Theoretical Physics Seminar

From the Manhatten Project to Number Theory: How Nuclear Physics helps us understand primes

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Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: what rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w Zeros of Functions.

Goals of the Talk

- Determine correct scale to study spacings.

- See similar behavior in different systems.

- Discuss tools / techniques needed to prove the results.

PART I

NORMALIZED SPACINGS

Normalized Spacing

Example: Fractional Parts

For $\alpha \notin \mathbb{Q}$, set $x_n = n\alpha \bmod 1$.

Order $x_1, \dots, x_N$: $0 \leq y_1 \leq \cdots \leq y_N \leq 1$.

Expect spacings between adjacent y's of size $\frac{1}{N}$.

Should study $\frac{y_{n+1}-y_n}{1/N}$.

PART II

RANDOM MATRIX THEORY

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

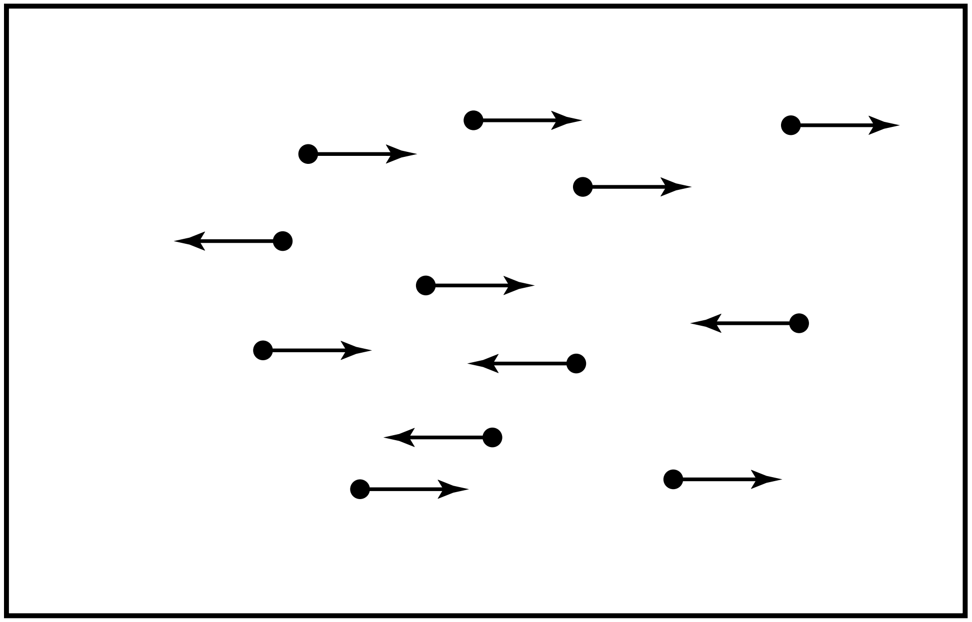
$$H\psi_n \ = \ E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins (continued)



Statistical Mechanics: for each configuration, calculate quantity (say pressure).

Average over all configurations – most configurations close to system average.

Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices.

Look at: Real Symmetric ($A = A^T$), Complex Hermitian ($\overline{A}^T = A$).

Random Matrix Ensembles

Real Symmetric Matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0.

To each A, attach a probability measure:

$$
\begin{aligned}
\mu_{A,N}(x) &= \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) \\
\int_a^b \mu_{A,N}(x)dx &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N} \\
\text{k}^{\text{th}}\text{ moment} &= \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.
\end{aligned}
$$

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$.

Semi-Circle Law: Assume p has mean 0, variance 1, other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Correct Scale

$$\text{Trace}(A^2) \; = \; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\text{Trace}(A^2) \; = \; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \; = \; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \; \sim \; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \; \sim \; N^2$$

Gives $N\text{Ave}(\lambda_i(A)^2) \; \sim \; N^2$ or $\text{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Method of Moments

$$\text{Trace}(A^2) \ = \ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \ = \ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2.$$

Substituting into expansion gives

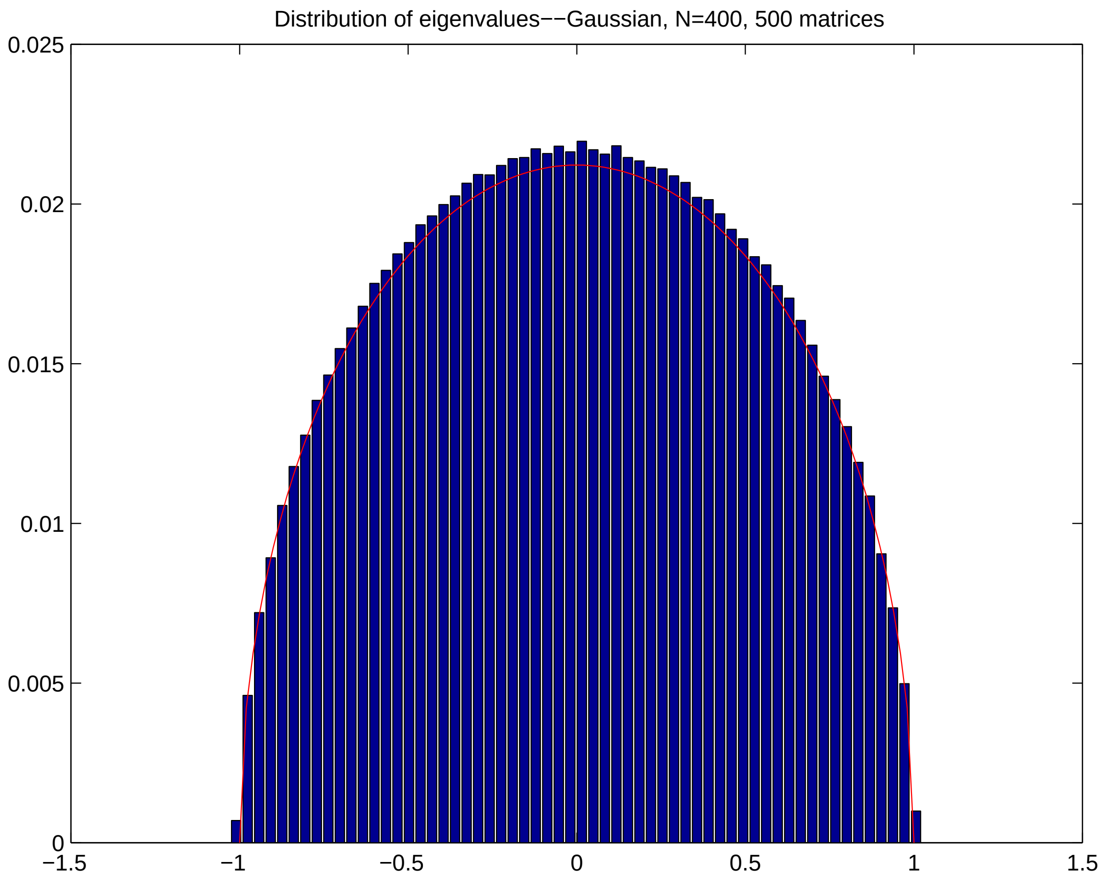
$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}^2 \cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij})da_{ij} \ \ \cdot \ \prod_{\substack{(k,l)\neq(i,j)\\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl})da_{kl} \ = \ 1.$$

Have N^2 summands, answer is $\frac{1}{4} = 2^{\text{nd}}$ moment of Semi-Circle.

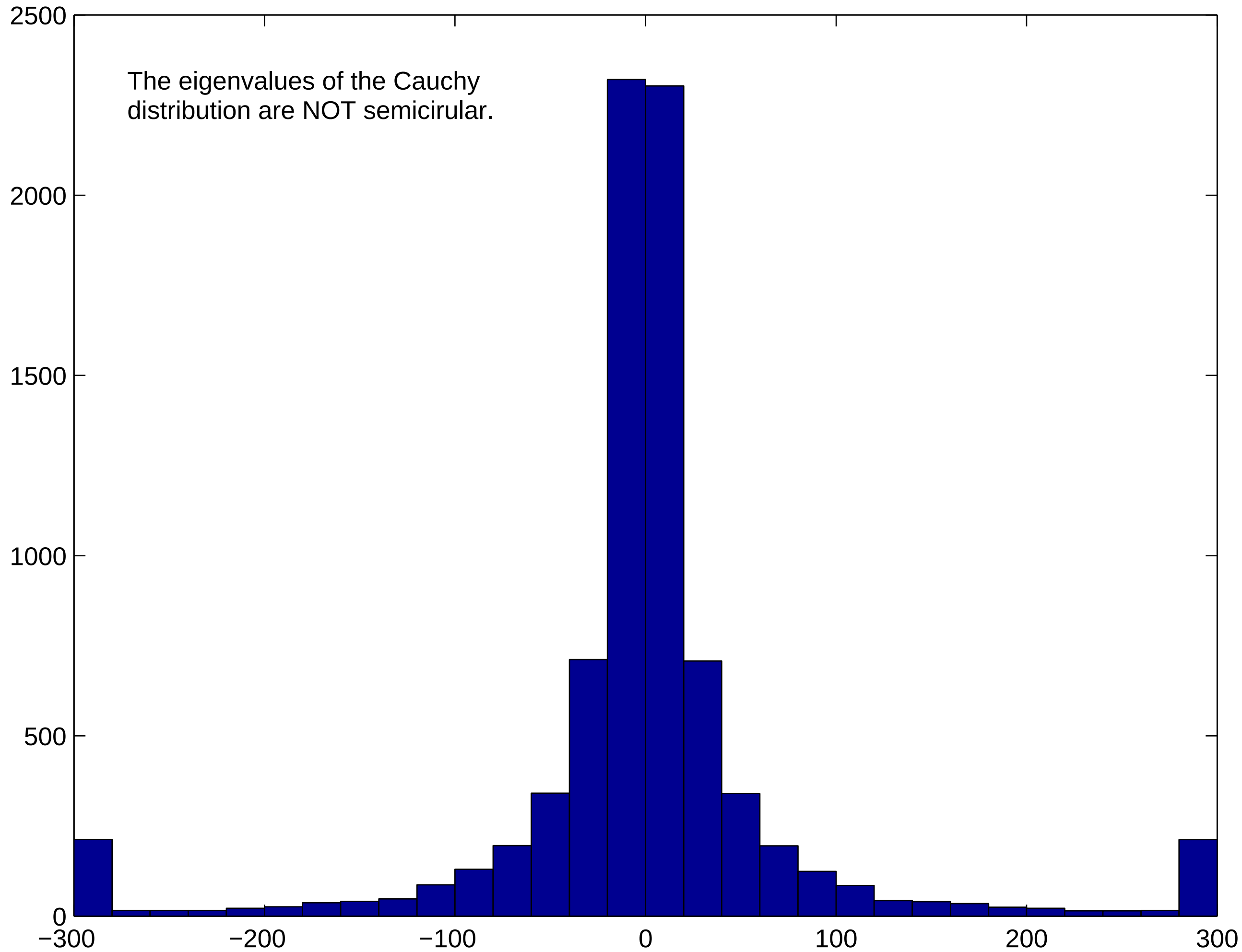
Random Matrix Theory: Semi-Circle Law



500 Matrices: Gaussian 400×400

$$p(x) \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

Random Matrix Theory: Semi-Circle Law



Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

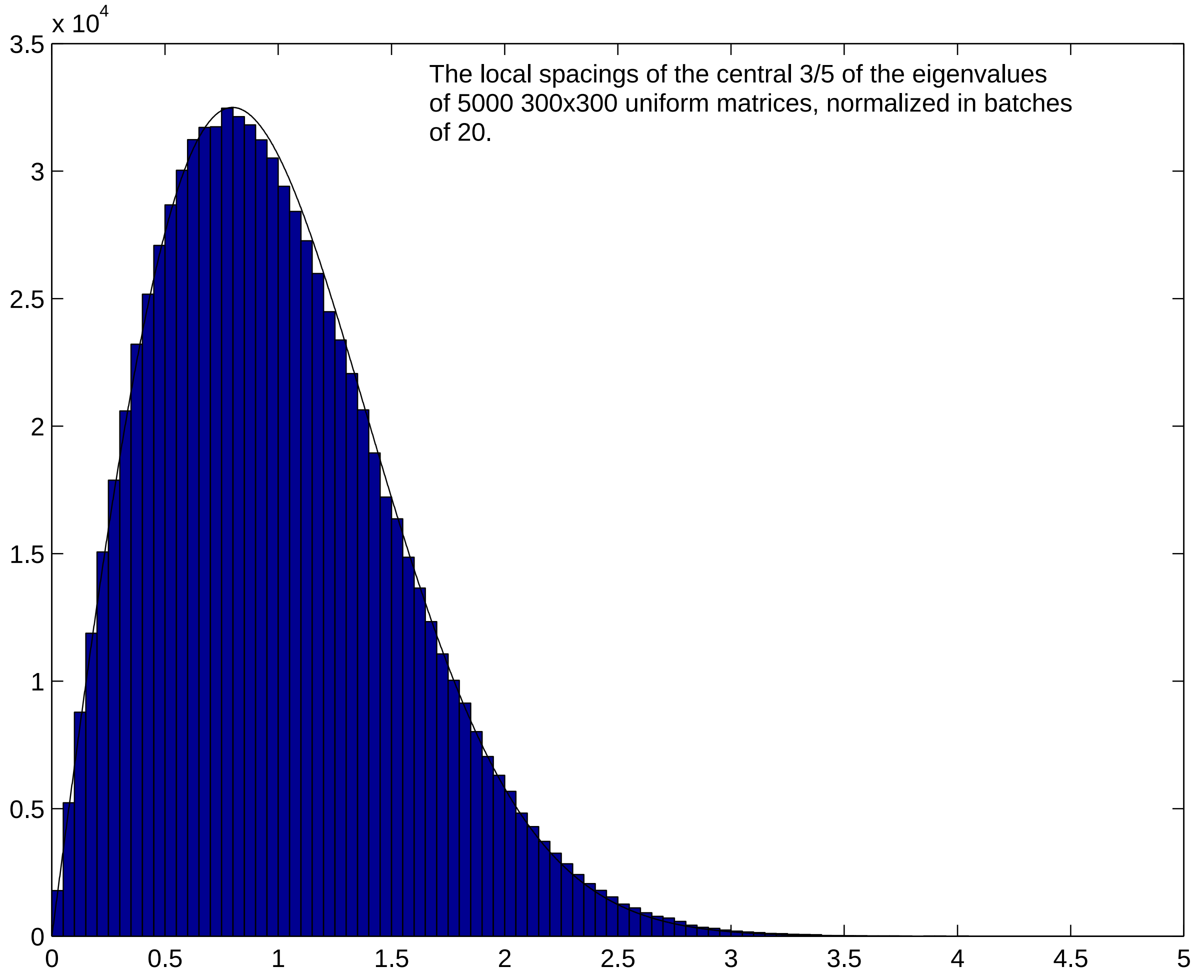
GOE Conjecture

GOE Conjecture: As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of p.

Only known if p is a Gaussian.

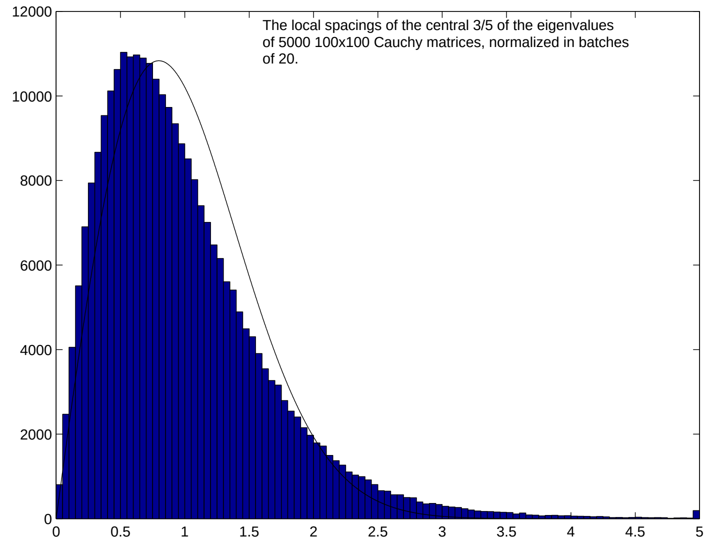
GOE$(x) \;\approx\; Axe^{-Bx^2}$.

Uniform Distribution: $p(x) = \frac{1}{2}$ for $|x| \leq 1$

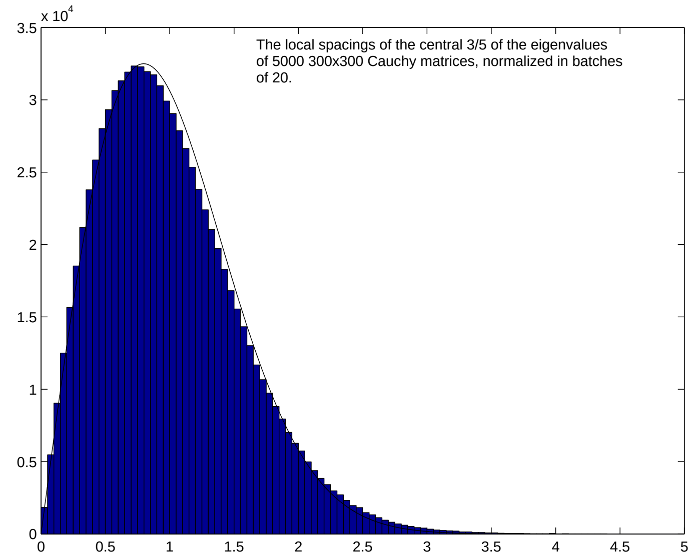


5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$



5000: 100×100 Cauchy



5000: 300×300 Cauchy

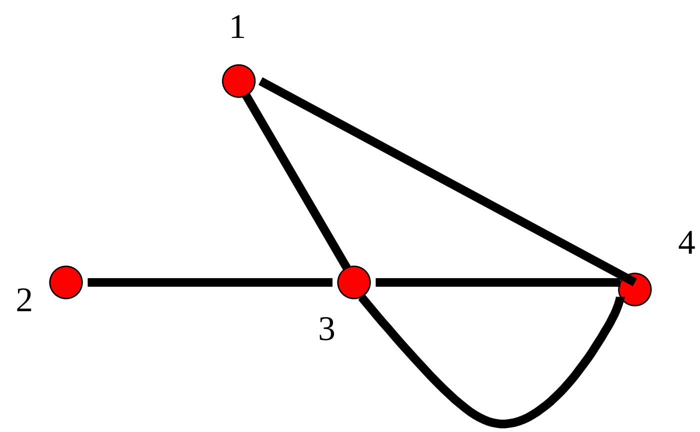
Fat Thin Families

Need a family ***F A T*** enough to do averaging.

Need a family ***THIN*** enough so that everything isn't averaged out.

Real Symmetric Matrices have $\frac{N(N+1)}{2}$ independent entries.

Random Graphs



Degree of a vertex = number of edges leaving the vertex.

Adjacency matrix: a_{ij} = number edges b/w Vertex i and Vertex j.

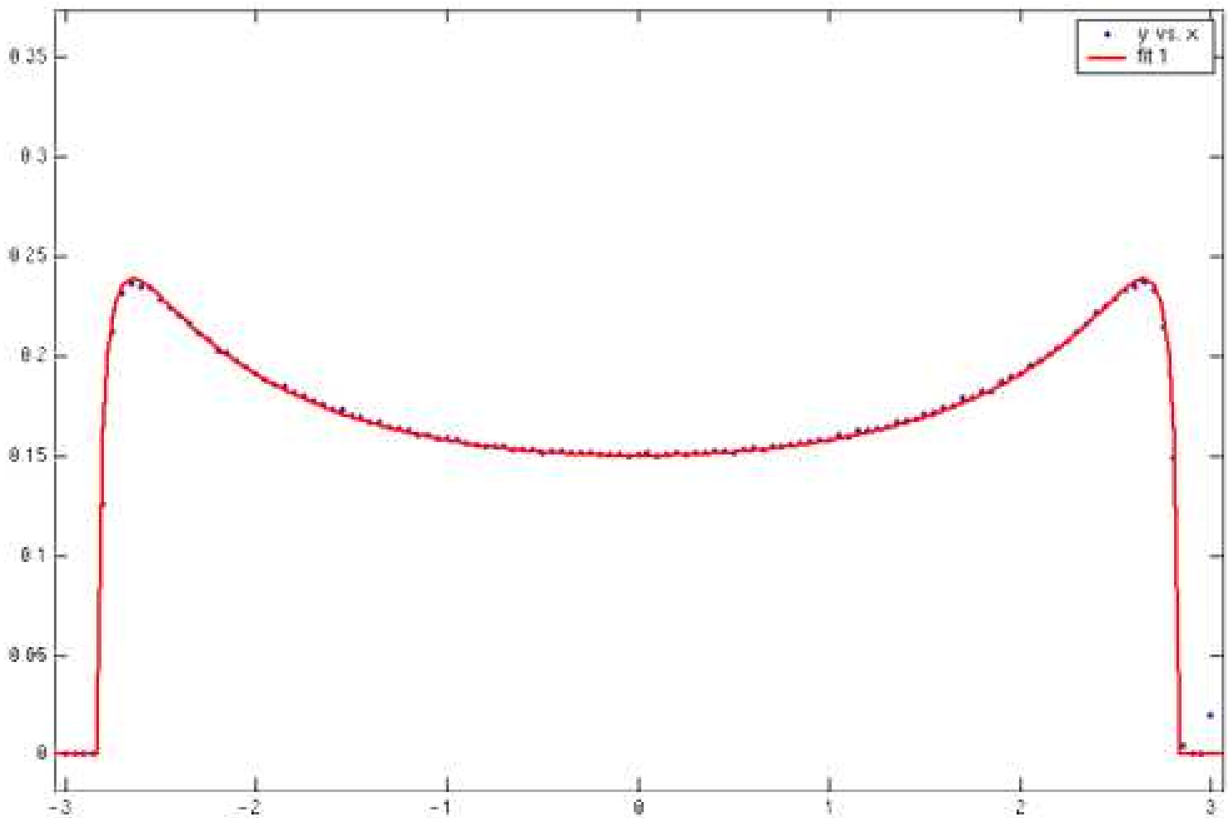
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

McKay's Law (Kesten Measure)

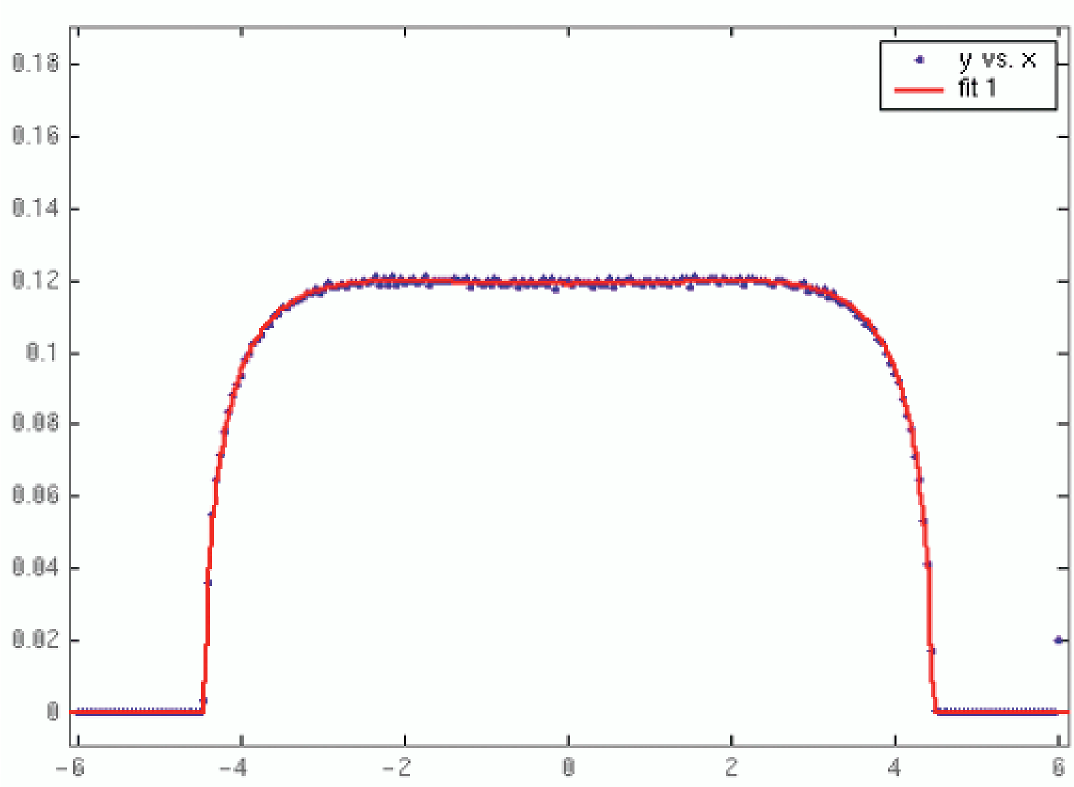
Density of Eigenvalues for d-regular graphs

$$f(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



$$d \;=\; 3.$$

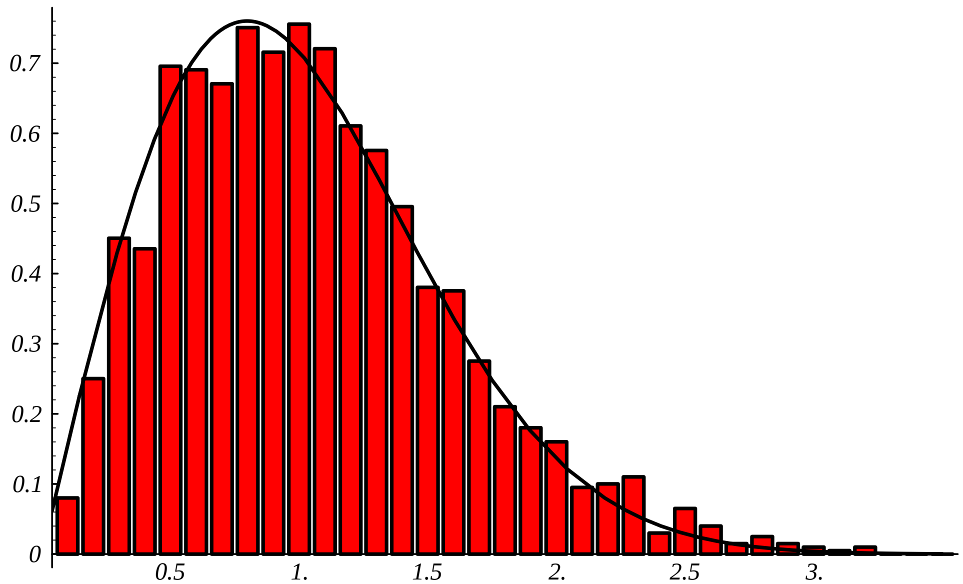
McKay's Law (Kesten Measure)



$$d = 6.$$

Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.

3-Regular, 2000 Vertices and GOE



PART III

NUMBER THEORY

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right]\cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont):

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$
$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.

- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

Riemann Zeta Function (cont):

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

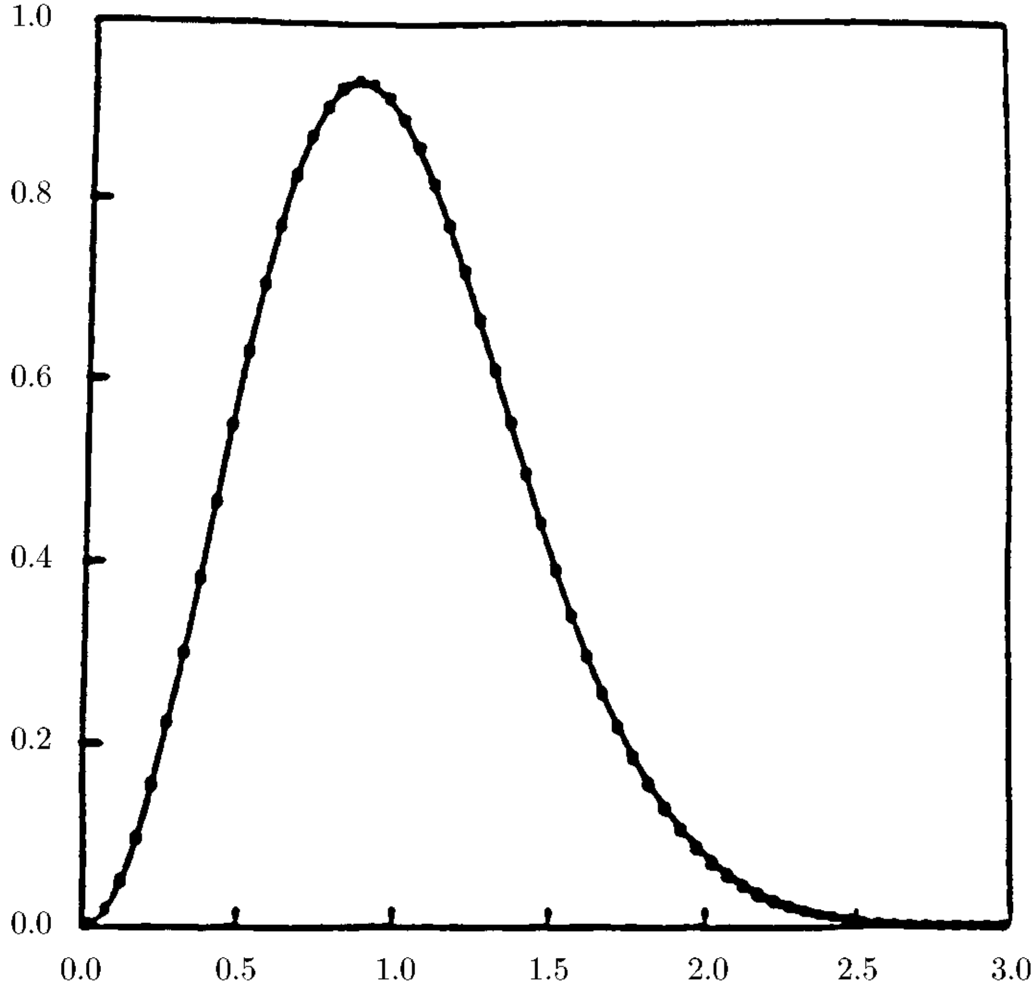
Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation:

Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $(\overline{A}^T =$

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko)

Explicit Formula: (Contour Integration)

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\ &= \sum_p \frac{\log p \ \cdot \ p^{-s}}{1-p^{-s}} \\ &= \sum_p \frac{\log p}{p^s} \ + \ \mathrm{Good}(s). \end{aligned}$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

Knowledge of zeros gives info on coefficients.

PART IV

ELLIPTIC CURVES

Elliptic Curves:

$$E_t : y^2 \; = \; x^3 + A(T)x + B(T), \;\; A(T), B(T) \in \mathbb{Z}(T).$$

$$a_t(p) \; = \; - \sum_{x \bmod p} \left(\frac{x^3 + A(t)x + B(t)}{p} \right) \; = \; a_{t+mp}(p)$$

$$L(E, s) \; = \; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \; = \; \prod_p L_p(E, s).$$

By GRH: All zeros on the critical line.

Rational solutions: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$.

Birch and Swinnerton-Dyer Conjecture:
Geometric rank $r =$ analytic rank (order of vanishing at central point).

Limiting Behavior

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_f^{(j)} \log N_f}{2\pi}\right) \ = \ \int \phi(x) W_{1,\mathcal{G}(\mathcal{F})}(x) dx.$$

Density Conjecture: Distribution of low zeros of L-functions agree with the distribution of eigenvalues near 1 of a classical compact group.

Tools to Study Low Zeros

- explicit formula relating zeros and Fourier coeffs;
  - ◇ Analogue of Eigenvalue Trace Lemma

- averaging formulas for the family;
  - ◇ Analogue of integration formulas for $\mathrm{Trace}(A^k)$.

Explicit Formula

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_j \phi\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) = \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\widehat{\phi}(0)+\phi_i(0)$$

$$-\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_p \frac{\log p}{\log N_E}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log N_E}\right)a_E(p)$$

$$-\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_p \frac{\log p}{\log N_E}\frac{1}{p^2}\widehat{\phi}\left(2\frac{\log p}{\log N_E}\right)a_E^2(p)$$

$$+O\left(\frac{\log\log N_E}{\log N_E}\right)$$

Want to move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$, Leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t \bmod p} a_t^r(p), \quad r = 1 \text{ or } 2.$$

One-Level Result

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E\in\mathcal{F}_N} \sum_j \phi\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) \ = \ \int \phi(x) W_{\mathcal{G}}(x) dx + \ r\phi(0),$$

where

$$\mathcal{G} \ = \ \begin{cases} \text{O} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Forced zeros seem independent.

Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}$.

Natural sub-families

- Curves of rank r.
- Curves of rank $r + 2$.

Question: Does the sub-family of rank $r + 2$ curves in a rank r family behave like the sub-family of rank $r + 2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r+2$?

Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model:

Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$:

$$d\varepsilon_{2r}(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

with $1 \leq j,k \leq N-r$.

Comparing the RMT Models

Small support, one-parameter families agree with $\rho_{r,\text{Indep}}$ and not $\rho_{r,\text{Inter}}$.

Curve E, conductor N_E, expect first zero $\frac{1}{2} + i\gamma_E^{(1)}$ with $\gamma_E^{(1)} \approx \frac{1}{\log N_E}$.

r zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log N_E}$, can detect in

$$\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_E^{(j)} \log N_E}{2\pi}\right).$$

Corrections of size

$$\phi\left(x_0 + c_r\right) - \phi(x_0) \;\approx\; \phi'\left(x(x_0, c_r)\right) \cdot c_r.$$

Testing Random Matrix Theory Predictions

First (Normalized) Zero above Central Point: Do extra zeros at the central point affect the distribution of zeros near the central point?

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

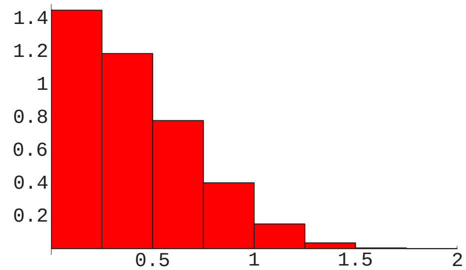


Figure 1a: First normalized eigenangle above 0: 23,040 SO(4) matrices
Mean $= .357$, Standard Deviation about the Mean $= .302$, Median $= .357$

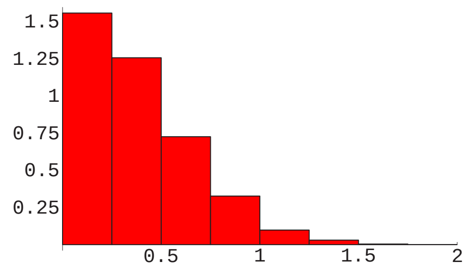


Figure 1b: First normalized eigenangle above 0: 23,040 SO(6) matrices
Mean $= .325$, Standard Deviation about the Mean $= .284$, Median $= .325$

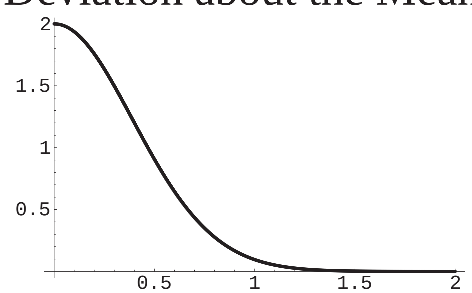


Figure 1c: First normalized eigenangle above 0:
$N \to \infty$ scaling limit of SO($2N$): Mean = .321.

Rank 0 Curves: 1st Normalized Zero above Central Point

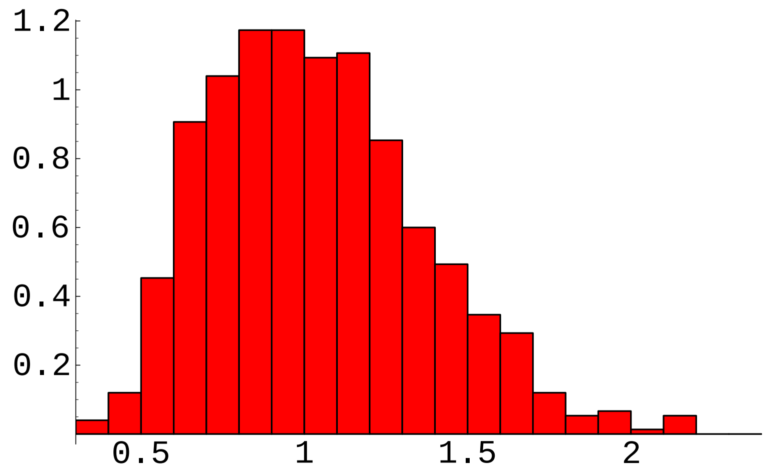


Figure 2a: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, $\sigma_\mu = .32$

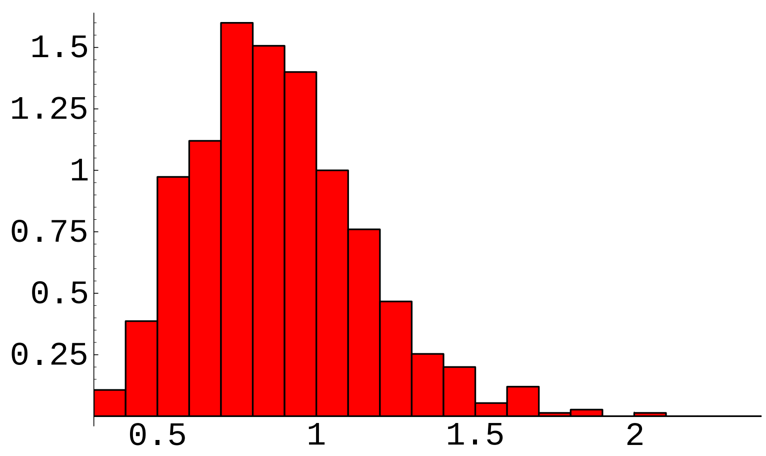


Figure 2b: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

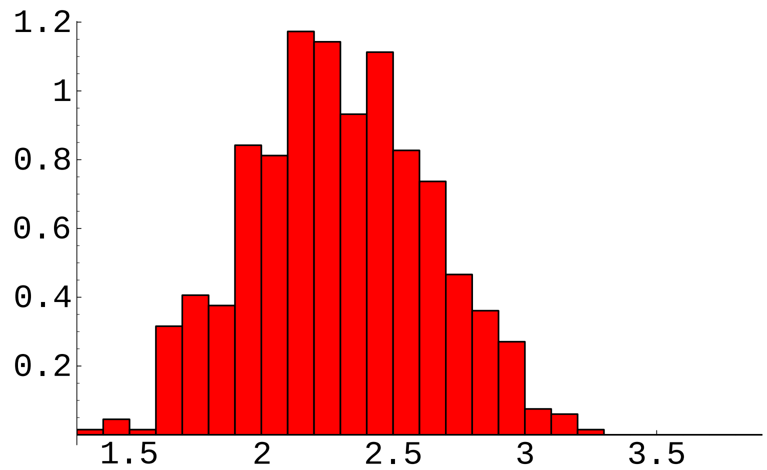


Figure 3a: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

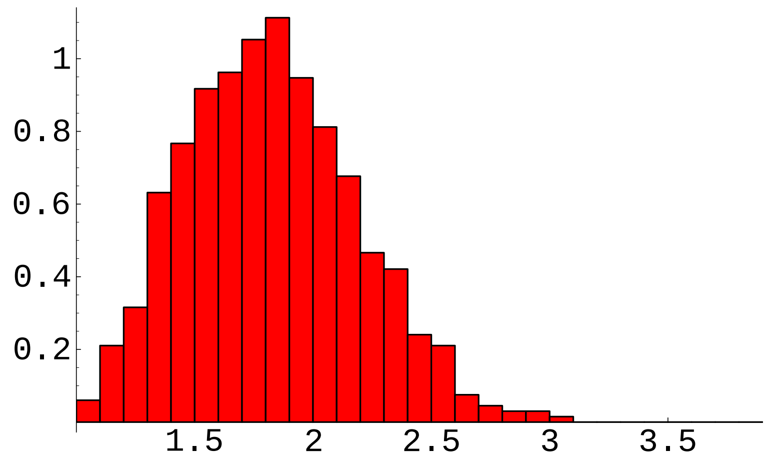


Figure 3b: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

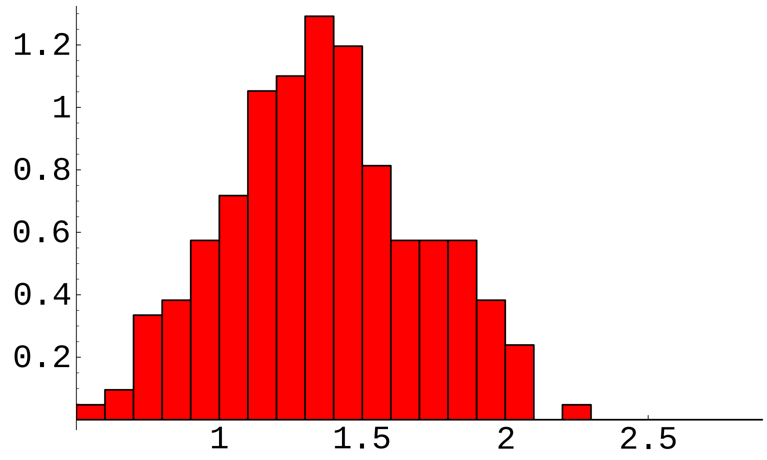


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

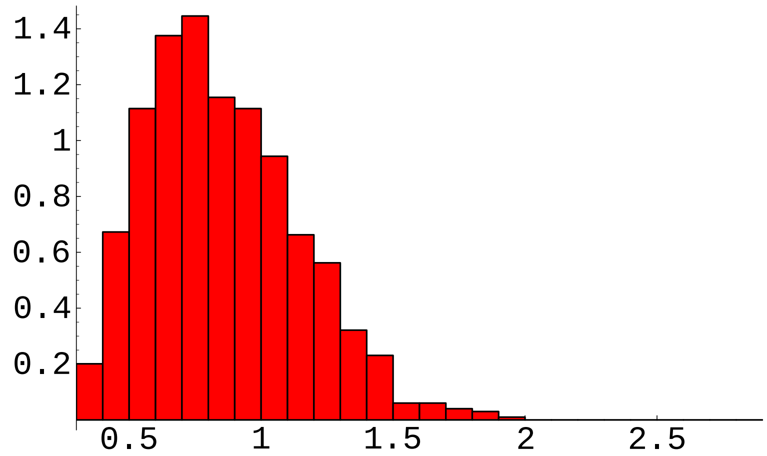


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	**Median** $\widetilde{\mu}$	**Mean** μ	**StDev** σ_μ	**log(conductor)**	**Number**
1: [0,1,1,1,T]	1.28	1.33	0.26	[3.93, 9.66]	7
2: [1,0,0,1,T]	1.39	1.40	0.29	[4.66, 9.94]	11
3: [1,0,0,2,T]	1.40	1.41	0.33	[5.37, 9.97]	11
4: [1,0,0,-1,T]	1.50	1.42	0.37	[4.70, 9.98]	20
5: [1,0,0,-2,T]	1.40	1.48	0.32	[4.95, 9.85]	11
6: [1,0,0,T,0]	1.35	1.37	0.30	[4.74, 9.97]	44
7: [1,0,1,-2,T]	1.25	1.34	0.42	[4.04, 9.46]	10
8: [1,0,2,1,T]	1.40	1.41	0.33	[5.37, 9.97]	11
9: [1,0,-1,1,T]	1.39	1.32	0.25	[7.45, 9.96]	9
10: [1,0,-2,1,T]	1.34	1.34	0.42	[3.26, 9.56]	9
11: [1,1,-2,1,T]	1.21	1.19	0.41	[5.73, 9.92]	6
12: [1,1,-3,1,T]	1.32	1.32	0.32	[5.04, 9.98]	11
13: [1,-2,0,T,0]	1.31	1.29	0.37	[4.73, 9.91]	39
14: [-1,1,-3,1,T]	1.45	1.45	0.31	[5.76, 9.92]	10
All Curves	1.35	1.36	0.33	[3.26, 9.98]	209
Distinct Curves	1.35	1.36	0.33	[3.26, 9.98]	196

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(conductor)	Number
1: [0,1,1,1,T]	0.80	0.86	0.23	[15.02, 15.97]	49
2: [1,0,0,1,T]	0.91	0.93	0.29	[15.00, 15.99]	58
3: [1,0,0,2,T]	0.90	0.94	0.30	[15.00, 16.00]	55
4: [1,0,0,-1,T]	0.80	0.90	0.29	[15.02, 16.00]	59
5: [1,0,0,-2,T]	0.75	0.77	0.25	[15.04, 15.98]	53
6: [1,0,0,T,0]	0.75	0.82	0.27	[15.00, 16.00]	130
7: [1,0,1,-2,T]	0.84	0.84	0.25	[15.04, 15.99]	63
8: [1,0,2,1,T]	0.90	0.94	0.30	[15.00, 16.00]	55
9: [1,0,-1,1,T]	0.86	0.89	0.27	[15.02, 15.98]	57
10: [1,0,-2,1,T]	0.86	0.91	0.30	[15.03, 15.97]	59
11: [1,1,-2,1,T]	0.73	0.79	0.27	[15.00, 16.00]	124
12: [1,1,-3,1,T]	0.98	0.99	0.36	[15.01, 16.00]	66
13: [1,-2,0,T,0]	0.72	0.76	0.27	[15.00, 16.00]	120
14: [-1,1,-3,1,T]	0.90	0.91	0.24	[15.00, 15.99]	48
All Curves	0.81	0.86	0.29	[15.00,16.00]	996
Distinct Curves	0.81	0.86	0.28	[15.00,16.00]	863

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$
(Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

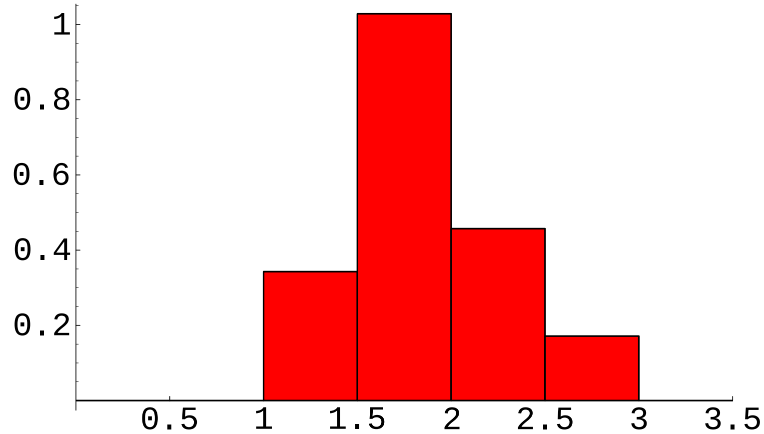


Figure $5a$:35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

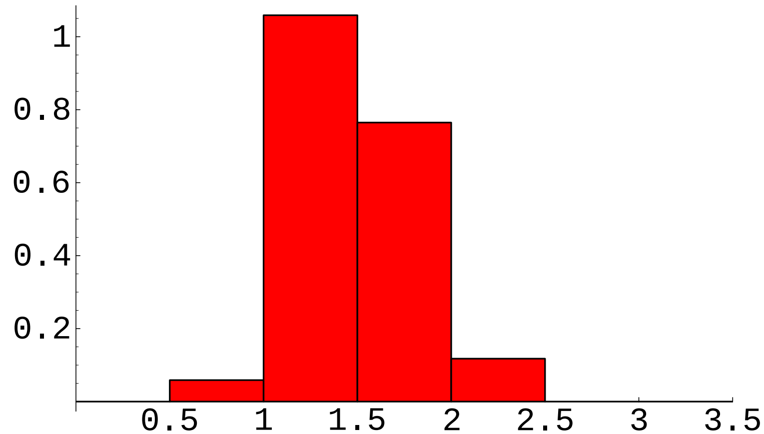


Figure $5b$:34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Repulsion or Attraction?

Conductors in $[15, 16]$; first set is rank 0 curves from 14 one-parameter families of rank 0 over $\mathbb{Q}$; second set rank 2 curves from 21 one-parameter families of rank 0 over $\mathbb{Q}$. The t-statistics exceed 6.

Family	**2nd vs 1st Zero**	**3rd vs 2nd Zero**	**Number**
Rank 0 Curves	2.16	3.41	863
Rank 2 Curves	1.93	3.27	701

The additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Comparison b/w One-Param Families of Different Rank

First normalized zero above the central point.

- The first family is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;
- the second family is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

Family	**Median**	**Mean**	**Std. Dev.**	**Number**
Rank 2 Curves from Rank 0 Families	1.926	1.936	0.388	701
Rank 2 Curves from Rank 2 Families	1.642	1.610	0.247	64

- t-statistic is 6.60, indicating the means differ.
- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on *how* we choose the curves.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- While the normalized zeros are repelled from the central point (and by different amounts for the two sets), the *differences* between the normalized zeros are statistically independent of this repulsion (t-statistics < 2).

- While for a given range of log-conductors the average second normalized zero of a rank 0 curve is close to the average first normalized zero of a rank 2 curve, they are not the same and the additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

PART V

CONCLUSIONS

Correspondences

Similarities b/w Nuclei and Primes:

Energy Levels $\longleftrightarrow$ Zeros of L-Functions

Neutron Energy $\longleftrightarrow$ Summation Lemmas (Test Fn Support)

Different Elements: U, Pu, ... $\longleftrightarrow$ Different L-Functions or Families

Summary

- Similar behavior in different systems.
- Find correct scale.
- Average over similar elements.
- Need a Trace Lemma.
- Thin subsets can exhibit very different behavior.

Open Problems

Real Symmetric Band Matrices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 & 0 & 0 \\ a_{1,2} & a_{2,2} & a_{2,3} & \cdots & 0 & 0 & 0 \\ 0 & a_{2,3} & a_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-2,N-2} & a_{N-2,N} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-2,N} & a_{N-1,N-1} & a_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-1,N} & a_{N,N} \end{pmatrix}$$

Real Symmetric Toeplitz Matrices

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_1 & b_0 & b_1 & \cdots & b_{N-2} \\ b_2 & b_1 & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & b_{N-3} & \cdots & b_0 \end{pmatrix}$$

Rates of Convergence