Distribution of Missing Sums in Sumsets

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Key Question: What is the structure of $A + A$?

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Note: Both theorems can be more naturally stated in terms of missing sums (independent of n).

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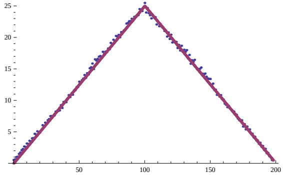

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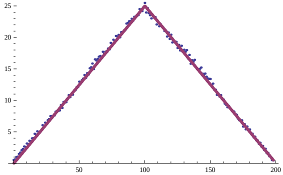

Figure: Comparison of predicted and observed number of representations of possible elements of the sumset.

- Key fact: if $k < n$, then $\mathrm{P}(k \notin A+A) \sim \left(\frac{3}{4}\right)^{k/2}$.

Results

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Theorem: Bounds on the distribution (Lazarev-Miller, 2011)

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Our main results are about $P(A : a_1, \cdots, \text{ and } a_m \notin A + A)$.
Main idea: Use graph theory.

More Results

Theorem: Variance (Lazarev-Miller)

$$\mathrm{Var}|A+A| = 4 \sum_{i<j\leq n-1} P(i \text{ and } j \notin A+A) - 40 \sim 35.98.$$

Theorem: Distribution of configurations (Lazarev-Miller)

For any fixed $a_1, \cdots, a_m$, exists $\lambda_{a_1,\cdots,a_m}$ such that

$$P(k+a_1, k+a_2, \cdots, \text{ and } k+a_m \notin A+A) = \Theta(\lambda^k_{a_1,\cdots,a_m}).$$

Theorem: Consecutive missing sums (Lazarev-Miller)

$$P(k, k+1, \cdots, \text{ and } k+m \notin A+A) = \left(\frac{1}{2}+o(1)\right)^{(k+m)/2}.$$

Bounds on the Distribution

Bound on Distribution: Lower Bound

Lower bound: $P(A + A \text{ has } k \text{ missing sums}) > 0.01 \cdot 0.70^k$
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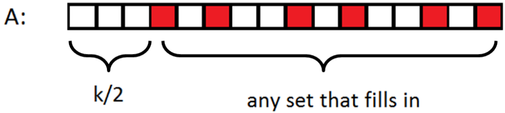

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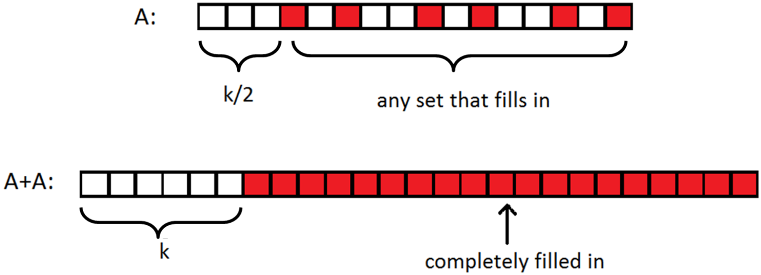

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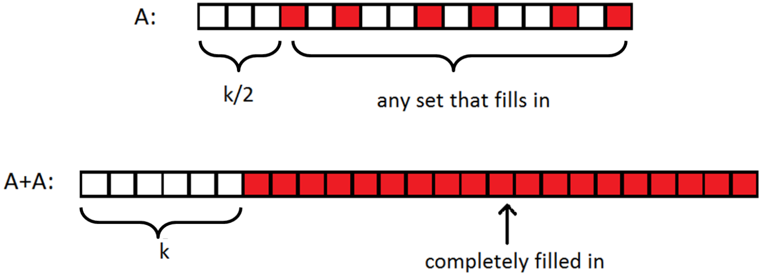

$$P(A + A \text{ has } k \text{ missing sums}) > 0.01\left(\frac{1}{2}\right)^{k/2} \sim 0.01 \cdot 0.70^k.$$

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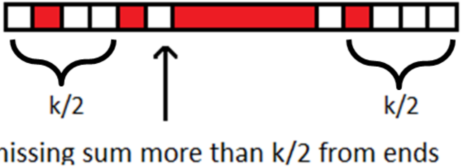

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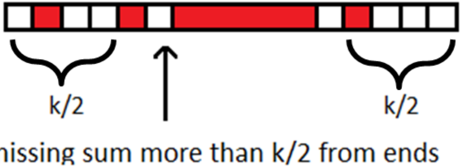

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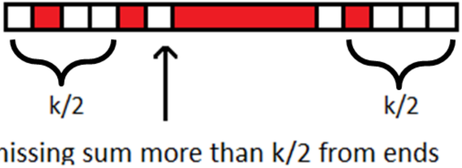

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Note: Bounds on $P(k + a_1, k + a_2, \cdots, \text{ and } k + a_m \notin A + A)$ yield upper bounds on $P(A + A \text{ has } k \text{ missing sums})$.

Variance

Problem: Dependent Random Variables

Variances reduces to $\sum_{0 \le i,j \le 2n-2} \mathrm{P}(A : i \text{ and } j \notin A + A)$.

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Example: $P(A : 3 \text{ and } 7 \notin A+A)$

- Conditions:

$i = 3$: 0 or $3 \notin A$ and 1 or $2 \notin A$

$j = 7$: 0 or $7 \notin A$ and 1 or $6 \notin A$ and 2 or $5 \notin A$ and 3 or $4 \notin A$.

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- Since there are common integers in both lists, the events $3 \notin A+A$ and $7 \notin A+A$ are dependent.

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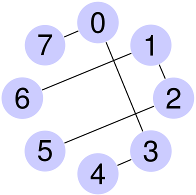

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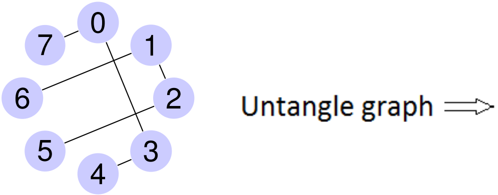

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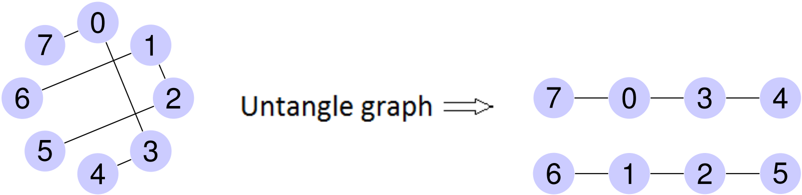

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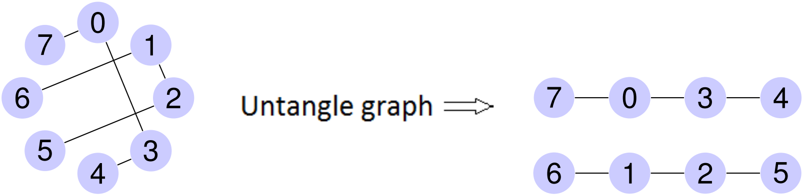

- One-to-one correspondence between conditions/edges (and integers/vertices).

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- Need to pick integers so that each condition is satisfied.
- Therefore, need to pick vertices so that each edge has a vertex chosen.
- So need to pick a vertex cover!

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If $7, 0, 4, 6, 2 \notin A$, then $3, 7 \notin A + A$

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If $7, 0, 4, 6, 2 \notin A$, then $3, 7 \notin A + A$

Lemma (Lazarev-Miller)

$$P(i, j \notin A + A) = P(\text{pick a vertex cover for graph}).$$

Number of Vertex Covers

Condition graphs are always ‘segment’ graphs. So we just need $g(n)$, the number of vertex covers for a ‘segment’ graph with n vertices.

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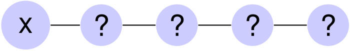

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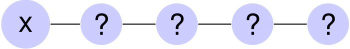

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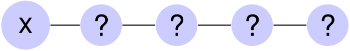

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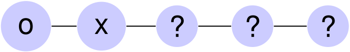

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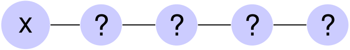

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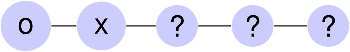

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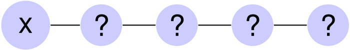

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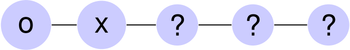

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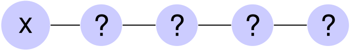

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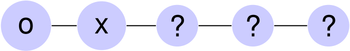

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$$\Longrightarrow g(n) = F_{n+2}$$

General i, j

- In particular

$$P(3 \text{ and } 7 \notin A + A) = \frac{1}{2^8} F_{4+2} F_{4+2} = \frac{1}{4}$$

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- In general $P(k \text{ and } k+1 \notin A+A) < C(\phi/2)^k \sim 0.81^k$, giving upper bound.

Variance Formula

$$Var|A+A| = -40 + 4 \sum_{i<j<n} P(i,j \notin A+A)$$

$$= -40 + O(c^n)$$

$$+4 \sum_{i,j\ odd} \frac{1}{2^{j+1}} F_{2\lceil \frac{i+1}{j-i} \rceil +2}^{\frac{1}{2}\left((j-i)\lceil \frac{i+1}{j-i}\rceil -(i+1)\right)} F_{2\lceil \frac{i+1}{j-i}\rceil +4}^{\frac{1}{2}\left(j+1-(j-i)\lceil \frac{i+1}{j-i}\rceil\right)}$$

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So clearly

$$Var|A+A| \sim 35.98.$$

Consecutive Missing Sums

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- Will study the particular case of $P(a_1, \cdots, a_j \notin A + A)$ of consecutive missing sums: $P(k, k+1, \cdots, k+i \notin A+A)$.

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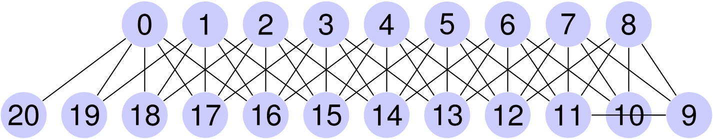

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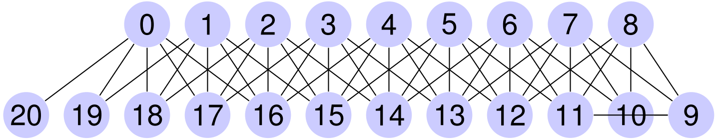

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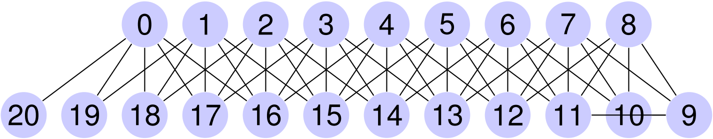

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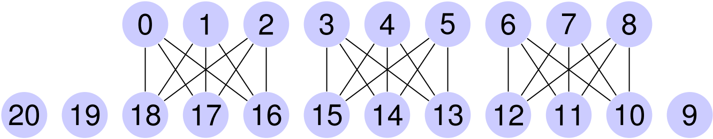

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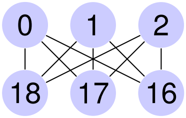
- So have 3 complete bipartite graphs like:



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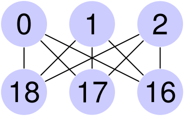

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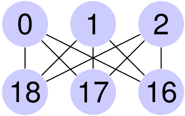

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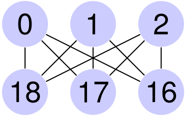

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- In general, $P(k, k+1, k+2, k+3, k+4) \leq \left(\frac{1}{4}\right)^{(k+4)/6} \sim 0.79^{k+4}$.

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- Why interesting? Bounds almost match!
- Essentially the only way to miss a block of i consecutive sums is to miss all elements before the block as well.

Summary

Use graph theory to study $P(a_1, \cdots, \text{ and } a_m \notin A + A)$.

Currently investigating:

- Is distribution of missing sums approximately exponential?
- Higher moments: third moment involves $P(i, j, k \notin A + A)$, with more complicated graphs.
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Thank you!

CANT 2012

Williams