Distribution of summands in generalized Zeckendorf decompositions

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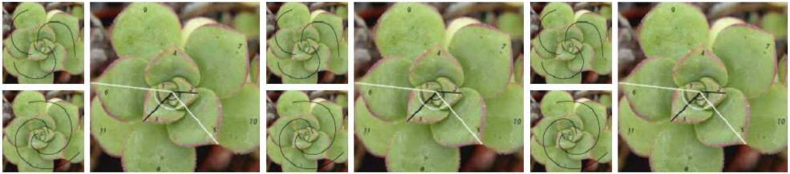
AMS Session on Number Theory, I
Room 12, Mezzanine Level, San Diego
Wednesday January 9, 2013, 3:30p.m.

Introduction

Goals of the Talk

- Explain consequences of combinatorial perspective.
- Perspective important: misleading proofs.
- Highlight techniques.
- Some open problems.



Joint work at SMALL (Undergraduate REU Program) at Williams College in 2010, 2011 and 2012.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Old Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian (normal).

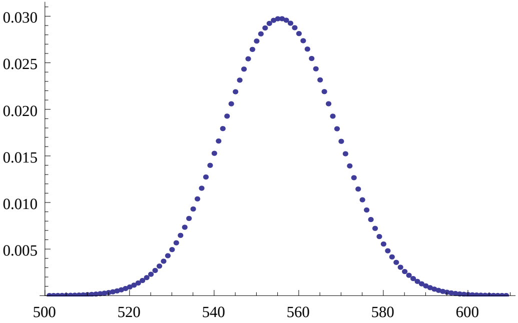

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m \;=\; \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) \;=\; \frac{1}{k(m)-1}\sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge almost surely to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

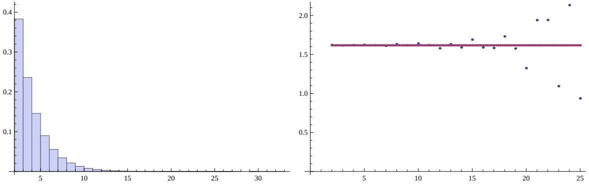

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

New Results: Longest Gap

Theorem (Longest Gap)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\mathrm{Prob}\left(L_n(m) \le f(n)\right) \;\approx\; e^{-e^{\log n - f(n)/\log \phi}}$$

Immediate Corollary: If $f(n)$ grows ***slower*** or ***faster*** than $\log \phi \cdot \log n$, then $\mathrm{Prob}(L_n(m) \le f(n))$ goes to **0** or ***1***, respectively.

Gaussian Behavior

From The Cookie Problem to Gaussian Behavior

Reinterpreting the Cookie (or Stars and Bars) Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

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$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

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$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

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Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1\ +\ k-1}{k-1} = \binom{n-k}{k-1}$.

Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \; n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \; n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the a_i's (e.g. cannot use the recurrence relation to remove any summand).
- Lekkerkerker
- Central Limit Type Theorem

Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$:
  $a_i \in \{0, 1, \dots, 9\}$ $(1 \leq i < m)$, $a_m \in \{1, \dots, 9\}$.
- For $N \in [H_n, H_{n+1})$, $m = n$, i.e., first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i.
  The A_i's are independent.
- For large n, the contribution of A_n is immaterial.
  A_i $(1 \leq i < n)$ are identically distributed random variables with mean 4.5 and variance 8.25.
- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \rightarrow$ Gaussian
  with mean $4.5n + O(1)$
  and variance $8.25n + O(1)$.

Far-difference Representation

Theorem (Alpert, 2009) (Analogue to Zeckendorf)

Every integer can be written uniquely as a sum of the $\pm F_n$'s, such that every two terms of the same (opposite) sign differ in index by at least 4 (3).

Example: $1900 = F_{17} - F_{14} - F_{10} + F_6 + F_2$.

K: # of positive terms, L: # of negative terms.

Generalized Lekkerkerker's Theorem

As $n \to \infty$, $E[K]$ and $E[L] \to n/10$. $E[K] - E[L] = \varphi/2 \approx .809$.

Central Limit Type Theorem

As $n \to \infty$, K and L converges to a bivariate Gaussian.

- $\text{corr}(K, L) = -(21 - 2\varphi)/(29 + 2\varphi) \approx -.551$, $\varphi = \frac{\sqrt{5}+1}{2}$.
- $K + L$ and $K - L$ are independent.

Gaps in the Bulk

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

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Can ask similar questions about binary or other expansions: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.

Main Results

Theorem (Base B Gap Distribution)

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, *and for* $k \geq 1$, $P(k) = c_B B^{-k}$, *with* $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution)

For Zeckendorf decompositions, $P(k) = \frac{\phi(\phi-1)}{\phi^k}$ *for* $k \geq 2$, *with* $\phi = \frac{1+\sqrt{5}}{2}$ *the golden mean.*

Main Results

- H_n: $H_{n+1} = c_1H_n + c_2H_{n-1} + \cdots + c_LH_{n+1-L}$ a positive linear recurrence of length L where $c_i \geq 1$ for all $1 \leq i \leq L$.

- $\lambda_1 > 1$: largest root (in absolute value) of characteristic polynomial of H_n.

- Generalized Binet: $H_n = a_1\lambda_1^n + \cdots$.

Theorem

Notation as above, probability of a gap of length j is

$$\begin{cases} 1 - (\frac{a_1}{C_{Lek}})(\lambda_1^{-n+2} - \lambda_1^{-n+1} + 2\lambda_1^{-1} + a_1^{-1} - 3) & j = 0 \\ \lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1 - 2a_1) + a_1) & j = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & j \geq 2 \end{cases}$$

Proof of Fibonacci Result

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

Let $X_{i,j} = \#\{m \in [F_n, F_{n+1})$: decomposition of m includes F_i, F_j, but not F_q for $i < q < j\}$.

$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Proof sketch of almost sure convergence

- $m = \sum_{j=1}^{k(m)} F_{i_j}$,
  $\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right)$.
- $\mu_{m,n}(t) \ = \ \int x^t \mathrm{d}\nu_{m;n}(x)$.
- Show $\mathbb{E}_m[\mu_{m;n}(t)]$ equals average gap moments, $\mu(t)$.
- Show $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^2]$ and $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^4]$ tend to zero.

Key ideas: (1) Replace $k(m)$ with average (Gaussianity); (2) use $X_{i,i+g_1,j,j+g_2}$.

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References

References

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