Generalized More-Sum-Than Difference Sets

Geoffrey Iyer[1], Liyang Zhang[2], Oleg Lazarev[3]

1. University of Michigan, 2. Williams College, 3. Princeton University

Advisor Steven J Miller (sjm1@williams.edu)

http://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

CANT 2012, CUNY Graduate Center

Motivation

We often care about the sum/difference of a set $A \subseteq \mathbb{Z}$.

Motivation

We often care about the sum/difference of a set $A \subseteq \mathbb{Z}$.

- Goldbach's Conjecture: $E \subseteq P + P$

Motivation

We often care about the sum/difference of a set $A \subseteq \mathbb{Z}$.

- Goldbach's Conjecture: $E \subseteq P + P$
- Fermat's Last Theorem: If A_n is the set of positive n-th powers, then $A_n + A_n \cap A_n = \emptyset$ for all $n \geq 3$

Motivation

We often care about the sum/difference of a set $A \subseteq \mathbb{Z}$.

- Goldbach's Conjecture: $E \subseteq P + P$
- Fermat's Last Theorem: If A_n is the set of positive n-th powers, then $A_n + A_n \cap A_n = \emptyset$ for all $n \geq 3$

Natural question: What are the sizes of the sum/difference sets?

Definitions

A finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

Definitions

A finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

Definition

Difference dominated: $|A - A| > |A + A|$
Balanced: $|A - A| = |A + A|$
Sum dominated (or MSTD): $|A + A| > |A - A|$.

History

What could cause a set to be sum-dominated?
Difference-dominated?

History

What could cause a set to be sum-dominated? Difference-dominated?

- $x + x = 2x$, but $x - x = 0$.

History

What could cause a set to be sum-dominated? Difference-dominated?

- $x + x = 2x$, but $x - x = 0$.
- $x + y = y + x$, but $x - y \neq y - x$.

History

What could cause a set to be sum-dominated? Difference-dominated?

- $x + x = 2x$, but $x - x = 0$.
- $x + y = y + x$, but $x - y \neq y - x$.

Nathanson, *Problems in Additive Number Theory*. "With the right way of counting the vast majority of sets satisfy $|A - A| > |A + A|$."

History

Martin-O'Bryant: If each set $A \subseteq [0, n-1]$ is equally likely, then a positive percentage of sets are sum-dominant in the limit. More precisely:

$$\lim_{n\to\infty} \frac{\#\{A \subseteq [0, n-1];\ A \text{ is sum-dominant}\}}{2^n} \approx 0.00045.$$

History

How is it possible for a positive percent of sets to be sum-dominant?

History

How is it possible for a positive percent of sets to be sum-dominant?

Martin-O'Bryant: We have the expected values

- $|A+A| \sim 2n-11$,
- $|A-A| \sim 2n-7$.

History

Say $A \subseteq [0, n-1]$, $x \in A+A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

History

Say $A \subseteq [0, n-1]$, $x \in A + A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

If x is near n there are many possibilities for a_1, a_2.

History

Say $A \subseteq [0, n-1]$, $x \in A + A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

If x is near n there are many possibilities for a_1, a_2.

With high probability, the middle will be full.

History

Say $A \subseteq [0, n-1]$, $x \in A + A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

If x is near n there are many possibilities for a_1, a_2.

With high probability, the middle will be full.

The trick is to control the fringes.

Definitions

We are concerned with adding sets, not multiplying them, therefore we define

$$kA = \underbrace{A + \cdots + A}_{\text{k times}}.$$

Definitions

We are concerned with adding sets, not multiplying them, therefore we define

$$kA \;=\; \underbrace{A + \cdots + A}_{\text{k times}}.$$

Furthermore, we are only working with integers, therefore

$$[a, b] \;=\; \{a, a+1, \ldots, b\}.$$

Questions

- Can we find a set A such that $|kA + kA| > |kA - kA|$?

Questions

- Can we find a set A such that $|kA + kA| > |kA - kA|$?
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$?

Questions

- Can we find a set A such that $|kA + kA| > |kA - kA|$?
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$?
- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k?

Questions

- Can we find a set A such that $|kA + kA| > |kA - kA|$? YES!
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$? YES!
- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k? NO!

Questions

- Can we find a set A such that $|kA + kA| > |kA - kA|$? YES!
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$? YES!
- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k? NO! (No such set exists.)

Initial Observations

Question: Can we find a set A such that $|kA + kA| > |kA - kA|$?

Initial Observations

Question: Can we find a set A such that
$|kA + kA| > |kA - kA|$?

- One set is enough to show a positive percentage.

Initial Observations

Question: Can we find a set A such that $|kA + kA| > |kA - kA|$?

- One set is enough to show a positive percentage.
- How do we find one set?

Initial Observations

Question: Can we find a set A such that $|kA + kA| > |kA - kA|$?

- One set is enough to show a positive percentage.
- How do we find one set?
- Computer simulations? We couldn't find a set for $k = 2$; the probability of finding some of these sets is less than 10^{-16}.

Initial Observations

Question: Can we find a set A such that $|kA+kA| > |kA-kA|$?

- One set is enough to show a positive percentage.
- How do we find one set?
- Computer simulations? We couldn't find a set for $k=2$; the probability of finding some of these sets is less than 10^{-16}.

If A is symmetric ($A = c - A$ for some c) then

$$|A+A| = |A+(c-A)| = |A-A|.$$

$|2A + 2A| > |2A - 2A|$

Example: $|2A + 2A| > |2A - 2A|$

$|2A+2A| > |2A-2A|$

Example: $|2A+2A| > |2A-2A|$

$$A = \{0,1,3,4,5,9\} \cup [33,56] \cup \{79,83,84,85,87,88,89\}$$

$|2A+2A| > |2A-2A|$

Example: $|2A+2A| > |2A-2A|$

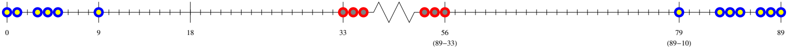
$$A \ = \ \{0,1,3,4,5,9\} \cup [33,56] \cup \{79,83,84,85,87,88,89\}$$

$|2A+2A| > |2A-2A|$

$$A + A = [0,9] \cup \{10,12,13,14,18\} \cup [33,145] \cup \{158,162,163,164,166,167\} \cup [168,178]$$

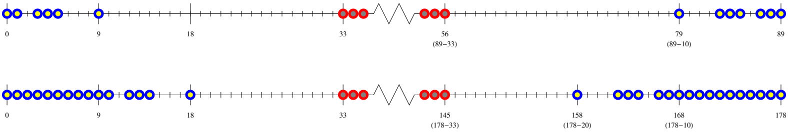
$|2A + 2A| > |2A - 2A|$

A



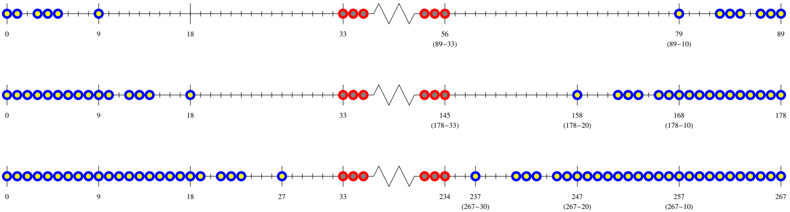
$|2A+2A| > |2A-2A|$

$A+A$


0
9
18
33
56
(89−33)
79
(89−10)
89
0
9
18
33
145
(178−33)
158
(178−20)
168
(178−10)
178

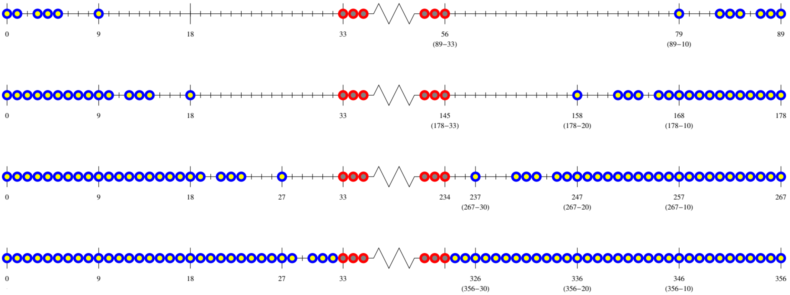
$|2A + 2A| > |2A - 2A|$

$A + A + A$



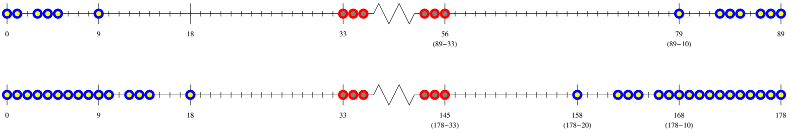
$|2A + 2A| > |2A - 2A|$

$A + A + A + A$



$|2A + 2A| > |2A - 2A|$

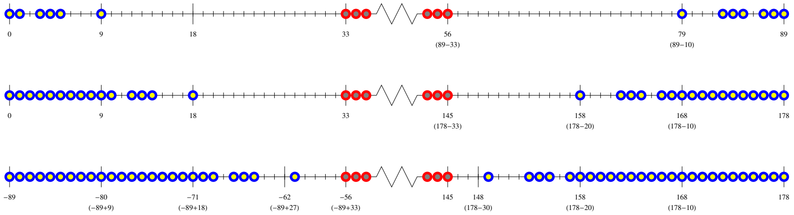
$A + A$



0
9
18
33
56
(89−33)
79
(89−10)
89
0
9
18
33
145
(178−33)
158
(178−20)
168
(178−10)
178

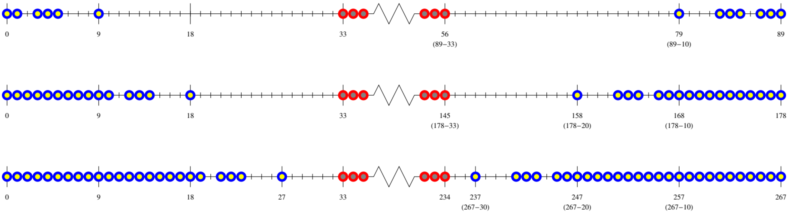
$|2A + 2A| > |2A - 2A|$

$A + A - A$



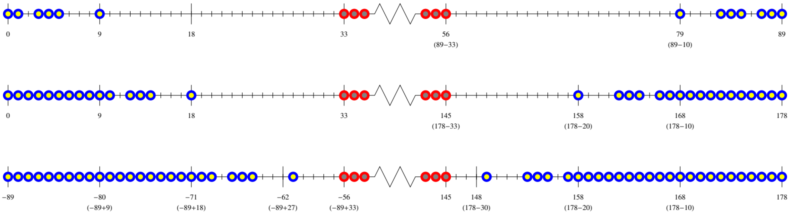
$|2A+2A| > |2A-2A|$

$A+A+A$



$|2A + 2A| > |2A - 2A|$

$A + A - A$



$|2A + 2A| > |2A - 2A|$

Say that L is the left fringe of A, R the right fringe.

$|2A+2A| > |2A-2A|$

Say that L is the left fringe of A, R the right fringe.

The left fringe of $A+A$ is $L+L$, the right fringe is $R+R$.

$|2A + 2A| > |2A - 2A|$

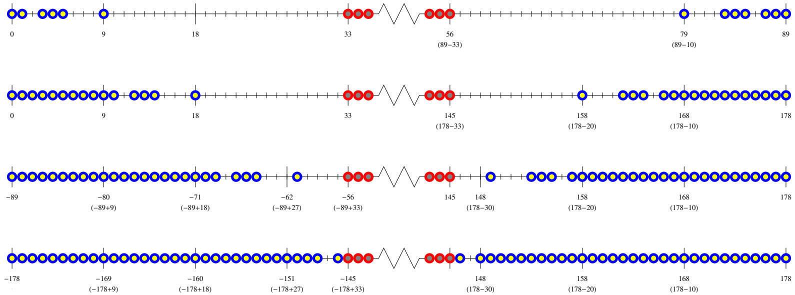
Say that L is the left fringe of A, R the right fringe.

The left fringe of $A + A$ is $L + L$, the right fringe is $R + R$.

The left fringe of $A - A$ is $L - R$, the right fringe is $R - L$.

$|2A + 2A| > |2A - 2A|$

$A + A - A - A$



Generalization

After dealing with some technical obstructions, we can generalize:

Generalization

After dealing with some technical obstructions, we can generalize:

For all nontrivial choices of s_1, d_1, s_2, d_2, $\exists\, A \subseteq \mathbb{Z}$ such that $|s_1A - d_1A| > |s_2A - d_2A|$.

Generalization

After dealing with some technical obstructions, we can generalize:

For all nontrivial choices of s_1, d_1, s_2, d_2, $\exists\, A \subseteq \mathbb{Z}$ such that $|s_1A - d_1A| > |s_2A - d_2A|$.

Example: We can have $|A+A+A+A| > |A+A+A-A|$:

$$A = \{0, 1, 3, 4, 5, 9, 33, 34, 35, 50, 54, 55, 56, 58, 59, 60\}$$

k-Generational Sets

Question: Does a set A exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$? If yes call it 2-generational.

k-Generational Sets

Question: Does a set A exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$? If yes call it 2-generational.

Equivalently, $A, 2A$ are sum-dominant.

k-Generational Sets

Question: Does a set A exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$? If yes call it 2-generational.

Equivalently, $A, 2A$ are sum-dominant.

More generally, A is k-generational if $|cA+cA| > |cA-cA|$ for all $1 \leq c \leq k$.

k-Generational Sets

Question: Does a set A exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$? If yes call it 2-generational.

k-Generational Sets

Question: Does a set A exist such that $|A + A| > |A - A|$ and $|A + A + A + A| > |A + A - A - A|$? If yes call it 2-generational.

Yes!

$$A = \{0, 1, 3, 4, 7, 26, 27, 29, 30, 33, 37, 38, 40, 41, 42, 43, 46, 49, 50, 52, 53, 54, 72, 75, 76, 78, 79, 80\}$$

In fact, we can find a k-generational set for all k.

k-Generational Sets

Idea of proof: We can find A_j such that $|jA_j + jA_j| > |jA_j - jA_j|$ for a specific $1 \leq j \leq k$.

k-Generational Sets

Idea of proof: We can find A_j such that $|jA_j + jA_j| > |jA_j - jA_j|$ for a specific $1 \leq j \leq k$.

Combine the A_j using the method of base expansion.

Base Expansion

Example: $A_1 = \{0, 1\}$, $A_2 = \{0, 2, 3, 4, 7, 11, 12, 14\}$

Base Expansion

Example: $A_1 = \{0, 1\}$, $A_2 = \{0, 2, 3, 4, 7, 11, 12, 14\}$

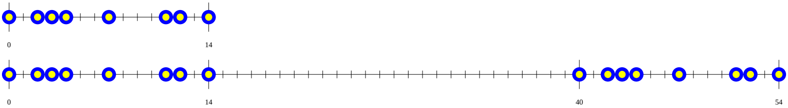


Figure : A_2 and $40 \cdot A_1 + A_2$

Base Expansion

$$A_1 = \{0, 1\},\ A_2 = \{0, 2, 3, 4, 7, 11, 12, 14\}$$

Figure : $A_2 + A_2$ and $(40 \cdot A_1 + A_2) + (40 \cdot A_1 + A_2)$

Base Expansion

$$A_1 = \{0, 1\},\ A_2 = \{0, 2, 3, 4, 7, 11, 12, 14\}$$

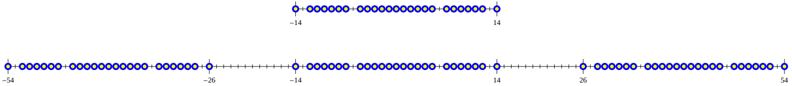


Figure : $A_2 - A_2$ and $(40 \cdot A_1 + A_2) - (40 \cdot A_1 + A_2)$

Base Expansion

Base Expansion: For sets A_1, A_2 and $m \in \mathbb{N}$ sufficiently large (relative to A_1, A_2) the set

$$A = m \cdot A_1 + A_2$$

behaves like the direct product $A_1 \times A_2 \subseteq \mathbb{Z} \times \mathbb{Z}$.

Base Expansion

Base Expansion: For sets A_1, A_2 and $m \in \mathbb{N}$ sufficiently large (relative to A_1, A_2) the set

$$A = m \cdot A_1 + A_2$$

behaves like the direct product $A_1 \times A_2 \subseteq \mathbb{Z} \times \mathbb{Z}$.
(here multiplication is the usual scalar multiplication)

Base Expansion

Base Expansion: For sets A_1, A_2 and $m \in \mathbb{N}$ sufficiently large (relative to A_1, A_2) the set

$$A = m \cdot A_1 + A_2$$

behaves like the direct product $A_1 \times A_2 \subseteq \mathbb{Z} \times \mathbb{Z}$.
(here multiplication is the usual scalar multiplication)

In particular:

$$|xA - yA| = |xA_1 - yA_1| \cdot |xA_2 - yA_2|$$

whenever $x + y$ is small relative to m.

Generalization

For nontrivial x_j, y_j, w_j, z_j $(2 \leq j \leq k)$, we can find an A such that $|x_jA - y_jA| > |w_jA - z_jA|$ for all j.

Generalization

For nontrivial x_j, y_j, w_j, z_j $(2 \leq j \leq k)$, we can find an A such that $|x_jA - y_jA| > |w_jA - z_jA|$ for all j.

Example: We can find an A such that

$$|A + A| > |A - A|$$
$$|A + A - A| > |A + A + A|$$
$$|5A - 2A| > |A - 6A|$$
$$\vdots$$
$$|1870A - 141A| > |1817A - 194A|$$

Limiting behavior of kA

Question: Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k?

Limiting behavior of kA

Question: Can we find a set A such that
$|kA + kA| > |kA - kA|$ for all k?

NO!

Limiting behavior of kA

Question: Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k?

NO! No such set exists!

Limiting behavior of kA

Question: Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k?

NO! No such set exists!

It turns out that all sets have a sort of limiting behavior.

Stabilizing Fringes

Example: $A = \{0, 3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 20\}$

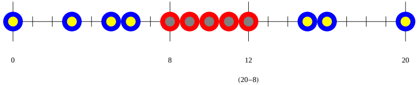


Figure : A

Stabilizing Fringes

Example: $A = \{0, 3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 20\}$

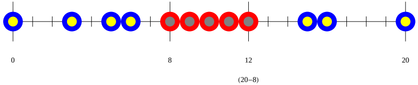


Figure : A

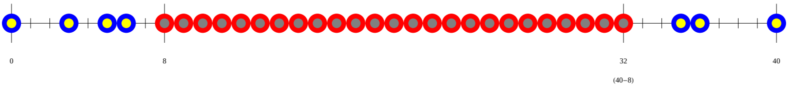


Figure : $A + A$

$|kA - kA|$ vs. $|kA + kA|$

Nathanson: For any set A, as k goes to infinity kA eventually becomes stabilized before k reaches $\max(A)^2 \cdot |A|$.

$|kA - kA|$ vs. $|kA + kA|$

Theorem

For any set A, as k goes to infinity kA eventually becomes stabilized before k reaches $\max(A)$.

$|kA - kA|$ vs. $|kA + kA|$

Theorem

For any set A, as k goes to infinity kA eventually becomes stabilized before k reaches $\max(A)$. Furthermore kA will become difference-dominated or balanced k reaches $2 \cdot \max(A)$.

$|kA - kA|$ vs. $|kA + kA|$

Theorem

For any set A, as k goes to infinity kA eventually becomes stabilized before k reaches max(A). Furthermore kA will become difference-dominated or balanced k reaches $2 \cdot \max(A)$.

Proof Idea:

- The middle will quickly become full, and the remaining fringes are finite.

$|kA - kA|$ vs. $|kA + kA|$

Theorem

For any set A, as k goes to infinity kA eventually becomes stabilized before k reaches $\max(A)$. Furthermore kA will become difference-dominated or balanced k reaches $2 \cdot \max(A)$.

Proof Idea:

- The middle will quickly become full, and the remaining fringes are finite.
- $kA \subseteq kA - kA$. Any sum can eventually be written as a difference.

$|kA - kA|$ vs. $|kA + kA|$

Theorem

For any set A, as k goes to infinity kA eventually becomes stabilized before k reaches $\max(A)$. Furthermore kA will become difference-dominated or balanced k reaches $2 \cdot \max(A)$.

Proof Idea:

- The middle will quickly become full, and the remaining fringes are finite.
- $kA \subseteq kA - kA$. Any sum can eventually be written as a difference.

Because the form stabilizes, this means $kA - kA \supseteq kA + kA$ when k large.

Other Results

Arbitrary Differences: We can create sets A where

$$|kA + kA| - |kA - kA| = m.$$

Other Results

Arbitrary Differences: We can create sets A where

$$|kA + kA| - |kA - kA| = m.$$

More generally, $|s_1 A - d_1 A| - |s_2 A - d_2 A| = m$.

Other Results

Arbitrary Differences: We can create sets A where

$$|kA + kA| - |kA - kA| = m.$$

More generally, $|s_1 A - d_1 A| - |s_2 A - d_2 A| = m$.

Simultaneous Comparison: We can create a set A where

$$|4A| > |3A - A| > |2A - 2A|.$$

Other Results

Arbitrary Differences: We can create sets A where

$$|kA + kA| - |kA - kA| = m.$$

More generally, $|s_1 A - d_1 A| - |s_2 A - d_2 A| = m$.

Simultaneous Comparison: We can create a set A where

$$|4A| > |3A - A| > |2A - 2A|.$$

More generally, any order and number of (nontrivial) comparisons.

Thanks

Thanks to:

- Williams College,

Thanks

Thanks to:

- Williams College,
- NSF Grant DMS0850577,

Thanks

Thanks to:

- Williams College,
- NSF Grant DMS0850577,
- CANT Coordinators.