# On the Limiting Distribution of Eigenvalues of Large Random d-Regular Graphs with Weighted Edges 

M. C. Khoury S. J. Miller

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## Spectra of Large $d$-Regular Graphs

- Let $\left\{G_{i}\right\}$ be an infinite sequence of $d$-regular graphs such that the number of cycles of a given length is growing slowly relative to the number of vertices. (The technical condition.)
- For each $G_{i}$ consider its spectral measure $\nu_{G_{i}}$, the uniform measure on the eigenvalues of its adjacency matrix.


## Theorem (McKay)

The limit $\nu_{d}=\lim _{i \rightarrow \infty} \nu_{G_{i}}$ exists and depends only on d. If we normalize so that the support of $\nu_{d}^{\prime}$ is $[-1,1]$, then $\lim _{d \rightarrow \infty} \nu_{d}^{\prime}$ is the semicircle measure.

## Weighting the Edges

- Now we begin with a probability distribution $\mu$ with finite moments. Fix $d$ and $\mu$.
- Take a sequence $\left\{G_{i}\right\}$ of $d$-regular graphs satisfying the same technical condition.
- For each graph, assign each edge a weight (independently, using distribution $\mu$ ) and form the uniform probability measure on the modified adjacency matrix.
- Average over all possible weights to get a spectral measure $\nu_{G_{i, \mu}}$.
- The limiting spectral measure $T_{d}(\mu)=\nu_{d, \mu}=\lim _{i \rightarrow \infty} \nu_{G_{i}, \mu}$ depends only on $d$ and $\mu$.
- We are interested in the relationship between $\mu$ and $T_{d}(\mu)$.


## Fixed Points

- A natural question is whether there are fixed points in any sense.
- Recall the notion of rescaling a probability measure on $\mathbb{R}$. $S_{\lambda}(\mu)$ is the measure defined by $S_{\lambda}(\mu)(I)=\mu(\lambda I)$ for any interval $I$.
- For all $\mu, d, \lambda, S_{\lambda}\left(T_{d}(\mu)=T_{d}\left(S_{\lambda}(\mu)\right)\right.$.
- Can we describe all the "eigendistributions"? That is, for which $\mu$ do we have $T_{d}(\mu)=S_{\lambda}(\mu)$ for some $\lambda$ ?


## Eigenmoments

- We work at the level of moments of distributions.
- $S_{\lambda}$ multiplies the $k$ th moment of $\mu$ by a factor of $\lambda^{-k}$.
- Write $\sigma_{k}$ for the $k$ th moment of $\mu$ and $\tilde{\sigma}_{k}$ for the $k$ th moment of $T_{d}(\mu)$.
- We seek sequences $\left\{\sigma_{k}\right\}$ so that $\tilde{\sigma_{k}}=\lambda^{-k} \sigma_{k}$.
- It turns out that $\tilde{\sigma}_{2}=d \sigma_{2}$, so $\lambda=d^{-1 / 2}$.
- We seek sequences $\left\{\sigma_{k}\right\}$ so that $\tilde{\sigma_{k}}=d^{k / 2} \sigma_{k}$.
- Without loss of generality, we can rescale so that $\sigma_{2}=\frac{1}{4}$.


## Key Ideas

- The sum of the $k$ th powers of the eigenvalues of $A$ is the trace of $A^{k}$.
- Thus $\tilde{\sigma}_{k}$ corresponds to an average diagonal element of $A^{k}$, where $A$ is the weighted adjacency matrix.
- Nonzero contributions to a given diagonal entry of $A^{k}$ come from closed paths starting and ending at a particular vertex in the graph.
- The size of the contribution of any particular path depends on the weights assigned, so on average the moments $\sigma_{k}$ will appear.
- The technical condition means that paths including cycles are negligible.


## Closed Acyclic Path Patterns

## Definition

The set $P_{2 k}$ of closed acyclic path patterns of length $2 k$ is defined as follows. For $k>0, P_{2 k}$ contains all (equivalence classes of) strings $\pi$ of $2 k$ symbols with the following properties.
(1) In the substring of symbols between any two consecutive instances of the same symbol, every symbol appears an even number of times.
(2) Every symbol appears an even number of times.

Two c.a.p.p. are the same if they differ only by a relabelling of the symbols.

## Closed Acyclic Path Patterns

- aabccbaa $\in P_{8}$, but aabccbcc $\notin P_{8}$
- $a a b c c b a a=b b d p p d b b=\alpha \alpha \beta \gamma \gamma \beta \alpha \alpha$
- Because we always count up to this equivalence, each $P_{k}$ is a finite set.
- $P_{2 k}$ parametrizes all possible types of closed paths of length $2 k$ starting (and ending) at a given vertex in a large tree (or locally treelike graph) with plenty of edges at each vertex.


## Closed Acyclic Path Patterns: Small Examples

- $P_{2}=\{a a\}$
- $P_{4}=\{$ aaaa, aabb, abba\}
- $P_{6}=\{$ aaaaaa, aaaabb, aabbaa, aabbbb, aabbcc, abbaaa, abbacc, aaabba, aabccb, abbbba, abbcca, abccba\}



## Moment Relations

- What we end up with is a sum of the following form.

$$
\tilde{\sigma}_{2 k}=\sum_{\pi \in P_{2 k}} m(\pi) w_{\sigma}(\pi)
$$

- $m(\pi)$ is a polynomial in $d$ corresponding to how many way the pattern can be realized in a $d$-ary tree.
- $w_{\sigma}(\pi)$ is product of moments $\sigma_{i}$ corresponding to how many times each symbol appears in the pattern.
- In particular:
- The odd moments vanish.
- For even $k, \tilde{\sigma}_{k}$ depends only on $\sigma_{2}, \sigma_{4}, \ldots, \sigma_{k}$.


## Moment Relations: Examples

- $\tilde{\sigma}_{2}=d \sigma_{2}$
- $\tilde{\sigma}_{4}=d \sigma_{4}+2 d(d-1) \sigma_{2}^{2}$
- $\tilde{\sigma}_{6}=d \sigma_{6}+6 d(d-1) \sigma_{4} \sigma_{2}+\left[3 d(d-1)^{2}+2 d(d-1)(d-2)\right] \sigma_{2}^{3}$

$$
\begin{aligned}
\tilde{\sigma}_{8}=d \sigma_{8} & +8 d(d-1) \sigma_{6} \sigma_{2}+6 d(d-1) \sigma_{4}^{2} \\
& +\left[16 d(d-1)^{2}+12 d(d-1)(d-2)\right] \sigma_{4} \sigma_{2}^{2} \\
+ & {\left[4 d(d-1)^{3}+8 d(d-1)^{2}(d-2)+\cdots\right.} \\
& \cdots+2 d(d-1)(d-2)(d-3)] \sigma_{2}^{4}
\end{aligned}
$$

## Theorem 1

## Theorem (Unique Existence)

For each $d \geq 2$, there exists a unique sequence of eigenmoments $\sigma_{k}^{\star}=\sigma_{k}^{\star}(d)$ satisfying $\sigma_{2}^{\star}=1 / 4$. Furthermore, $\sigma_{k}^{\star}(d)$ is a rational function of $d$.

- $\sigma_{k}^{\star}=0$ identically for odd $k$.
- The proof is straightforward by induction using the moment relations.


## Proof Sketch

- Substitute $\sigma_{k}=\sigma_{k}^{\star}$ and $\tilde{\sigma}_{k}=d^{k / 2} \sigma_{k}^{\star}$ into the moment relation.
- $d^{k / 2} \sigma_{k}^{\star}=\sum_{\pi \in P_{k}} m(\pi) w_{\sigma^{\star}}(\pi)$
- $\sigma_{k}^{\star}$ appears only once on the right hand side, for the path that uses only one edge.
- $\left(d^{k / 2}-d\right) \sigma_{k}^{\star}=\sum_{\pi \in P_{k}^{\prime}} m(\pi) w_{\sigma^{\star}}(\pi)$, where only strictly smaller moments now appear on the right hand side.


## Small Examples

- We begin with $\sigma_{2}^{\star}=1 / 4$ and solve recursively.
- $d^{2} \sigma_{4}^{\star}=d \sigma_{4}^{\star}+2 d(d-1)\left(\sigma_{2}^{\star}\right)^{2}$
- This gives $\sigma_{4}^{\star}=1 / 8$.
- $d^{3} \sigma_{6}^{\star}=$ $d \sigma_{6}^{\star}+6 d(d-1) \sigma_{4}^{\star} \sigma_{2}^{\star}+\left[3 d(d-1)^{2}+2 d(d-1)(d-2)\right]\left(\sigma_{2}^{\star}\right)^{3}$
- This gives $\sigma_{6}^{\star}=5 / 64$.
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- This gives $\sigma_{6}^{\star}=5 / 64$.
- Note the appearance of the moments of the semicircle distribution and the absence of $d$.
- However, $\sigma_{8}^{\star}=\frac{7}{128}+\frac{1}{128\left(d^{2}+d+1\right)}$.


## Theorem 2

## Theorem (Limiting Moments)

$$
\lim _{d \rightarrow \infty} \sigma_{2 k}^{\star}(d)=\frac{1}{4^{k}(k+1)}\binom{2 k}{k} .
$$

This agrees with the moments of the semicircle distribution.

- Notice the presence of the Catalan numbers.
- The only patterns that contribute to the highest-degree term in the moment relations are those in which no edge is repeated. The Catalan numbers count these patterns.


## Theorem 3

## Theorem (Improving the Error Estimate)

$$
\sigma_{2 k}^{\star}(d)=\frac{1}{4^{k}(k+1)}\binom{2 k}{k}+O\left(1 / d^{2}\right)
$$

The implied constant may depend on $k$.

- To get this estimate, we need the next-highest-degree term in the moment relations.
- One contribution is from the paths with exactly one repetition.
- Another contribution comes from the structure of the paths with no repetition.


## Proof Sketch

- It turns out that the $d^{-1}$ is proportional to $2 A-B$, where $A$ is the number of c.a.p.p. with exactly one repetition, and $B$ is the number of c.a.p.p. with no repetition and a distinguished pair of adjacent edges.
- There is an explicit two-to-one correspondence between c.a.p.p. with exactly one repeated edge and c.a.p.p. with no repetition and a distinguished pair of adjacent edges. (That is, there are two ways to "split" the double edge into two single edges.)
- Without actually computing $A$ or $B$, we conclude that the $d^{-1}$ term vanishes.



## Summary

- For fixed $d$, there is a unique (up to scale) eigendistribution of edge weights, which leads to a limiting spectral measure which is a rescaling of itself. For large $d$, these eigendistributions converge to the semicircular distribution.
- The convergence is faster than one should expect because of a combinatorial miracle, which we understand in terms of closed acyclic path patterns.
- One can understand random graph theory better by better understanding and enumerating closed acyclic path patterns.


## Future Directions

- Improving the Error Estimate Further. One could get a theorem of the form $\sigma_{n}^{\star}=c_{n}+\frac{\alpha_{n}}{d^{2}}+O\left(d^{-3}\right)$ by comparing the numbers of different types of paths in which at most two repetitions are allowed.
- Closed Acyclic Path Patterns. Many basic things are still not known about their enumeration. What can we say about the rate of growth in the number of patterns? Is the ratio between consecutive terms in the sequence bounded? (OEIS A094149)
- General Acyclic Path Patterns. What if we relax the condition that each symbol appear an even number of times in total? Almost nothing seems to be known about enumerating these. The sequence is not in OEIS.


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