An Elliptic Curve Test of the L-functions Ratios Conjecture

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The Riemann-Zeta Function:

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Extends uniquely to the rest of $\mathbb{C}$.

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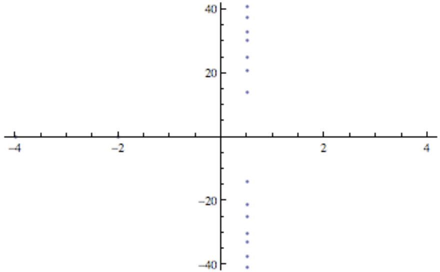

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What does the distribution of zeros looks like?

The Big Three

The Statistic:

The One-Level Density

The L-functions:

Quadratic Twists of Elliptic Curve L-functions

How We Look At Them:

Ratios Conjecture

The Statistic: The One-Level Density

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$$\lim_{Q\to\infty} \frac{1}{|\mathcal{F}(Q)|} \sum_{f\in\mathcal{F}(Q)} \sum_{\gamma_f} \phi\left(\frac{\gamma_f L}{\pi}\right)$$

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The L-functions we consider:

$$L_E(s,\chi) = \sum_{n=1}^{\infty} \frac{\lambda(n)\chi(n)}{n^s} = \prod_p \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\psi_M(p)\chi(p)^2}{p^{2s}}\right)^{-1}$$

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- Predicts the main term of one-level density as well as lower order terms in MANY situations

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We'll focus on testing the Ratios Conjecture for families of quadratic twists of elliptic curve L-functions

Testing the Ratios Conjecture for this Convolution

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How to Test the Ratios Conjecture?

- Calculate the Ratios prediction (easier, already done)
- Do the explicit number theory calculation (harder)
- See if the two expressions are equal up to an error term

Comparing the Two (Sans Error Terms)

NT:

$$\frac{g(0)}{2} + \frac{1}{2LX^*}\int_{-\infty}^{\infty} g(\tau) \sum_{d\in\mathcal{F}(X)} \left[2\log\left(\frac{\sqrt{M}|d|}{2\pi}\right) + \frac{\Gamma'}{\Gamma}\left(1+i\frac{\pi\tau}{L}\right) + \frac{\Gamma'}{\Gamma}\left(1-i\frac{\pi\tau}{L}\right)\right] d\tau$$
$$+\frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\left(-\frac{\zeta'}{\zeta}\left(1+\frac{2\pi i\tau}{L}\right) + \frac{L_E'}{L_E}\left(\mathrm{sym}^2, 1+\frac{2\pi i\tau}{L}\right) - \sum_{\ell=1}^{\infty}\frac{(M^\ell-1)\log M}{M^{(2+\frac{2\pi i\tau}{L})\ell}}\right) d\tau$$
$$-\frac{1}{L}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty} g(\tau)\frac{\log M}{M^{(k+1)(1+\frac{\pi i\tau}{L})}} d\tau + \frac{1}{L}\int_{-\infty}^{\infty} g(\tau)\sum_{p\nmid M}\frac{\log p}{(p+1)}\sum_{k=0}^{\infty}\frac{\lambda(p^{2k+2})-\lambda(p^{2k})}{p^{(k+1)(1+\frac{2\pi i\tau}{L})}} d\tau$$

Ratios:

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$$\left.\times A_E\left(-\frac{i\pi\tau}{L}, \frac{i\pi\tau}{L}\right)\right] d\tau$$

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$$
\begin{aligned}
A_E(\alpha,\gamma) \quad = \quad & \frac{\zeta(1+\alpha+\gamma)L_E(\mathrm{sym}^2, 1+\alpha+\gamma)}{\zeta(1+2\gamma)L_E(\mathrm{sym}^2, 1+2\alpha)} \times \prod_{p|M}\left(\sum_{m=0}^{\infty}\left(\frac{\lambda(p^m)\omega_E^m}{p^{m(1/2+\alpha)}} - \frac{\lambda(p)}{p^{1/2+\gamma}}\frac{\lambda(p^m)\omega_E^{m+1}}{p^{m(1/2+\alpha)}}\right)\right. \\
& \times \prod_{p\nmid M}\left(1+\frac{p}{p+1}\left(\sum_{m=1}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\lambda(p)}{p^{1+\alpha+\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}}\sum_{m=0}^{\infty}\frac{\lambda(p^{2m})}{p^{m(1+2\alpha)}}\right)\right.
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Big Result

Theorem

The Ratios Conjecture for quadratic twists of elliptic curve L-functions of prime conductor is correct up to errors of size $O(X^{-(1-\sigma)/2})$.

Thanks

- AMS/MAA
- Steven J. Miller and Duc Khiem Huynh
- Williams College and UC Berkeley
- Y'all