Elliptic Surfaces & Fibrations
AMS Sectional Meeting
(Lawrenceville, NJ)

The effect of zeros of elliptic curve L-functions at the central point on nearby zeros

Steven J. Miller

The Ohio State University

(Joint with Eduardo Dueñez and Aaron Lint)

April 18th, 2004
http://www.math.ohio-state.edu
/∼sjmiller/math/talks/talks.html

Collaborators

Theory

- Eduardo Dueñez

Programs

- Jon Hsu
- Leo Goldmakher
- Aaron Lint
- Stephen Lu
- Atul Pokharel
- Michael Rubinstein

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Info by shooting high-energy neutrons into nucleus.

Fundamental Equation:Quantum Mechanics

$$H\psi_n \;=\; E_n\psi_n$$

Similar to stat mech, leads to considering eigenvalues of ensembles of matrices.

Real Symmetric (GOE), Complex Hermitian (GUE), Classical Compact Groups.

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ be an increasing sequence of numbers, $B \subset \mathbf{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1} - \alpha_{j_2}, \dots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k \leq N\right\}}{N}$$

Results on Zeros (Assuming GRH):

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak)
4. n-level correlations for the classical compact groups (Katz-Sarnak)
5. **insensitive to any finite set of zeros**

Measures of Spacings: n-Level Density and Families

Let $\phi(x) = \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions, $\widehat{\phi}$ compactly supported.

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

$L_f =$ Conductor, Scale factor for low zeros.

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

$$D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} D_{n,f}(\phi).$$

Limiting Behavior

As $N \to \infty$,

$$\begin{aligned} & \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{1,f}(\phi) \\ = \ & \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_f^{(j)} \log L_f}{2\pi}\right) \\ \to \ & \int \phi(x) W_{1,\mathcal{G}(\mathcal{F})}(x) dx \\ \to \ & \int \widehat{\phi}(y) \widehat{W}_{1,\mathcal{G}(\mathcal{F})}(y) dy. \end{aligned}$$

Conj: Distribution of Low Zeros agrees with a classical compact group.

1-Level Densities

Fourier Transforms for 1-level densities:

$$\begin{aligned}
\widehat{W}_{1,SO(\text{even})}(u) &= \delta_0(u) + \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,SO}(u) &= \delta_0(u) + \tfrac{1}{2} \\
\widehat{W}_{1,SO(\text{odd})}(u) &= \delta_0(u) - \tfrac{1}{2}\eta(u) + 1 \\
\widehat{W}_{1,Sp}(u) &= \delta_0(u) - \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,U}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Elliptic Curves:

$$E_t : y^2 \ = \ x^3 + A(t)x + B(t), \ \ A(t), B(t) \in \mathbb{Z}(t).$$

$$a_t(p) \ = \ -\sum_{x \bmod p} \left(\frac{x^3 + A(t)x + B(t)}{p} \right) \ = \ a_{t+mp}(p)$$

$$L(E,s) \ = \ \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \ = \ \prod_p L_p(E,s).$$

By GRH: All zeros on the critical line.

Rational solutions: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$.

Birch and Swinnerton-Dyer Conjecture:
Geometric rank r equals the analytic rank (order of vanishing at central point).

Tools to Study Low Zeros

- explicit formula relating zeros and Fourier coeffs;
- averaging formulas for the family;
- conductors easy to control (constant or monotone)

1-Level Expansion

$$\begin{aligned} D_{1,\mathcal{F}}(\phi) &= \frac{1}{|\mathcal{F}|} \sum_{E\in\mathcal{F}} \sum_{j} \phi\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right) \\ &= \frac{1}{|\mathcal{F}|} \sum_{E\in\mathcal{F}} \widehat{\phi}(0) + \phi_i(0) \\ &\quad - \frac{2}{|\mathcal{F}|} \sum_{E\in\mathcal{F}} \sum_{p} \frac{\log p}{\log N_E} \frac{1}{p} \widehat{\phi}\left(\frac{\log p}{\log N_E}\right) a_E(p) \\ &\quad - \frac{2}{|\mathcal{F}|} \sum_{E\in\mathcal{F}} \sum_{p} \frac{\log p}{\log N_E} \frac{1}{p^2} \widehat{\phi}\left(2\frac{\log p}{\log N_E}\right) a_E^2(p) \\ &\quad + O\left(\frac{\log\log N_E}{\log N_E}\right) \end{aligned}$$

Want to move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$, Leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t \bmod p} a_t^r(p), \quad r = 1 \text{ or } 2.$$

Input

For many families

$$(1): A_{1,\mathcal{F}}(p) \ = \ -rp \ + \ O(1)$$
$$(2): A_{2,\mathcal{F}}(p) \ = \ p^2 \ + \ O(p^{3/2})$$

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(t)$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} -A_{1,\mathcal{F}}(p) \log p = r$$

Surfaces with $j(t)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p^2 + O\left(p^{3/2}\right).$$

One-Level Result

For small support, one-param family of rank r over $\mathbb{Q}(t)$:

$$\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{j}\phi\left(\frac{\log N_E}{2\pi}\gamma_E^{(j)}\right)$$
$$\longrightarrow \int \phi(x)W_{\mathcal{G}}(x)dx + r\phi(0),$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Forced zeros seem independent

Testing Random Matrix Theory Predictions

1. **Excess Rank:** In limit, 50% rank r, 50% rank $r + 1$.

2. **First (Normalized) Zero above Central Point:** Do extra zeros at the central point affect the distribution of zeros near the central point?

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(t)$.

RMT $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r = 32%
Percent with rank r+1 = 48%
Percent with rank r+2 = 18%
Percent with rank r+3 = 2%

Problem: small data sets, sub-families, convergence rate $\log$(conductor).

Data on Excess Rank

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a6$$

Family: $a_1 : 0$ to 10, rest -10 to 10.

Percent with rank 0 = 28.60%
Percent with rank 1 = 47.56%
Percent with rank 2 = 20.97%
Percent with rank 3 = 2.79%
Percent with rank 4 = .08%

14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).

Data on Excess Rank

$$y^2 + y = x^3 + tx.$$

Each data set 2000 curves from start.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{11}, but on logarithmic scale still small.

Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j$$

First Model:

The conditional eigenvalue measure for the sub-ensemble of $SO(2N)$ with the last $2n$ of the $2N$ eigenvalues equal $+1$ is

$$d\varepsilon_{2n}(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2n} \prod_j d\theta_j,$$

where $\theta = (\theta_1, \ldots, \theta_{N-n})$ and the indices j, k range from 1 to $N-n$.

Second Model:

$$\mathcal{A}_{2N,2n} = \left\{ \begin{pmatrix} g & \\ & I_{2n} \end{pmatrix} : g \in SO(2N-2n) \right\}$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2,\mathrm{Ind}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\mathrm{Int}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Testing RMT Models

For small support, 1-level densities for Elliptic Curves agree with $\rho_{r,\text{Indep}}$ and not $\rho_{r,\text{Interaction}}$.

Curve E, conductor N_E, expect first zero $\frac{1}{2} + i\gamma_E^{(1)}$ with $\gamma_E^{(1)} \approx \frac{1}{\log N_E}$.

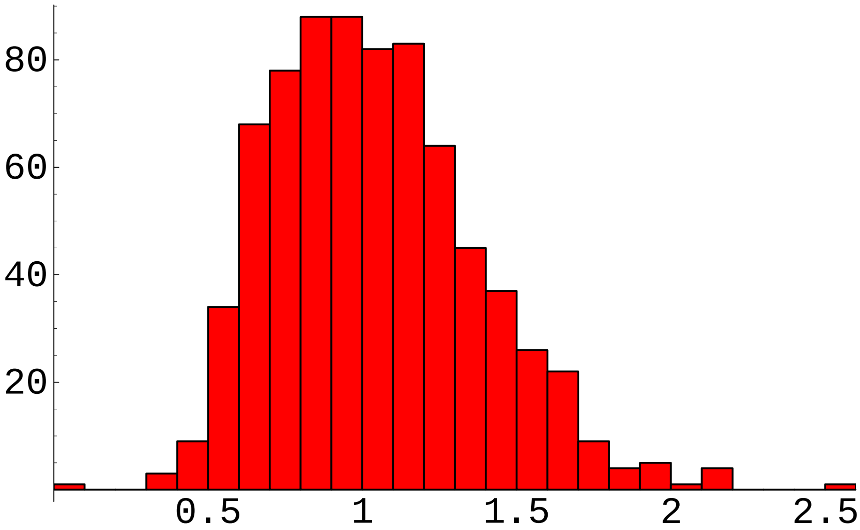
If r zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log N_E}$, might detect in 1-level density:

$$\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_E^{(j)} \log N_E}{2\pi}\right).$$

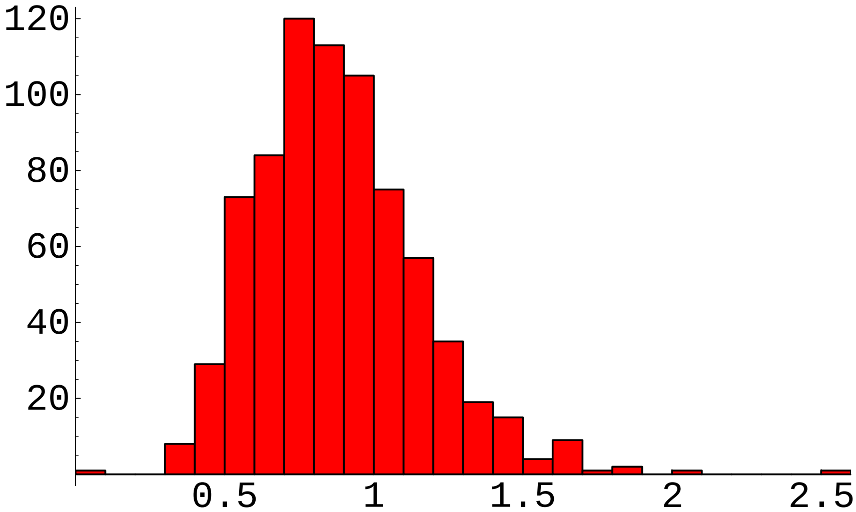
Corrections of size

$$\phi\left(x_0 + c_r\right) - \phi(x_0) \approx \phi'\left(x(x_0, c_r)\right) \cdot c_r.$$

Rank 0 Curves: 1st Normalized Zero

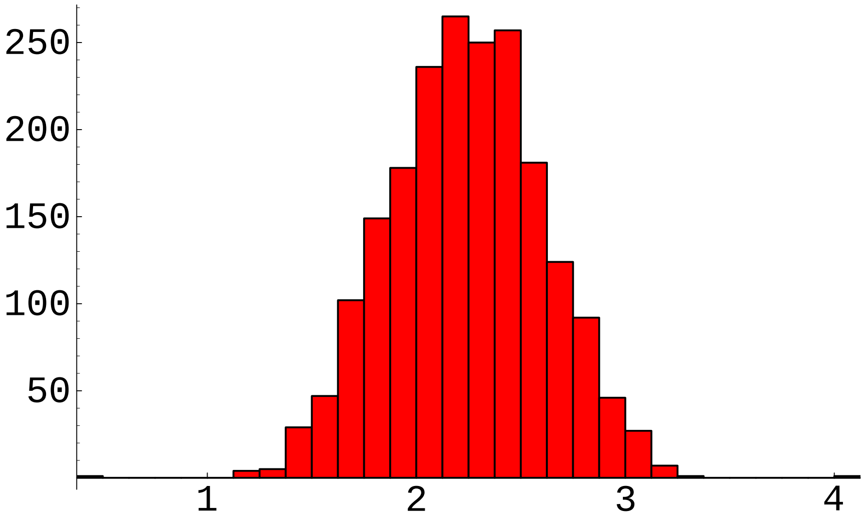


750curves, $\log($cond$) \in [3.2, 12.6]$; mean $= 1.04$

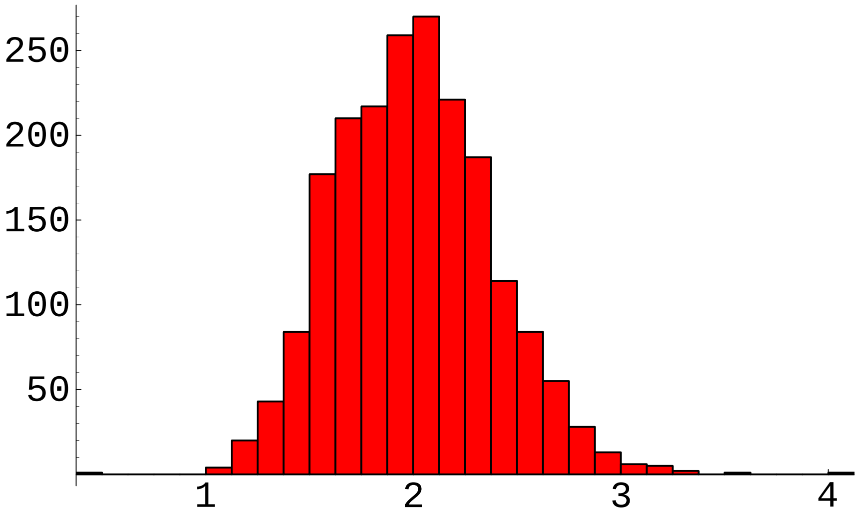


750curves, $\log($cond$) \in [12.6, 14.9]$; mean $= .88$

Rank 2 Curves: 1st Normalized Zero

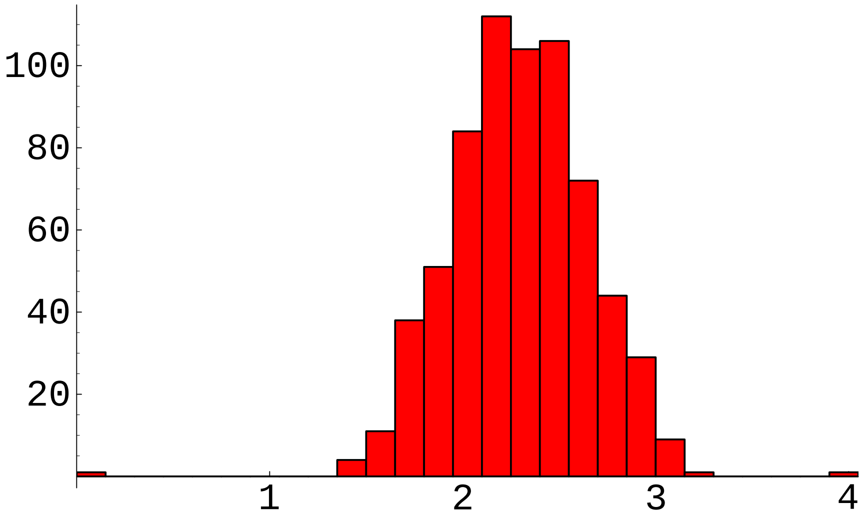


2000 **curves**, $\log(\textbf{cond}) \in [6.3, 12.1]$;
mean $= 2.24$

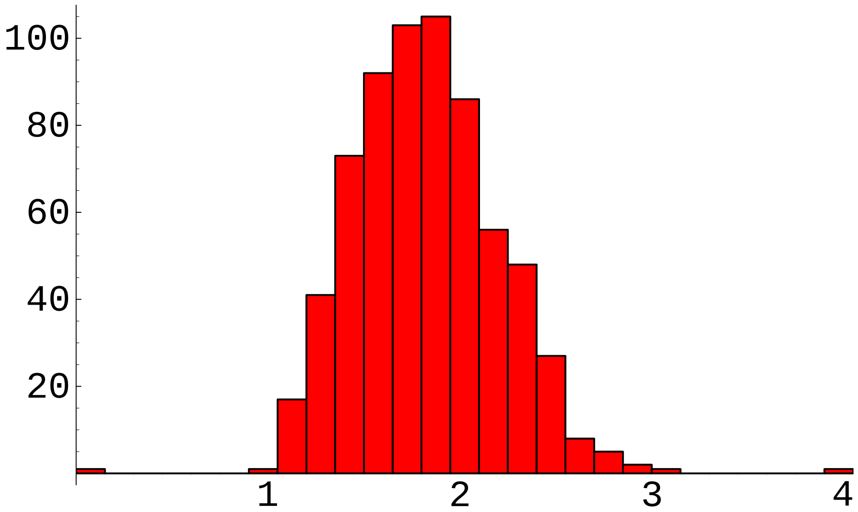


2000 **curves**, $\log(\textbf{cond}) \in [12.1, 15.0]$;
mean $= 2.00$

Rank 2 Curves: 1st Normalized Zero

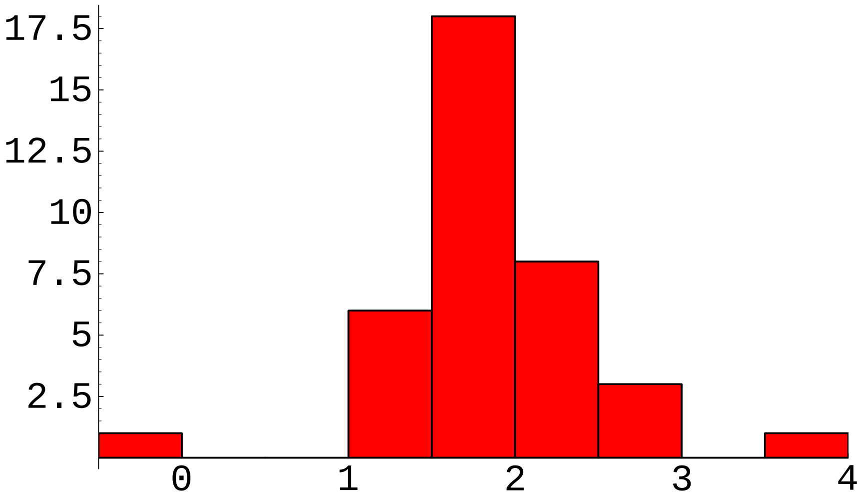


665 curves, $\log(\text{cond}) \in [10, 10.3125]$;
mean $= 2.30$

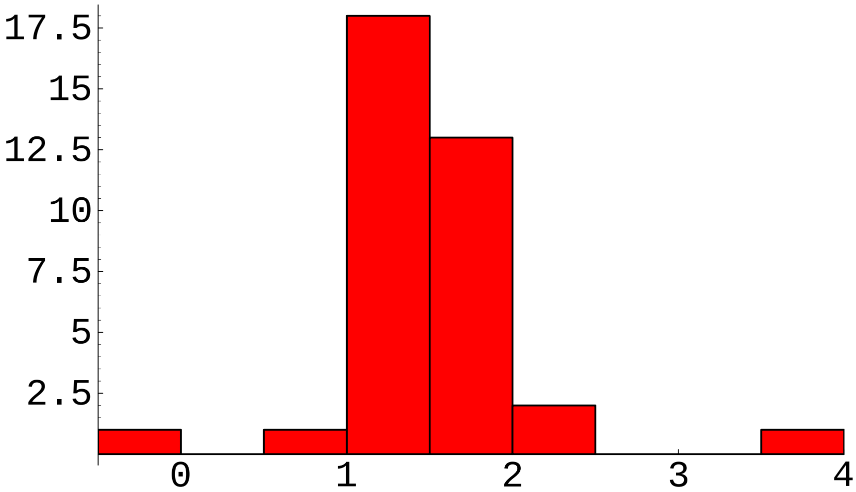


665 curves, $\log(\text{cond}) \in [16, 16.5]$;
mean $= 1.82$

Rank 2 Curves: $[0, 0, 0, -t^2, t^2]$
1st Normalized Zero



35 curves, $\log(\text{cond}) \in [7.8, 16.1]$; mean $= 2.24$



34 curves, $\log(\text{cond}) \in [16.2, 23.3]$; mean $= 2.00$

Points at -.25 and 3.75 added for formatting purposes.

Summary

- Evidence for B-SD, RMT interpretation of zeros.

- Theoretical results (n-level density) agrees with RMT model of forced zeros independent.

- Experimental results agrees with RMT model of forced zeros interact (repel); repulsion seems to decrease as conductors increase, could be artifact of small data.

- **Need more data.**

Appendices

The first appendix list various standard conjectures. The second appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor N_E, one needs about $\sqrt{N_E} \log N_E$ Fourier coefficients.

Appendix I: Standard Conjectures

Generalized Riemann Hypothesis (for Elliptic Curves) *Let* $L(s, E)$ *be the (normalized)* L*-function of the elliptic curve* E*. Then the non-trivial zeros of* $L(s, E)$ *satisfy* $Re(s) = \frac{1}{2}$.

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let* E *be an elliptic curve of geometric rank* r *over* $\mathbb{Q}$ *(the Mordell-Weil group is* $\mathbb{Z}^r \oplus T$*,* T *is the subset of torsion points). Then the analytic rank (the order of vanishing of the* L*-function at the central point) is also* r.

Tate's Conjecture for Elliptic Surfaces [Ta] *Let* $\mathcal{E}/\mathbb{Q}$ *be an elliptic surface and* $L_2(\mathcal{E}, s)$ *be the* L*-series attached to* $H^2_{\acute{e}t}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$*. Then* $L_2(\mathcal{E}, s)$ *has a meromorphic continuation to* **C** *and satisfies* $-\text{ord}_{s=2} L_2(\mathcal{E}, s) = \text{rank } NS(\mathcal{E}/\mathbb{Q})$*, where* $NS(\mathcal{E}/\mathbb{Q})$ *is the* $\mathbb{Q}$*-rational part of the Néron-Severi group of* $\mathcal{E}$*. Further,* $L_2(\mathcal{E}, s)$ *does not vanish on the line* $Re(s) = 2$.

Most of the 1-param families we investigate are rational surfaces, where Tate's conjecture is known. See [RSi].

Appendix II: Numerically Approximating Ranks: Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \quad \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use transformations.

Algorithm for $L^r(s, E)$: I

$$\begin{aligned}\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_1^\infty f(iy/\sqrt{N})(y^{s-1} + \epsilon y^{1-s})dy.\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1} + \epsilon(-1)^k y^{1-s})dy.$$

At $s = 1$,

$$\Lambda^{(k)}(E,1) = (1 + \epsilon(-1)^k)\int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s, E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy \\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi n y/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-r}}{n^r\cdot n!}x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t;\\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12};\\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3};\\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160};\\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\,e^{-2\pi ny/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

Bibliography

[BEW] B. Berndt, R. Evans and K. Williams, *Gauss and Jacobi Sums*, Canadian Mathematical Society Series of Monographs and Advanced Texts, vol. **21**, Wiley-Interscience Publications, John Wiley & Sons, Inc., New York, 1998.

[Bi] B. Birch, *How the number of points of an elliptic curve over a fixed prime field varies*, J. London Math. Soc. **43**, 1968, $57 - 60$.

[BS] B. Birch and N. Stephens, *The parity of the rank of the Mordell-Weil group*, Topology **5**, 1966, $295 - 299$.

[BSD1] B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. I*, J. reine angew. Math. **212**, 1963, $7 - 25$.

[BSD2] B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. reine angew. Math. **218**, 1965, $79 - 108$.

[BCDT] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over **Q**: wild 3-adic exercises*, J. Amer. Math. Soc. **14**, no. 4, 2001, $843 - 939$.

[Br] A. Brumer, *The average rank of elliptic curves I*, Invent. Math. **109**, 1992, $445 - 472$.

[BHB3] A. Brumer and R. Heath-Brown, *The average rank of elliptic curves III*, preprint.

[BHB5] A. Brumer and R. Heath-Brown, *The average rank of elliptic curves V*, preprint.

[BM] A. Brumer and O. McGuinness, *The behaviour of the Mordell-Weil group of elliptic curves*, Bull. AMS **23**, 1991, $375 - 382$.

[CW] J. Coates and A. Wiles, *On the conjecture of Birch and Swinnterton-Dyer*, Invent. Math. **39**, 1977, $43 - 67$.

[Cr] Cremona, *Algorithms for Modular Elliptic Curves*, Cambridge University Press, 1992.

[Di] F. Diamond, *On deformation rings and Hecke rings*, Ann. Math. **144**, 1996, $137 - 166$.

[Fe1] S. Fermigier, *Zéros des fonctions L de courbes elliptiques*, Exper. Math. **1**, 1992, $167 - 173$.

[Fe2] S. Fermigier, *Étude expérimentale du rang de familles de courbes elliptiques sur* $\mathbb{Q}$, Exper. Math. **5**, 1996, $119 - 130$.

[FP] E. Fouvrey and J. Pomykala, *Rang des courbes elliptiques et sommes d'exponentelles*, Monat. Math. **116**, 1993, $111 - 125$.

[GM] F. Gouvéa and B. Mazur, *The square-free sieve and the rank of elliptic curves*, J. Amer. Math. Soc. **4**, 1991, $45 - 65$.

[Go] D. Goldfeld, *Conjectures on elliptic curves over quadratic fields*, Number Theory (Proc. Conf. in Carbondale, 1979), Lecture Notes in Math. **751**, Springer-Verlag, 1979, $108 - 118$.

[Gr] Granville, *ABC Allows Us to Count Squarefrees*, International Mathematics Research Notices **19**, 1998, $991 - 1009$.

[He] H. Helfgott, *On the distribution of root numbers in families of elliptic curves*, preprint.

[Ho] C. Hooley, *Applications of Sieve Methods to the Theory of Numbers*, Cambridge University Press, Cambridge, 1976.

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91**, 2000, $55 - 131$.

[Kn] A. Knapp, *Elliptic Curves*, Princeton University Press, Princeton, 1992.

[KS1] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.

[KS2] N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36**, 1999, $1-26$.

[Ko] V. Kolyvagin, *On the Mordell-Weil group and the Shafarevich-Tate group of modular elliptic curves*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, $429-436$.

[Mai] L. Mai, *The analytic rank of a family of elliptic curves*, Canadian Journal of Mathematics **45**, 1993, $847-862$.

[Mes1] J. Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Mathematica **58**, 1986, $209-232$.

[Mes2] J. Mestre, *Courbes elliptiques de rang $\geq$ 11 sur* $\boldsymbol{Q}(t)$, C. R. Acad. Sci. Paris, ser. 1, **313**, 1991, $139-142$.

[Mes3] J. Mestre, *Courbes elliptiques de rang $\geq$ 12 sur* $\boldsymbol{Q}(t)$, C. R. Acad. Sci. Paris, ser. 1, **313**, 1991, $171-174$.

[Mi] P. Michel, *Rang moyen de familles de courbes elliptiques et lois de Sato-Tate*, Monat. Math. **120**, 1995, $127-136$.

[Mil] S. J. Miller, 1- *and* 2-*Level Densities for Families of Elliptic Curves: Evidence for the Underlying Group Symmetries*, P.H.D. Thesis, Princeton University, 2002, http://www.math.princeton.edu/∼sjmiller/thesis/thesis.pdf.

[Mor] Mordell, *Diophantine Equations*, Academic Press, New York, 1969.

[Na1] K. Nagao, *On the rank of elliptic curve* $y^2 = x^3 - kx$, Kobe J. Math. **11**, 1994, $205-210$.

[Na2] K. Nagao, *Construction of high-rank elliptic curves*, Kobe J. Math. **11**, 1994, $211-219$.

[Na3] K. Nagao, $\mathbb{Q}(t)$-*rank of elliptic curves and certain limit coming from the local points*, Manuscr. Math. **92**, 1997, $13-32$.

[Ri] Rizzo, *Average root numbers for a non-constant family of elliptic curves*, preprint.

[Ro] D. Rohrlich, *Variation of the root number in families of elliptic curves*, Compos. Math. **87**, 1993, $119 - 151$.

[RSi] M. Rosen and J. Silverman, *On the rank of an elliptic surface*, Invent. Math. **133**, 1998, $43 - 67$.

[Ru] M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L-functions*, P.H.D. Thesis, Princeton University, 1998, http://www.ma.utexas.edu/users/miker/thesis/thesis.html.

[RS] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Journal of Math. **81**, 1996, $269 - 322$.

[Sh] T. Shioda, *Construction of elliptic curves with high-rank via the invariants of the Weyl groups*, J. Math. Soc. Japan **43**, 1991, $673 - 719$.

[Si1] J. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **106**, Springer-Verlag, Berlin - New York, 1986.

[Si2] J. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **151**, Springer-Verlag, Berlin - New York, 1994.

[Si3] J. Silverman, *The average rank of an algebraic family of elliptic curves*, J. reine angew. Math. **504**, 1998, $227 - 236$.

[St1] N. Stephens, *A corollary to a conjecture of Birch and Swinnerton-Dyer*, J. London Math. Soc. **43**, 1968, $146 - 148$.

[St2] N. Stephens, *The diophantine equation $X^3 + Y^3 = DZ^3$ and the conjectures of Birch and Swinnerton-Dyer*, J. reine angew. Math. **231**, 1968, $16 - 162$.

[ST] C. Stewart and J. Top, *On ranks of twists of elliptic curves and power-free values of binary forms*, Journal of the American Mathematical Society **40**, number 4, 1995.

[Ta] J. Tate, *Algebraic cycles and the pole of zeta functions*, Arithmetical Algebraic Geometry, Harper and Row, New York, 1965, $93 - 110$.

[TW] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. Math. **141**, 1995, $553 - 572$.

[Wa] L. Washington, *Class numbers of the simplest cubic fields*, Math. Comp. **48**, number 177, 1987, $371-384$.

[Wi] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. Math **141**, 1995, $443-551$.