}\left(\frac{cx}{p}\right), \tag{0.0.5}$$

where $G_p = \sum_{a(p)} \left(\frac{a}{p}\right) \mathbf{e}\left(\frac{a}{p}\right)$, which equals $\sqrt{p}$ for $p \equiv 1(4)$ and $i\sqrt{p}$ for $p \equiv 3(4)$. See, for example, [BEW].

For the curve $y^2 = f_E(x)$, $a_E(p) = -\sum_{x(p)} \left(\frac{f_E(x)}{p}\right)$. We expand the $x$-sum by using Gauss sums, namely

$$a_E(p) = G_p^{-1} \sum_{x(p)} \sum_{c=1}^{p} \left(\frac{c}{p}\right) \mathbf{e}\left(\frac{cf_E(x)}{p}\right). \tag{0.0.6}$$

For the family of all elliptic curves, $y^2 = x^3 + ax + b$, when we need to calculate high moments of the $U$ sum we will be led to

$$\sum_{a,b} \sum_{t_1(p_1)} \cdots \sum_{t_{2k}(p_{2k})} \sum_{x_1(p_1)} \cdots \sum_{x_{2k}(p_{2k})} \prod_{i=1}^{2k} \left(\frac{t_i}{p_i}\right) e_{p_i}(t_i x_i^3 + t_i x_i a + t_i b). \tag{0.0.7}$$

We have complete sums as $a$ and $b$ run through $p_1 \cdots p_{2k}$. $|a| \leq T^{\frac{1}{3}}$, each tuple of $x_i$s will give a $\prod_i p_i$, and we will be left with having to bound

$$T^{\frac{1}{3}} \prod_i p_i \sum_{t_1(p_1)} \cdots \sum_{t_{2k}(p_{2k})} \left| \sum_b e\left(\left[\frac{t_1}{p_1} + \cdots + \frac{t_{2k}}{p_{2k}}\right] b\right) \right|. \tag{0.0.8}$$

The main contribution will be when there is at least one prime that occurs only once. In this case, the exponent is not an integer, since its denominator will be divisible by that prime. We then have a geometric series, which can be bounded well (where the bound will depend on the fractional part of $\frac{t_1}{p_1} + \cdots + \frac{t_{2k}}{p_{2k}}$.

It is crucial here that we have something of the form $y^2 = x^3 + ax + b$, and we can average over $b$. This allows us to use the Gauss sum expansion and obtain good cancellation. We are being very crude in estimating the $x_i$ sums. For a general family of elliptic curves, we will not have such a separation of parameters. Thus, in general we would have a 1-parameter family, say $y^2 = x^3 + A(t)x + B(t)$, and we would not be able to execute the summation over $t$ without having to worry about the values of the $x_i$s.

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