Classification of All Crescent Configurations on Four and Five Points

Rebecca F. Durst, Max Hlavacek, Chi Huynh
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rfd1@williams.edu, *nhuynh30@gatech.edu*,
mhlavacek@g.hmc.edu

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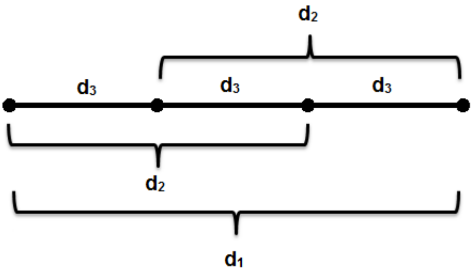
Motivation

- **Erdős' Distinct Distances Problem**: Starting with n points, what is the minimum number of distinct distances determined by these points?

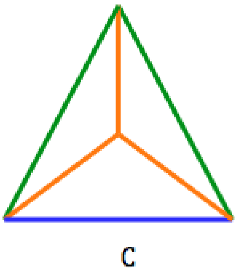
- **Distances of specified multiplicities:** Given a set of $n-1$ distinct distances, can n points be arranged such that for each $1 \leq i \leq n-1$, there is exactly one of $n-1$ distances occuring i times?

Motivation

Construction on $n = 4$ with no restrictions:



What if we impose restrictions to avoid "uninteresting" cases?



Crescent Configurations

Crescent Configuration (Burt et. al. 2015): We say n points are in crescent configuration (in $\mathbb{R}^d$) if they lie in *general position in* $\mathbb{R}^d$ and determine $n-1$ *distinct distances*, such that for every $1 \leq i \leq n-1$ there is a distance that occurs exactly i times.

General Position: We say that n points are in general position in $\mathbb{R}^d$ if no d+1 points lie on the same hyperplane and no d+2 lie on the same hypersphere.

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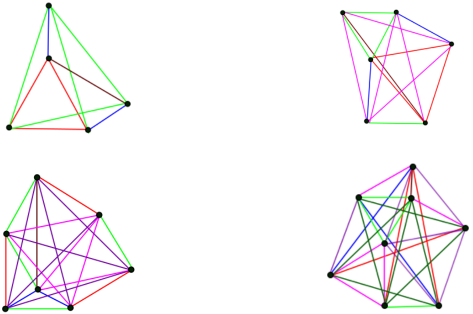
Erdős' Conjecture (1989): There exists N sufficiently large such that no crescent configuration exists on N points.

Constructions for $n = 5, 6, 7$ and 8

Due to Erdős, Pomerance and Palásti (1989)



The Approach

- **Distance Coordinate:** Given a set of points $\mathcal{P}$, the distance coordinate, D_A, of a point $A \in \mathcal{P}$ is the set of all distances, counting multiplicity, between A and the other points in $\mathcal{P}$.

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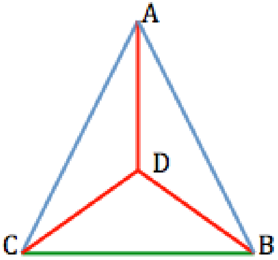


(1)

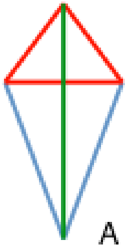
$$\begin{aligned} D_A &= \{d_2, d_2, d_3\}; \\ D_B &= \{d_1, d_2, d_3\}; \\ D_C &= \{d_1, d_2, d_3\}; \\ D_D &= \{d_3, d_3, d_3\}; \\ \mathcal{D} &= \{D_A, D_B, D_C, D_D\}. \end{aligned}$$

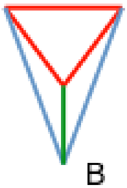
Graph Isomorphism of Crescent Configurations

Theorem (Durst-Hlavacek-Huynh-Miller-Palsson 2016)

Let A and B be two crescent configurations on the same number of points n. If A and B have the same distance sets, then there exists a graph isomorphism $A \to B$.

Graph Isomorphism Example





$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ d_1 & d_2 & 0 & d_2 \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ d_2 & d_1 & d_2 & 0 \end{pmatrix}$$

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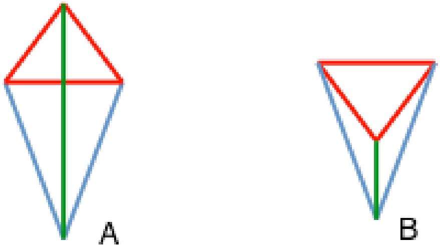

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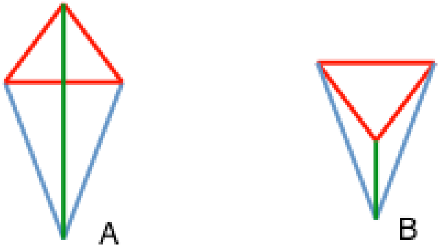

$$\begin{pmatrix} 0 & d_3 & d_1 & d_3 \\ d_3 & 0 & d_2 & d_3 \\ \color{red}{d_1} & \color{red}{d_2} & \color{red}{0} & \color{red}{d_2} \\ d_3 & d_3 & d_2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & d_3 & d_3 & d_2 \\ d_3 & 0 & d_3 & d_1 \\ d_3 & d_3 & 0 & d_2 \\ \color{red}{d_2} & \color{red}{d_1} & \color{red}{d_2} & \color{red}{0} \end{pmatrix}$$

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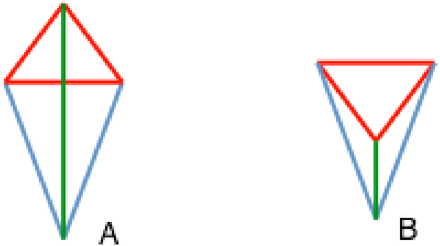

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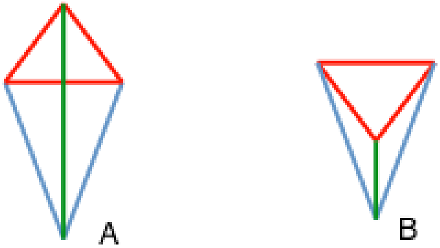

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Initial Result

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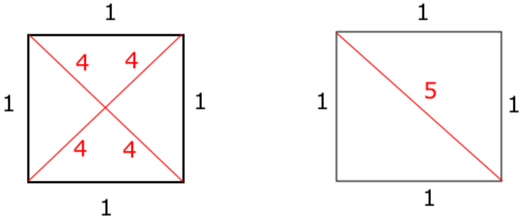
- For a set of 3 distinct distances on 4 points: Generation of 60 possible adjacency matrices $\xrightarrow[isomorphism]{graph}$ decrease to 4 potential configuration classes
- For a set of 4 distinct distances on 5 points: Generation of $12,600$ adjacency matrices $\xrightarrow[isomorphism]{graph}$ decrease to 85 potential configuration classes.

Which candidate is geometrically realizable?

The Question of Geometric Realizability

- **Distance Geometry Problem:** If we are given a set of distances between points, what can we find out about the relative position of these points?



Cayley-Menger Matrices

Cayley-Menger Matrix: The Cayley-Menger matrix for a set of n points $\{P_1, P_2, \dots P_n\}$ is an $(n+1) \times (n+1)$ matrix of the following form:

$$\begin{pmatrix} 0 & d_{1,2}^2 & \dots & d_{1,n}^2 & 1 \\ d_{2,1}^2 & 0 & \dots & d_{2,n}^2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{n,1}^2 & d_{n,2}^2 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

where $d_{i,j}$ is the distance between P_i and P_j.

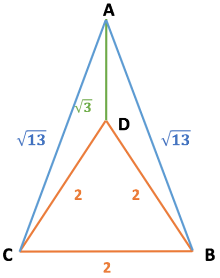
Cayley-Menger and Geometric realizability

Theorem (Sommerville 1958)

A distance set corresponding to 4 points is geometrically realizable in $\mathbb{R}^2$ if and only if the Cayley-Menger matrix is not invertible.

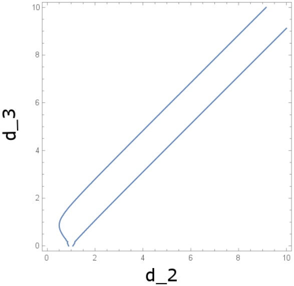
Example

$$
\begin{array}{c c}
 & \begin{array}{ccccc} A & B & C & D & \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ \end{array} &
\begin{pmatrix}
0 & 13 & 13 & 3 & 1 \\
13 & 0 & 4 & 4 & 1 \\
13 & 4 & 0 & 4 & 1 \\
3 & 4 & 4 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$



Solutions for a Given Crescent Configuration Type

- Suppose we are given a distance set with the multiplicities of the distances specified, but we are not given values for the distances.
- We can fix one of the unknown distances and use Cayley-Menger determinants to find a system of equations that yields geometrically realizable distances.



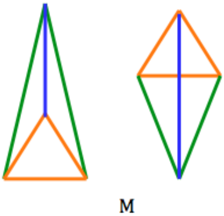


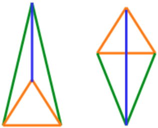
Figure: Possible values for d_2, d_3 for the M-type when $d_1 = 1$

All Configurations on Four and Five Points

Theorem (Durst-Hlavacek-Huynh-Miller-Palsson 2016)

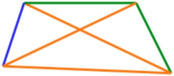
Given a set of three distinct distances, $\{d_1, d_2, d_3\}$, on four points , there are only three allowable crescent configurations up to graph isomorphism.

- We label these M-type, C-type, and R-type, respectively.



M

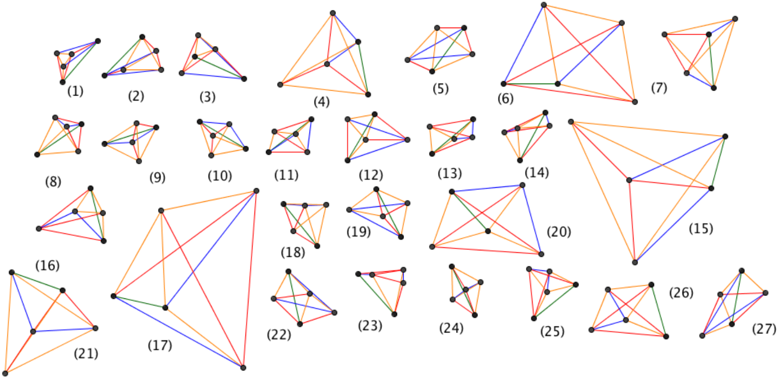
C



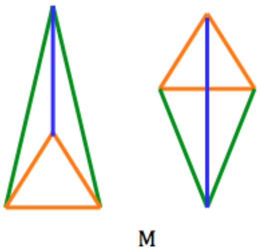
R

Theorem (Durst-Hlavacek-Huynh-Miller-Palsson 2016)

Given a set of four distinct distances, $\{d_1, d_2, d_3, d_4\}$, on five points , there are only 27 allowable crescent configurations up to graph isomorphism.



The Uniqueness Question



Given a particular isomorphism class of crescent configurations on n points, how many realizations of the associated distance set could we construct?

Inspiration from the Molecule Problem

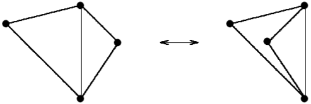


Figure: Two Realizations of a Flexible Graph[1]

- The Molecule Problem: given a set of distance measurements between points in Euclidean space, can we find the appropriate realization? $\rightarrow$ NP-hard
- More generally: Graph realization (how many arrangements?) and rigidity (can we distort the arrangements?)

[1] B. Hendrickson. Conditions for Unique Graph Realization. SIAM Journal of Computing . 21(1). 64–84, Feb. 1992

Laman's Condition for Graph Rigidity

Definition (Graph rigidity - Asimow and Roth 1978)

Let G be a graph (V, E) on v vertices in $\mathbb{R}^n$ then $G(p)$ is G together with the point $p = (p_1, p_2, \ldots, p_v) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ldots \mathbb{R}^n = \mathbb{R}^{nv}$. Let K be the complete graph on v vertices. The graph $G(p)$ is *rigid* in $\mathbb{R}^n$ if there exists a neighbordhood $\mathbf{U}$ of p such that

$$e_K^{-1}(e_K(p)) \cap \mathbf{U} = e_G^{-1}(e_G(p)) \cap \mathbf{U},$$

where e_K and e_G are the edge functions of K and G, which return the distances of edges of the associated graphs.

Laman's Condition (1970)

A graph with $2n - 3$ edges is rigid in two dimensions if and only if no subgraph G' has more than $2n' - 3$ edges.

Crescent Configurations are Rigid

- For each n, any crescent configurations on n points is a complete graph $\rightarrow$ It suffices to show that K_n is rigid for every $n \in \mathbb{N}$

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However, does the rigidity ranking differ between configurations?

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- **Frameworks**: Flexible vs. Rigid vs. Redundantly Rigid

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- **The Rigidity Matrix**: Given a graph G with m vertices and n edges, the d-dimensional rigidity matrix M_G is a $n \times dm$ matrix such that:
  - The columns are indexed by the entries in the coordinate of each vertex $v = (v_1, v_2, \ldots, v_d)$.
  - The rows are indexed by the edges
  - The entry in row e and column v_i is:

$$\begin{cases} v_i - w_i \text{ if } e = vw \text{ is incident to } v \\ 0 \qquad \text{otherwise} \end{cases}$$

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- **Example**: K_3 with $v_1 = (0,1)$, $v_2 = (-1,0)$ and $v_3 = (1,0)$

$$\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 & 2 & 0 \end{bmatrix}$$

Theorem (Hendrickson 1992)

A framework $f(G)$ is rigid if and only if its rigidity matrix has rank exactly equal to $S(n, d)$, which is the number of allowed motions, where:

$$S(n,d) = \begin{cases} nd - \frac{d(d+1)}{2} & \text{for } n \geq d \\ \frac{n(n-1)}{2} & \text{otherwise} \end{cases}$$

Note: $S(n, d)$ can also be used to determine whether a graph is *redundantly rigid*, which in turn can be used to determine if there exists a unique realization

Type R Realization

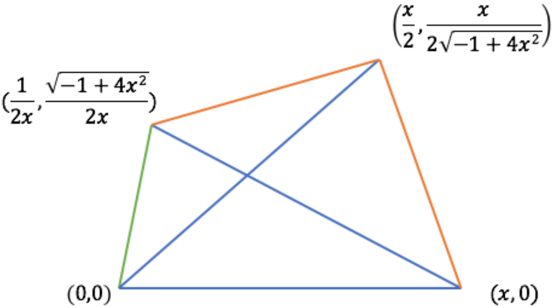

Figure: Realization obtained by fixing $d_1 = 1$

Rigidity Analysis for Type R

Letting $y = \sqrt{-1+4x^2}$, we get the rigidity matrix A_R:

$$\begin{bmatrix} -x & 0 & x & 0 & 0 & 0 & 0 & 0 \\ \frac{-x}{2} & \frac{-x}{y} & 0 & 0 & \frac{x}{2} & \frac{x}{y} & 0 & 0 \\ \frac{-1}{2x} & \frac{-y}{2x} & 0 & 0 & 0 & 0 & \frac{1}{2x} & \frac{y}{2x} \\ 0 & 0 & x-\frac{x}{2} & \frac{-x}{2y} & -x+\frac{x}{2} & \frac{x}{2y} & 0 & 0 \\ 0 & 0 & x-\frac{1}{2x} & \frac{-y}{2x} & 0 & 0 & -x+\frac{1}{2x} & \frac{y}{2x} \\ 0 & 0 & 0 & 0 & \frac{x}{2}-\frac{1}{2x} & \frac{x}{2y}-\frac{y}{2x} & \frac{-x}{2}+\frac{1}{2x} & \frac{-x}{2y}+\frac{y}{2x} \end{bmatrix}$$

$\text{Rank}(A_R) = 6 > S(4,2)$ but when removing any row, rank of remaining matrix is $5 \to$ redundantly rigid

Type M Realizations

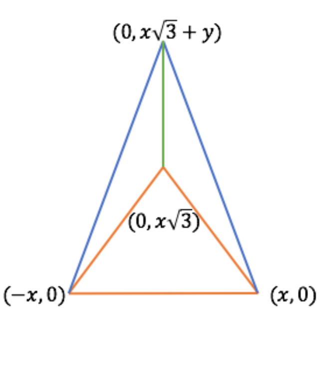

Figure: Two Realizations of Type M: M_1 and M_2

Rigidity Analysis for Type M

Rigidity matrix A_{M_1}

$$\begin{bmatrix} -2x & 0 & 2x & 0 & 0 & 0 & 0 & 0 \\ -x & -x\sqrt{3} & 0 & 0 & x & x\sqrt{3} & 0 & 0 \\ -x & -x\sqrt{3}-y & 0 & 0 & 0 & 0 & x & x\sqrt{3}+y \\ 0 & 0 & x & -x\sqrt{3} & -x & x\sqrt{3} & 0 & 0 \\ 0 & 0 & x & -x\sqrt{3}-y & 0 & 0 & -x & x\sqrt{3}+y \\ 0 & 0 & 0 & 0 & 0 & -y & 0 & y \end{bmatrix}$$

$\text{Rank}(A_{M_1}) = 5 = S(4,2) \rightarrow \text{rigid}$
Same results for M_2

Questions to explore

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- Which distance sets can be realized in higher dimensions?
- In addition to rigidity, which other properties of crescent configurations can we explore?
- Given a rigidity ranking, can we use the rigidity matrix to generate a crescent configuration?

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