Limiting Behavior in Missing Sums of Sumsets

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Introduction

Given $A \subseteq \mathbb{Z}$, define its sumset

- $A + A := \{a_1 + a_2 \mid a_1, a_2 \in A\}$.

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Sumsets are fundamental objects in number theory.

- Fermat's Last Theorem: if G_k is the set of k^{th} powers of $\mathbb{Z}_{>0}$, $(G_k + G_k) \cap G_k = \emptyset$ for $k > 2$.
- Goldbach Conjecture: for the set of primes P, $P + P \supseteq \{4, 6, 8, \ldots\}$.

Setting

- Fix $N \geq 0$. Fix $p \in (0, 1)$, and let $q := 1 - p$.
- Select $A \subseteq [0, N]$ by a Bernoulli process: for each $k \in [0, N]$, independently include k in A with probability p.

Setting

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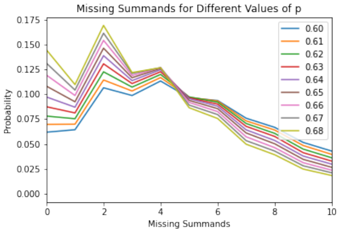
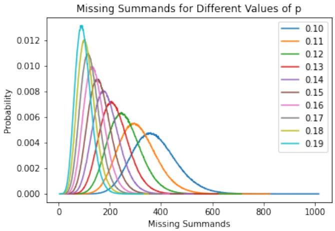
- Select $A \subseteq [0, N]$ by a Bernoulli process: for each $k \in [0, N]$, independently include k in A with probability p.
- Recent research in $|A + A|$ as a random variable.
- Martin and O'Bryant's formative paper [MO] compared $|A + A|$ to $|A - A|$ when $p = 1/2$.

Numerically observed behavior



(a) Large p



(b) Small p

Figure: Point distribution function $\mathbb{P}\left(|(A+A)^c| = m\right)$ for several values of p.

Motivating Questions

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- What are $\mathbb{E}\left(|A+A|^c\right)$ and $\mathrm{Var}\left(|A+A|^c\right)$?
- Why are the "divots" here?

Prior Work

Theorem (Martin and O'Bryant '06)

If $p = \frac{1}{2}$, *then* $\mathbb{E}[|(A+A)^c|] = 10 + O((3/4)^{N/2})$.

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Theorem (Lazarev, Miller, and O'Bryant '13 [LMO])

If $p = \frac{1}{2}$, then for $i < j \leq N$ with i, j odd,

$$\mathbb{P}(i \text{ and } j \notin A+A) = \frac{1}{2^{j+1}} F_{q+2}^{r} F_{q+4}^{r'}$$

for q, r, r' depending on i and j, and similar formulations hold for the other 3 parity cases.

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- Chu, King, Luntzlara, Martinez, Miller, Shao, Sun, and Xu [CKLMMSSX] study sumsets for generic p.
- [CKLMMSSX] and [LMO] both use graph-theoretic approaches, particularly the notion of a *condition graph*.

Prior Work

Theorem (King, Martinez, Miller, Sun '19)

For $p \in [0,1]$ and $q := 1-p$,

$$\mathbb{E}[|A+A|] = \sum_{r=0}^{n} p^r q^{n-r} \binom{n}{r} \left(2 \sum_{k=0}^{n-1} \left(1 - \frac{f(k)}{\binom{n}{r}} \right) - \left(1 - \frac{f(n-1)}{\binom{n}{r}} \right) \right),$$

where $n = N+1$ and

$$f(k) = \begin{cases} \sum_{i=\frac{k+1}{2}}^{k+1} 2^{k+1-i} \binom{\frac{k+1}{2}}{i-\frac{k+1}{2}} \binom{n-k-1}{r-i} & \text{for } k \text{ odd} \\ \sum_{i=\frac{k}{2}}^{k} 2^{k-i} \binom{\frac{k}{2}}{i-\frac{k}{2}} \binom{n-k-1}{r-1-i} & \text{for } k \text{ even.} \end{cases}$$

In particular, where the LHS holds for $p > \frac{1}{2}$,

$$2n-1-2\,\frac{1}{1-\sqrt{2q}} - (2q)^{\frac{n-1}{2}} \leq \mathbb{E}[|A+A|] \leq 2n-1-2\,\frac{1-q^{\frac{n-1}{2}}}{1-\sqrt{q}}.$$

Prior Work

Theorem (King, Martinez, Miller, Sun '19)

For $p \in (0,1)$ and $q := 1-p$,

$$\begin{aligned}\mathrm{Var}(|A+A|) &= \sum_{r=0}^{n}\binom{n}{r}p^r q^{n-r} \\ &\quad \times \left(2\sum_{0\le i<j\le 2n-2} 1 - P_r(i,j) + \sum_{0\le i\le 2n-2} 1 - P_r(i)\right) \\ &\quad - \mathbb{E}[|A+A|]^2,\end{aligned}$$

where $n = N+1$,

$$P_r(i) = \mathbb{P}(i \notin A+A \mid |A| = r),$$

and

$$P_r(i,j) = \mathbb{P}(i \text{ and } j \notin A+A \mid |A| = r).$$

The Infinite Case

Setup

- Instead of considering $A \subseteq [0, N]$ for some natural number N, consider $\mathbb{A} \subseteq \mathbb{Z}_{\geq 0}$ chosen randomly via a Bernouli process.
- For any $k \in \mathbb{Z}_{\geq 0}$, include k in $\mathbb{A}$ with probability p.

Setup

- Instead of considering $A \subseteq [0, N]$ for some natural number N, consider $\mathbb{A} \subseteq \mathbb{Z}_{\geq 0}$ chosen randomly via a Bernouli process.
- For any $k \in \mathbb{Z}_{\geq 0}$, include k in $\mathbb{A}$ with probability p.
- With probability 1, $\mathbb{A}$ and $\mathbb{A}^c$ both include infinitely many elements.
- How do $\mathbb{A} + \mathbb{A}$ and $\mathbb{A} - \mathbb{A}$ behave?

Motivation

- In general, sum sets are “almost full” in the middle, and missing elements are only on the fringes. In the “infinite case,” there is only one fringe to worry about.

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Motivation

- In general, sum sets are "almost full" in the middle, and missing elements are only on the fringes. In the "infinite case," there is only one fringe to worry about.
- Having "$N = \infty$" allow us to properly use asymptotic notation for the number of missing summands.
- Studying the "infinite case" will help us understand the "finite case."

Analysis of $\mathbb{A} - \mathbb{A}$

Proposition

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Proof.

- For $j \in \mathbb{Z}_{\geq 0}$, each pair $(j, 0), (2j + 1, j + 1)$, $(3j + 2, 2j + 2), \dots$ has probability p^2 of occurring.
- Second Borel-Cantelli lemma: with probability 1, infinitely many of these pairs are in $\mathbb{A}$.
- Reverse pairs to get $-j$.

$\square$

Analysis of $\mathbb{A} + \mathbb{A}$

- Unlike $\mathbb{A} - \mathbb{A}$, where there are infinitely many pairs that can lead to j, there are only finitely many for $\mathbb{A} + \mathbb{A}$.
- To check if $n \in \mathbb{A} + \mathbb{A}$, only need to know about the first $n + 1$ elements: $\{0, 1, 2, \ldots, n\}$.
- Focus on the sumset $\mathbb{A} + \mathbb{A}$.

Expected Value

Probability of Missing a Specific Summand

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- For each $i \geq 0$, let $\mathbb{X}_i$ be the indicator variable for $i \notin \mathbb{A} + \mathbb{A}$:

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- The upshot is that

$$\mathbb{Y} = \sum_{i=0}^{\infty} \mathbb{X}_i.$$

- To calculate $\mathbb{E}(\mathbb{Y})$, need $\mathbb{E}(\mathbb{X}_i) = \mathbb{P}(i \notin \mathbb{A} + \mathbb{A})$.

Probability of Missing a Specific Summand

Like [LMO], for odd n,

$$\{n \notin \mathbb{A}+\mathbb{A}\} = \{(0 \notin \mathbb{A} \text{ or } n \notin \mathbb{A}) \text{ and } \cdots \text{ and } (\frac{n-1}{2} \notin \mathbb{A} \text{ or } \frac{n+1}{2} \notin \mathbb{A})\}$$

and for even n,

$$\{n \notin \mathbb{A} + \mathbb{A}\} = \{(0 \notin \mathbb{A} \text{ or } n \notin \mathbb{A}) \text{ and } \cdots \text{ and } n/2 \notin \mathbb{A}\}.$$

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Hence,

$$\mathbb{P}(n \notin \mathbb{A}+\mathbb{A}) = \begin{cases} (1-p^2)^{\frac{n+1}{2}} & n \text{ odd} \\ (1-p)(1-p^2)^{\frac{n}{2}} & n \text{ even.} \end{cases}$$

Calculating $\mathbb{E}(\mathbb{Y})$

- By the Monotone Convergence Theorem,

$$\mathbb{E}(\mathbb{Y}) = \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{X}_n) = \sum_{n \text{ odd}} (1-p^2)^{(n+1)/2} + \sum_{n \text{ even}} (1-p)(1-p^2)^{n/2}.$$

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Proposition

For $p \in (0,1)$,

$$\mathbb{E}(\mathbb{Y}) = \frac{2}{p^2} - \frac{1}{p} - 1.$$

Higher Moments

A Problem with Dependencies

- To calculate $\mathbb{E}\left(\mathbb{Y}^2\right)$, need $\mathbb{P}\left(i, j \notin \mathbb{A} + \mathbb{A}\right)$.
- Unlike $\mathbb{P}\left(i \notin \mathbb{A} + \mathbb{A}\right)$, $\mathbb{P}\left(i, j \notin \mathbb{A} + \mathbb{A}\right)$ is laden with dependencies.
- Example: $\mathbb{P}\left(0 \notin \mathbb{A} + \mathbb{A}\right) = 1 - p$ and $\mathbb{P}\left(1 \notin \mathbb{A} + \mathbb{A}\right) = 1 - p^2$, but $\mathbb{P}\left(0, 1 \notin \mathbb{A} + \mathbb{A}\right) = 1 - p^2$.
- For higher moments, $\mathbb{E}\left(\mathbb{Y}^k\right)$, even more dependency.

A Workaround

- Instead of an exact expression, we find a bound:

$$\begin{aligned}\mathbb{E}\left(\mathbb{Y}^k\right) &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \mathbb{P}(n_1, \dots, n_k \notin \mathbb{A}+\mathbb{A}) \\ &\leq \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \mathbb{P}(\max\{n_1, \dots, n_k\} \notin \mathbb{A}+\mathbb{A}).\end{aligned}$$

- We know the probability of $n \notin \mathbb{A}+\mathbb{A}$:

$$\mathbb{E}\left(\mathbb{Y}^k\right) \leq \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} (1-p^2)^{(\max\{n_1,\dots,n_k\}+1)/2}.$$

- Intuitively may not be too much loss; if $\max\{n_1, \dots, n_k\} \notin \mathbb{A}+\mathbb{A}$, many elements are missing from $\mathbb{A}$, so other values are probably also missing from $\mathbb{A}+\mathbb{A}$.

The bound

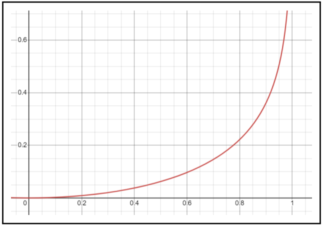
- Evaluating the “almost-geometric” sum yields

$$\mathbb{E}\left(\mathbb{Y}^k\right) \leq \left(1 + \frac{\alpha}{\sqrt{2\pi}}\right)\frac{k!}{\alpha^k},$$

where

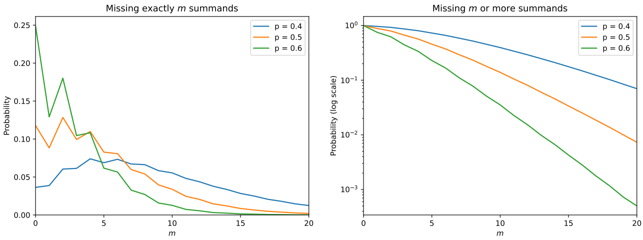
$$\alpha := \log\frac{1}{\sqrt{1-p^2}} = \left|\log\sqrt{1-p^2}\right|.$$

- $O(k!/\alpha^k)$ moments correspond to $f(x) = e^{-\alpha x}$.



The Distribution of $\mathbb{Y}$

Empirical results



- Asymptotic rate of decay seems "approximately exponential."
- $\mathbb{Y}$ appears more likely to be even than odd.
- Question: how does this distribution relate to the "finite case"?

Proving exponential decay

- Since $\mathbb{E}\left(\mathbb{Y}^k\right) = O(k!/\alpha^k)$, Chernoff's inequality yields

$$\mathbb{P}\left(\mathbb{Y} \geq n\right) \;=\; O\left(n\left(1-p^2\right)^{n/2}\right).$$

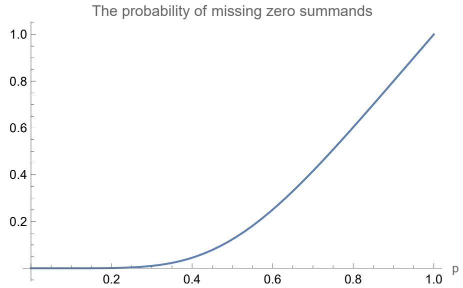
- If $0, \ldots, n/2$ are missing from $\mathbb{A}$, then $0, \ldots, n$ are missing from $\mathbb{A}+\mathbb{A}$. Therefore,

$$\mathbb{P}\left(\mathbb{Y} \geq n\right) \geq (1-p)^{n/2+1}.$$

- Bounded above and below by exponential functions, $\mathbb{P}\left(\mathbb{Y} \geq n\right)$ is "approximately exponential."

Can we determine the details?

- At large, the distribution is exponential.
- Determining the exact details is *very* hard.
- A “simple” question: what is the probability $\mathbb{P}(\mathbb{Y} = 0)$ of missing zero summands, e.g., of having $\mathbb{A} + \mathbb{A} = \mathbb{Z}_{\geq 0}$?



Speculation about $\mathbb{P}(\mathbb{Y} = n)$

- Proposition: $\mathbb{P}(\mathbb{Y} = 0)$ is asymptotically less than every polynomial.
- Conjecture: $\mathbb{P}(\mathbb{Y} = 0) > 0$ for every $p \neq 0$.
- Conjecture: $\mathbb{P}(\mathbb{Y} = 0)$ cannot be an analytic function of p.
- Conjecture: $\mathbb{P}(\mathbb{Y} = n)$ has no closed-form expression in elementary functions.

The Limit of the Finite Case

Review of the finite case

- $A \subseteq [0, N]$ selected at random such that $\mathbb{P}(i \in A) = p$ for all i independently.
- Define $Y := 2N + 1 - |A + A|$ and $X_i := [i \notin A + A]$.
- Object of interest: random variable $Y_{N\to\infty}$,

$$\mathbb{P}(Y_{N\to\infty} = n) := \lim_{N\to\infty} \mathbb{P}(Y = n).$$

- What we will compute: the k-th moment

$$\mathbb{E}\left(Y_{N\to\infty}^k\right) = \lim_{N\to\infty} \mathbb{E}\left(Y^k\right).$$

The k-th moment of Y as a corner sum

- $\mathbb{E}\left(Y^k\right) = \sum_{i_1,\dots,i_k=0}^{2N} \mathbb{E}\left(X_{i_1}\dots X_{i_k}\right)$ is a sum over a k-dimensional hypercube.
- Observation: $A + A$ is “almost full” in the middle.
- Conclusion: To compute $\mathbb{E}\left(Y^k\right)$, we just need to sum over the corners of the hypercube.

Summing over the corners

- Observation: When $j - i > N$, events $i \notin A + A$ and $j \notin A + A$ are independent. Therefore, the corners are independent.
- Result of calculations: the k-th moment of $Y_{N\to\infty}$ is

$$\lim_{N\to\infty} \mathbb{E}\left(Y^k\right) = \sum_{s=0}^{k} \binom{k}{s} \mathbb{E}\left(\mathbb{Y}^s\right) \mathbb{E}\left(\mathbb{Y}^{k-s}\right).$$

The finite case, reduced

- Observation: The moments $\lim_{N\to\infty} \mathbb{E}\left(Y^k\right)$ are the same as those of $\mathbb{Y} + \mathbb{Y}'$. Apply Carleman's condition.

Theorem

The probability distribution of $Y_{N\to\infty}$ is the same as that of $\mathbb{Y} + \mathbb{Y}'$, where $\mathbb{Y}'$ is a copy of $\mathbb{Y}$ independent of it.

- Intuition: Summands can be missing from the left and right fringes, and these are independent for large N.

Future Work

- Use Euler's identity to calculate the even-odd disparity: $\mathbb{P}(\mathbb{Y} \text{ even}) - \mathbb{P}(\mathbb{Y} \text{ odd}) = \mathbb{E}\left(e^{i\pi\mathbb{Y}}\right)$.
- Get tighter bounds on the asymptotic decay rate of $\mathbb{P}(\mathbb{Y} \geq n)$.
- Investigate A^{+k}, the k-th additive power of A, as well as $A^{+\infty} = \{0\} \cup A \cup A^{+2} \ldots$, the set of all possible sums resulting from A.

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