Distribution of Eigenvalues of Weighted, Structured Matrix Ensembles

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Random Matrices and their Limiting Spectral Measure

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Applications:

- Nuclear Physics
- Number Theory

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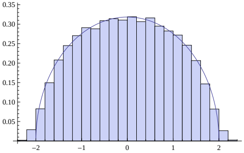

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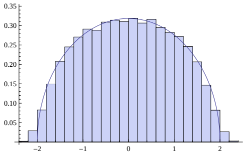

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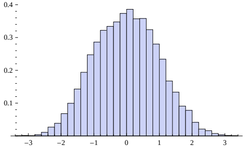

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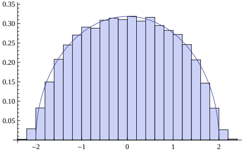

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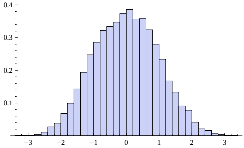

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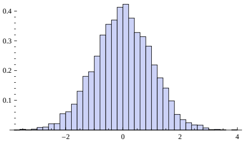
Palindromic Toeplitz



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Our Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

Multiply each entry of a (Palindromic) Toeplitz matrix by

$$\epsilon_{ij} = \epsilon_{ji} = \begin{cases} 1 & \text{with prob. } p \\ -1 & \text{with prob. } 1\text{-}p \end{cases} \quad \text{so} \quad a_{ij} = a_{ji} = \epsilon_{ij} b_{|i-j|}$$

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What is the eigenvalue distribution of these signed ensembles?

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$$\int x^k \mu_{A,N}(x)\,dx = \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i(A)^k}{\left(2\sqrt{N}\right)^k} = \frac{\mathrm{Trace}\left(A^k\right)}{2^k N^{\frac{k}{2}+1}}$$

The average k^{th} moment, $M_k(N) = \mathbb{E}[M_{N,k}(A_N)]$ is thus:

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \ldots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \ldots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

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Preliminary Result:

The b's must be matched in pairs to contribute in the limit.

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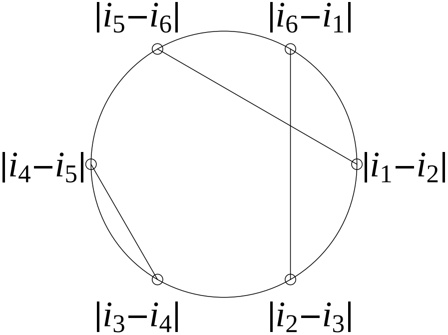
- For the even moments M_{2k} we can represent each contributing term as a pairing of $2k$ vertices on a circle as follows:



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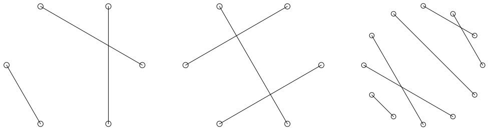

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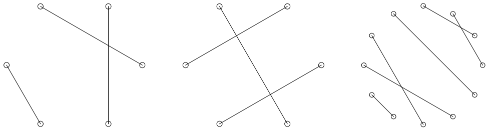

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Semicircle: Only non-crossing configurations contribute 1
Gaussian: All configurations contribute 1

Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing $(\mathrm{Cross}_{2k,2m})$?

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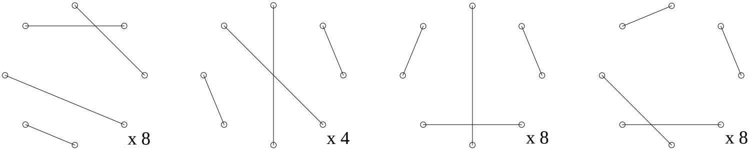

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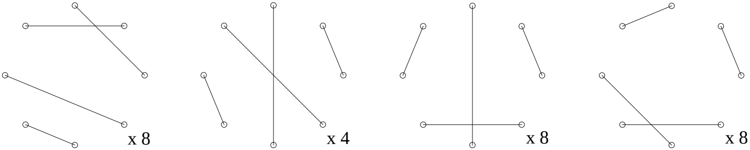

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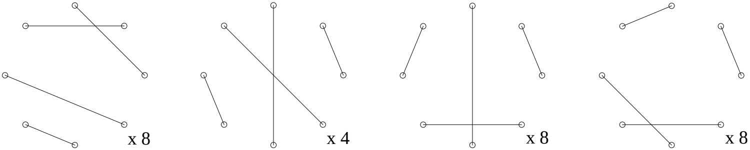

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What about for higher m?

Counting Crossing Configurations: Non-Crossing Regions

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Suppose $2m$ vertices are already paired in some configuration. The number of ways to pair and place the remaining $2k - 2m$ vertices such that none of them are involved in a crossing is $\binom{2k}{k-m}$.

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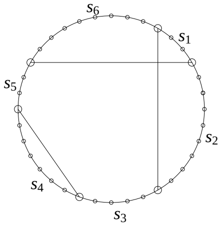

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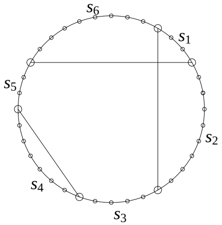

Proof: $\sum_{s_1+\cdots+s_{2m}=2k-2m} C_{s_1} C_{s_2} \cdots C_{s_{2m}} = \binom{2k}{k-m}$.

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We solved for small m, then applying our Non-Crossing Regions Theorem gives:

- $\mathrm{Cross}_{2k,4} = \binom{2k}{k-2}$
- $\mathrm{Cross}_{2k,6} = 4\binom{2k}{k-3}$
- $\mathrm{Cross}_{2k,8} = 31\binom{2k}{k-4} + \frac{1}{2}\sum_{i=0}^{k-5}\binom{2k}{i}(2k-2i)$
- $\mathrm{Cross}_{2k,10} = 288\binom{2k}{k-5} + 4\sum_{i=0}^{k-6}\binom{2k}{i}(2k-2i)$

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- Limiting behavior of the mean and variance of the moments, giving bounds for the moments

Many thanks to:

- AMS/MAA
- Williams College, SMALL 2011
- National Science Foundation
- Olivia Beckwith, Professor Steven J Miller