Random Matrix Theory, L-Functions, and Virus Dynamics

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1. Random Matrix Theory

1.1 Introduction

An important problem in random matrix theory involves investigating the distribution of eigenvalues of random matrix ensembles. Such a study has applications from nuclear physics to number theory. Previous work has given the eigenvalue distribution of real symmetric matrices and Toeplitz matrices.

We provide a way to investigate behavior between these two previously studied ensembles by looking at the Signed Toeplitz matrix ensemble, which are constant along the diagonal up to a randomly chosen sign for each entry:

$$\begin{pmatrix} \epsilon_{11}b_0 & \epsilon_{12}b_1 & \epsilon_{13}b_2 & \epsilon_{14}b_3 \\ \epsilon_{21}b_1 & \epsilon_{22}b_0 & \epsilon_{23}b_1 & \epsilon_{24}b_2 \\ \epsilon_{31}b_2 & \epsilon_{32}b_1 & \epsilon_{33}b_0 & \epsilon_{34}b_1 \\ \epsilon_{41}b_3 & \epsilon_{42}b_2 & \epsilon_{43}b_1 & \epsilon_{44}b_0 \end{pmatrix}$$

where $\epsilon_{ij} = \epsilon_{ji} \in \{1, -1\}$ and $p = \mathbb{P}\left(\epsilon_{ij}\right) = 1$.

1.2 Methods

Markov's Method of Moments We attempt to show a typical eigenvalue measure $\mu_{A,N}(x)$ converges to a probability distribution P by controlling convergence of average moments of the measures as $N \to \infty$ to the moments of P.

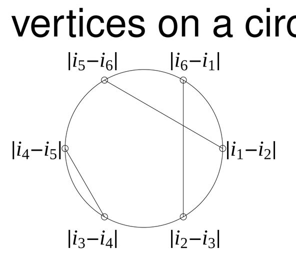
In order to calculate the moments of the eigenvalue distribution, we use the:
Eigenvalue Trace Lemma For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then

$$\mathrm{Trace}\left(A^k\right) = \sum_{i=1}^{N} \lambda_i(A)^k .$$

We get the following formula for the average k^{th} moment, $M_k(N) = \mathbb{E}\left[M_k(A_N)\right]$, is:

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \ldots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

Circle Configurations We represent the different terms in the sums as ways of pairing vertices on a circle. For example, a configuration of the 6^{th} moment:



1.3 Results

Weighted Contributions
$p = 1/2$: semi-circle distribution (bounded support)
$p \neq 1/2$: Each configuration weighted by $(2p-1)^m$, where m is the number of points on the circle whose edge crosses another edge. (unbounded support)
Counting Crossing Configurations
For:

- $m = 0$, well-known to be the Catalan numbers.
- $m = 4$, we proved there are $\binom{2k}{k-2}$ such pairings.
- $m = 6$, we proved there are $4\binom{2k}{k-3}$ such pairings.

For higher m, we were unable to find closed form expressions, but were able to prove that as $k \to \infty$, $\mathbb{E}(m) \to 2k - 2$ and $\mathrm{Var}(m) = 4$, which allows us to get very reasonable bounds on the moments.

2. L-functions

2.1 Introduction

An L-function is a Dirichlet series$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, $s \in \mathbb{C}$ An simple example is the Riemann zeta function where $a_n = 1$ for all n. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $Re(s) > 1$.

Though Random Matrix Theory was developed to explain the energy levels of heavy nuclei, later it was observed that similar answers are found for zeros of L-functions, and since then RMT has modeled their behavior. These zeros are connected to many problems in number theory, from the prime number theorem to the class number problem. The Katz-Sarnak Density Conjecture states that the behavior of zeros of a family of L-functions near the central point (as the conductors tend to zero) agree with the behavior of eigenvalues near 1 of a classical compact group (as the matrix size tends to infinity).

Maass forms are smooth functions on the upper half plane, are invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$, are eigenfunctions of the non-Euclidean Laplacian, and are a natural generalization of the Riemann zeta function. While they arise in a variety of problems in number theory, they are significantly harder to work with then their cousins (the holomorphic cusp forms) as the averaging formula here is significantly more unwieldy. We study the distribution of zeros near the central point of L-functions of level 1 Maass forms; this is essentially summing a smooth test function whose Fourier transform is compactly supported over the scaled zeros.

2.2 Methods and Results

First we define the one level density by

$$D_1(\phi) = \sum_j h(t_j) \sum_k \phi(\tilde{\gamma}_{j,k})$$

With prime number theorem

$$\sum_{p \le x} \log p = x + O\left(x \exp(-c\sqrt{\log x})\right)$$

and partial summations we are able to get

$$D_1(\phi) \;=\; \phi(0) + \hat{\phi}(0)\frac{\log(t_j)}{\log R} + O\left(\frac{\log\log R}{\log R}\right)$$
$$- \sum_p \frac{2\lambda_f(p)\log p}{p^{\frac{1}{2}}\log R} g\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_f^2(p)\log p}{p\log R} g\left(\frac{2\log p}{\log R}\right)$$

To handle the λ terms, we use the Kuznetsov trace formula

$$\sum_j h(t_j)\lambda_j(m)\overline{\lambda_j(n)}/\|u_j\|^2 + \frac{1}{4\pi}\int_{\mathbb{R}} \overline{\tau(m,r)}\tau(n,r)\frac{h(r)}{\cosh(\pi r)}dr$$
$$= \frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} r\tanh(r)h(r)dr + \frac{2i}{\pi}\sum_{c \ge 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr$$

In particular we get a term

$$\frac{1}{\sum_j h_T(t_j)/\|u_j\|^2}\sum_p \frac{2\log p}{p^{\frac{1}{2}}\log R} g\left(\frac{\log p}{\log R}\right)\sum_j h_T(t_j)\lambda_j(p)$$
$$= O(T^{3\sigma/2 - 1/4 + \epsilon} + T^{-1+\epsilon})$$

which forces the support to be $\frac{1}{6}$.

3. Virus Dynamics

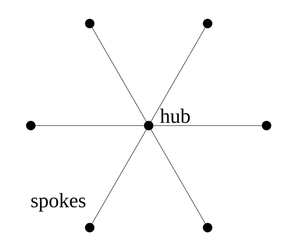
3.1 Introduction

A common way to model virus propagation is via network graphs. Such a model has applications from biology to the study of other networks, such as electronic or airline networks. We study the SIS model on the star topology.

SIS (Susceptible Infected Susceptible) model

- Each node is either Susceptible (S) or Infected (I). At each time step, susceptible nodes can be infected by their neighbors, while infected neighbors can be cured and go back to being susceptible.
- At any time step, an infected node tries to infect its neighbors with probability β and has probability δ of being cured.

Star Topology



If we let n = the number of nodes, $x = \mathbb{P}$(hub is infected), $y = \mathbb{P}$(a spoke node is infected), $a = 1 + \delta$ and $b = \beta$, we can model the system with the following equation:

$$f_n\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 - (1-ax)(1-by)^n \\ 1 - (1-ay)(1-bx) \end{pmatrix}.$$

We study $f_n : [0,1]^2 \to [0,1]^2$.

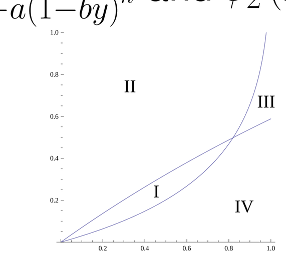
3.2 Questions

- Given an initial state, does the system reach a steady state?
- If the system does reach a steady state, what are its characteristics?

3.3 Results

Our primary method for attacking these questions was to analyze the behavior of the system in the four regions created by the curves $x = \phi_1(y)$ and $y = \phi_2(x)$ which represent the points where x and y are fixed on iteration, respectively. If these curves intersect, then we have found a fixed point, or steady state.

$$\phi_1(y) = \frac{1-(1-by)^n}{1-a(1-by)^n} \text{ and } \phi_2(x) = \frac{bx}{1-a+abx}$$



- There exists a unique nontrivial steady state.
- Initial conditions in regions I and III (excluding $(0,0)$) eventually reach the nontrivial steady state.

From numerical simulations, we are also able to conjecture the behavior of conditions in regions II and IV.
Conjecture: Depending on a, b, n, points in region II and IV will exhibit either "flipping" or "non-flipping" behavior until the points either reach the steady state, or enter regions I and III where they will then eventually reach the steady state.

Theorem: If $b > (1-a)/\sqrt{n}$ then all initial configurations save $(0,0)$ converge to the unique, non-trivial fixed point; for all other a, b the system converges to the fixed point $(0,0)$.