# Newman's Conjecture in Function Fields

Joint work with: Tomer Reiter, Dylan Yott
Advisors: Steve Miller, Alan Chang

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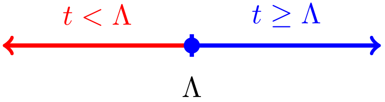
Presenters:

David Mehrle




Joseph Stahl

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**Conjecture: Riemann Hypothesis**

$$\xi(s) = 0 \implies \Re(s) = \frac{1}{2}$$

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**Theorem (De Bruijn, Newman)**

*There exists $\Lambda \in \mathbb{R}$ such that if $t < \Lambda$, $\Xi_t$ has a nonreal zero, and if $t \geq \Lambda$, $\Xi_t$ has only real zeros.*

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**Conjecture (Newman)**

$$\Lambda \geq 0.$$

"The new conjecture is a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so." - Newman

# Newman's Conjecture

**Fact**

It is known that $\Lambda \geq -1.2 \cdot 10^{-11}$.

**Perspective**

"Experiments have Diracs number at 1.00115965221 (with an uncertainty of about 4 in the last digit); the theory puts it at 1.00115965246 (with an uncertainty of about five times as much). To give you a feeling for the accuracy of these numbers, it comes out something like this: If you were to measure the distance from Los Angeles to New York to this accuracy, it would be exact to the thickness of a human hair. Thats how delicately quantum electrodynamics has, in the past fifty years, been checked-both theoretically and experimentally." - R. Feynman

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| | Riemann Hypothesis | Weil Conjectures |

# Newman setup in Function Fields

**Idea:** Mimic Pólya's setup

$$q = p^n, \quad D \in \mathbb{F}_q[x], \quad L(s, \chi_D) := \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s}$$

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$$\Xi_t(x,\chi_D) := \Phi_0 + \sum_{n=1}^{g} \Phi_n e^{tn^2}\left(e^{inx} + e^{-inx}\right)$$

**Lemma (Andrade, Chang, Miller 2013)**

*If $\Xi_t(x, \chi_D)$ has only real zeros for some $t \in \mathbb{R}$, then for all $t' > t$, $\Xi_{t'}(x, \chi_D)$ has only real zeros.*

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**Lemma (Andrade, Chang, Miller 2013)**

*There exists $\Lambda_D \in [-\infty, 0]$ such that*

1. *if $t \geq \Lambda$, then $\Xi_t(x, \chi_D)$ has only real zeros*
2. *if $t < \Lambda$, then $\Xi_t(x, \chi_D)$ has a non-real zero.*

# Examples

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$$\Lambda_D = -\infty$$

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## Conjecture

Fix $q$ a power of an odd prime. Then

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Fix $g \in \mathbb{N}$. Then

$$\sup_{\substack{D \text{ good}, \ \deg D = 2g+1 \\ q = p^k, \ p \geq 3}} \Lambda_D \geq 0.$$

# Newman's Conjecture in Function Fields

**Conjecture**

Fix $D \in \mathbb{Z}[T]$ square-free. Let $p$ be prime, and let $D_p \in \mathbb{F}_p[T]$ be the polynomial obtained by reducing $D$ modulo $p$. Then

$$\sup_{\substack{D_p \text{ good} \\ p \geq 3}} \Lambda_{D_p} \geq 0.$$

# Previous Work

**Theorem (Andrade, Chang, Miller 2013)**

*Let $D \in \mathbb{Z}[x]$ be square-free with $\deg D = 3$. For each odd prime $p$, we can reduce $D$ to $D_p \in \mathbb{F}_p[x]$. Then $\sup_p \Lambda_{D_p} = 0$.*

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**Proof sketch.**

Step 1: Show that

$$\Lambda_{D_p} = \log \frac{|a_p(D)|}{2\sqrt{p}}$$

where $a_p(D)$ is the trace of Frobenius.

Step 2: Use the Sato–Tate conjecture. $\square$

# Motivation For Our Strategy

- If $q = p^n$ is a square, then $2\sqrt{q} \in \mathbb{Z}$, so $|a_q(D)|$ can actually equal $2\sqrt{q}$. In this case, $\Lambda_D = \log 1 = 0$.

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- Judicious choices of $D$ and $p$ (such that $y^2 = D(x)$ has $p+1$ points over $\mathbb{F}_p$) will give us Newman's conjecture in certain cases!

# The Weil Conjectures for Curves

**Theorem (Weil Conjectures)**

*Let $X$ be a curve over $\mathbb{F}_q$. The Hasse-Weil zeta function of $X$ is defined as $Z(X, s) = \exp\left(\sum_{m\geq 1} \frac{N_m}{m} (q^{-s})^m\right)$, where $N_m$ is the number of points of $X$ over $\mathbb{F}_{q^m}$.*

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2. $Z(X, n-s) = \pm q^{f(X)} Z(X,s)$.

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**Example:**

$$Z(\mathbb{P}^1, s) = \frac{1}{(1-T)(1-qT)}$$

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$L(s, \chi_D)$ is the numerator of the zeta function $Z(X, s)$,
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**Proof (Idea)**

Use the Euler products for $Z(X, s)$, $L(s, \chi_D)$.

$$Z(X, s) = \prod_{\pi \text{ monic, irred.}} (1 - N(\pi))^{-s}$$

$$L(s, \chi_D) = \prod_{\pi} \left(1 - \chi(\pi)N(\pi)^{-s}\right)^{-1}$$

# Results

**Theorem**

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## Proof Sketch

The curve $X : y^2 = x^q - x$ carries an action of $\mathbb{F}_q$ that commutes with Frobenius. These actions reduce to actions at the level of cohomology $H^*_\ell(X)$. For $X : y^2 = x^q - x$, $Z(X, s) = Z(\mathbb{P}^1, s)L(s, \chi_D)$. Next, recall that the $L$-function is defined as a Gauss sum. A result of Nick Katz
$\implies L(s, \chi_D) = (T^2 q \pm 1)^g$.

# Results

**Corollary**

*If $\mathcal{F}$ is a family of good polynomials over various finite fields, and contains at least one polynomial over $\mathbb{F}_q$ of the form $x^q - x$ for some $q$, then*

$$\sup_{D \in \mathcal{F}} \Lambda_D = 0.$$

*In particular, $\mathcal{F} = \{D \in \mathbb{F}_q[T] \mid D \text{ good}\}$ and $\mathcal{F} = \{D \mid \deg D = 2g+1, 2g+1 = p^k \text{ for some } p\}$ are such families.*

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This sup is really a max!

# Continuing Previous Results

**Theorem**

*Let $D \in \mathbb{Z}[T]$ be a square-free monic cubic polynomial. Then there exists a number field $K/\mathbb{Q}$ such that*

$$\sup_{\mathfrak{p} \subseteq \mathcal{O}_K} \Lambda_{D_\mathfrak{p}} = \max_{\mathfrak{p} \subseteq \mathcal{O}_K} \Lambda_{D_\mathfrak{p}} = 0,$$

*where $D_\mathfrak{p}$ denotes reduction modulo the prime ideal $\mathfrak{p}$.*

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- Does Newman's conjecture hold for the family $\mathcal{F} = \{D \mid \deg D = 2g+1, g \in \mathbb{N}\}$ when $2g+1$ is not a power of a prime?

# Acknowledgements

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