Low-lying zeros of GL(2) L-functions

Steven J Miller
Williams College

Steven.J.Miller@williams.edu

http://web.williams.edu/Mathematics/sjmiller/public_html/jmm2013.html/

Maass forms joint with Levent Alpoge, holomorphic cusp forms with Chris Hughes, Geoff Iyer and Nicholas Triantafillou, elliptic curves with Eduardo Dueñez, Duc Khim Huynh, Jon Keating and Nina Snaith

AMS Special Session on Arithmetic Statistics, I
Room 31A, Upper Level, San Diego
Thursday January 10, 2013, 8:00am

Introduction

Summary

- Quick review of Katz-Sarnak philosophy.
- Symmetry type of compound families of L-functions.
- New model (excised orthogonal ensemble) for zeros of elliptic curve L-functions near the central point for finite conductors.
- Increasing support for n-level densities of holomorphic cuspidal newforms: agree up to $\left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$ (previous was $\frac{1}{n-1}$).
- 1-level densities of level 1 Maass forms: up to $\left(-\frac{4}{3}, \frac{4}{3}\right)$ (in progress, possibly larger when incorporate non-trivial Kloosterman results).

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

Measures of Spacings: *n*-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

Katz-Sarnak Conjecture

For a 'nice' family of *L*-functions, the *n*-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1}\sum_{f\in\mathcal{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}|\to\infty$,

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

Katz-Sarnak Conjecture

As $C_f\to\infty$ the behavior of zeros near $1/2$ agrees with $N\to\infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Some Number Theory Results

- Orthogonal: Iwaniec-Luo-Sarnak; Ricotta-Royer; ...: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign); Miller; Young; ...: One and two-parameter families of elliptic curves.

- Symplectic: Rubinstein; Gao; Levinson-Miller; Entin, Roddity-Gershon and Rudnick; Ozluk-Snyder; ...: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.

- Unitary: Fiorilli-Miller; Hughes-Rudnick; ...: Families of Primitive Dirichlet Characters.

Applications of *n*-level density

Order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Applications of n-level density

Order of vanishing at the central point.
$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.

Bound percentage vanish to order r:

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.
- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation:
- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake parameters $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu}\widehat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} = \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F}\times\mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Breaks analysis of compound families into simple ones.

1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\begin{aligned}
\lambda_f(p^\nu) &= \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu \\
S_1(\mathcal{F}) &= -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right] \\
S_2(\mathcal{F}) &= -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]
\end{aligned}$$

The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

Takeaways

Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.
- First moment zero save for families of elliptic curves.
- Higher moments control convergence and can depend on arithmetic of family.

Elliptic Curves: First Zero Above Central Point
(With E. Dueñez, D. K. Huynh, J. P. Keating, N. Snaith)

Theoretical results

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t\in\mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

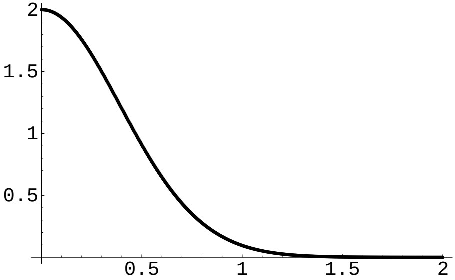
$$\text{where} \quad \mathcal{G} = \begin{cases} \text{SO(odd)} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{weighted average} & \text{otherwise.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

Independent Model:

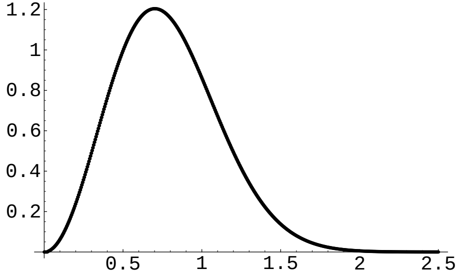
$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

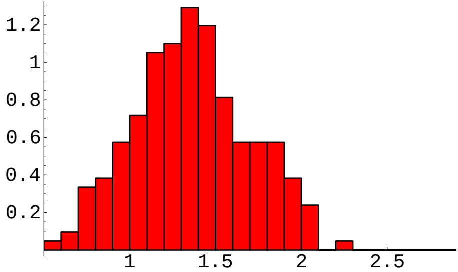


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

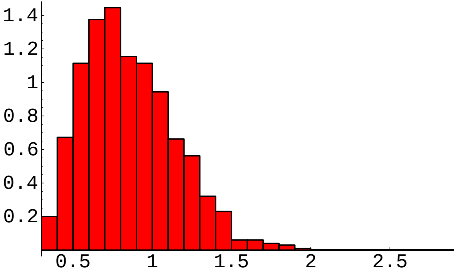


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i.e., shifted by the same amount).

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	23 Rank 4 Curves	t-Statistic
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	64 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

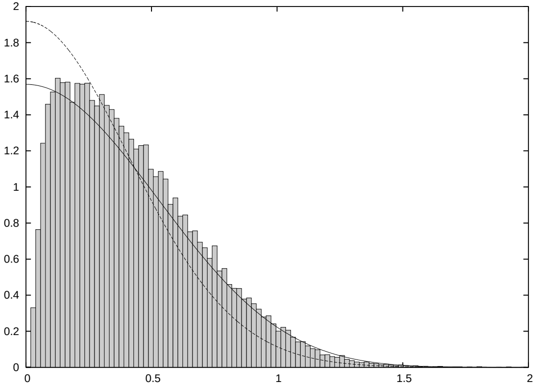
New Model for Finite Conductors

- Replace conductor N with $N_{\text{effective}}$.
  - ⋄ Arithmetic info, predict with L-function Ratios Conj.
  - ⋄ Do the number theory computation.

- Excised Orthogonal Ensembles.
  - ⋄ $L(1/2, E)$ discretized.
  - ⋄ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.

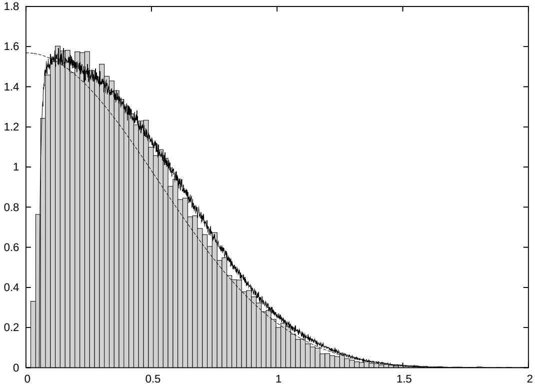
- Painlevé VI differential equation solver.
  - ⋄ Use explicit formulas for densities of Jacobi ensembles.
  - ⋄ Key input: Selberg-Aomoto integral for initial conditions.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_{\rm eff}$ (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Cuspidal Maass Forms
(Joint with Levent Alpoge)

Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a function $f : \mathcal{H} \to \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. f vanishes at the cusps of Γ, and
3. $\Delta f = \lambda f$ for some $\lambda = s(1 - s) > 0$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on $\mathcal{H}$.

- Coefficients contain information about partitions.
- For full modular group, $s = 1/2 + it_j$ with $t_j \in \mathbb{R}$.

L-function associated to Maass forms

Write Fourier expansion of Maass form u_j as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi|n|y) e^{2\pi inx}.$$

Define L-function attached to u_j as

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1}$$

where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1.$
Also,

$$\lambda_j(p) \ll p^{7/64}.$$

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)} + \frac{1}{4\pi}\int_{\mathbb{R}} \overline{\tau(m,r)}\tau(n,r)\frac{h(r)}{\cosh(\pi r)}dr =$$

$$\frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} r\tanh(r)h(r)dr + \frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr$$

where
$\tau(m,r) = \pi^{\frac{1}{2}+ir}\Gamma(1/2+ir)^{-1}\zeta(1+2ir)^{-1}n^{-\frac{1}{2}}\sum_{ab=|m|}\left(\frac{a}{b}\right)^{ir}$.

$$S(n,m;c) = \sum_{0\leq x\leq c-1,\gcd(x,c)=1} e^{2\pi i(nx+mx^*)/c}$$

$$J_{ir}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+ir+1)}\left(\frac{1}{2}x\right)^{2m+ir}.$$

Main Result (in progress)

Theorem (Alpoge-Miller 2012)

For $h_T(r) = \frac{r}{T} h(ir/T)/\sinh(\pi r/T)$ the Katz-Sarnak conjecture holds for level 1 cuspidal Maass forms for test functions whose Fourier transform is supported in $(-4/3, 4/3)$.

- Write $\int_{-\infty}^{\infty} J_{2ir}(X) \frac{r h_T(r)}{\cosh(\pi r)} dr$ as sum of $J_{2k+1}(X)$ and h_T at imaginary arguments and $J_{2kT}(X)$ and h at k.
- Bound contributions from sums. Apply Poisson summation, analyze, and Poisson summation again.
- Key steps: Taylor expanding, Fourier transform identities relating differentiation and multiplication.

Cuspidal Newforms
(Joint with C. Hughes, G. Iyer, N. Triantafillou)

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc = 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \;\; f(\gamma z) = (cz + d)^k f(z).$$

- Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z) \overline{g(z)} y^{k-2} dxdy$.
- Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}} \frac{1}{||f||} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) \ = \ \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) \ = \ 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi \frac{\sqrt{mn}}{c}\right) + \ \delta(m,n).$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{ccccccc} u_2 & = & x_1x_2 & \qquad & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1}\end{array}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

Theorem (Iyer-Miller-Triantafillou):

The n-level densities agree for $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$.

Philosophy:

Number theory harder - adapt tools to get an answer.
Random matrix theory easier - manipulate known answer.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\text{supp}(\widehat{\Psi}) \subset (-2, 2)$. Then

$$\sum_{m \le N^\epsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2, b)R(1, b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right) \frac{\mathrm{d}y}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x)\frac{\sin 2\pi x}{2\pi x}\mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k \log\log kN}{\log kN}\right),$$

where $R = k^2N$, φ is Euler's totient function, and $R(n, q)$ is a Ramanujan sum.

Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Sequence of Lemmas - New Contributions Arise

1. Apply Petersson Formula
2. Restrict Certain Sums
3. Convert Kloosterman Sums to Gauss Sums
4. Remove Non-Principal Characters
5. Convert Sums to Integrals

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Theorem

Fix $n \geq 4$ and let ϕ be an even Schwartz function with $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$. Then, the nth centered moment of the 1-level density for holomorphic cusp forms is

$$\frac{1+(-1)^n}{2}(-1)^n(n-1)!!\left(2\int_{-\infty}^{\infty}\hat{\phi}(y)^2|y|dy\right)^{n/2}$$
$$\pm(-2)^{n-1}\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx-\frac{1}{2}\phi(0)^n\right)$$
$$\mp(-2)^{n-1}n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1}dx_1dx_2-\frac{1}{2}\phi^n(0)\right).$$

Agrees with RMT.

n-Level Density: Katz-Sarnak Determinant Expansions

Example: SO(even)

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_1) \cdots \widehat{\phi}(x_n) \det \left(K_1(x_j, x_k) \right)_{1 \le j,k \le n} dx_1 \cdots dx_n,$$

where

$$K_1(x, y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

Problem: n-dimensional integral - looks very different.

Preliminaries

Easier to work with cumulants.

$$\sum_{n=1}^{\infty} C_n \frac{(it)^n}{n!} = \log \widehat{P}(t),$$

where P is the probability density function.

$$\mu_n' = C_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m \mu_{n-m}',$$

where μ_n' is uncentered moment.

Preliminaries

Manipulating determinant expansions leads to analysis of

$$K(y_1,\dots,y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\dots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \sum_{\epsilon_1,\dots,\epsilon_n=\pm 1} \prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n}\eta(\ell,j)\epsilon_j y_j\right|\leq 1\right\}},$$

where

$$\eta(\ell,j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^{\ell}\lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell}\lambda_k. \end{cases}$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\text{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Hughes-Miller solved for $\text{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-1}, \frac{1}{n-1})$.

New Complications: If $\text{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-2}, \frac{1}{n-2})$,

1. $\eta(\ell, j)\epsilon_j y_j$ need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed m, λ_j, ϵ_j).

Solution: Double count terms and subtract a correcting term ρ_j.

New Result: Iyer-Miller-Triantafillou: Large Support: $\mathrm{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

After Fourier transform identities:

$$\begin{aligned} C_n^{\mathrm{SO}}(\phi) &= \frac{(-1)^{n-1}}{2}\left(\int_{-\infty}^{\infty}\phi(x)^n \frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\phi(0)^n\right) \\ &\quad + \frac{n(-1)^n}{2}\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\right. \\ &\quad\quad \left.\frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1}dx_1dx_2 - \frac{1}{2}\phi^n(0)\right). \end{aligned}$$

Agrees with number theory!

Conclusion and Bibliography

Recap

- Understand compound families in terms of simple ones.
- Choose combinatorics to simplify calculations.
- Extending support often related to deep arithmetic questions.

References

- 1- *and* 2*-level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **140** (2004), 952–992. `http://arxiv.org/pdf/math/0310159`
- *Investigations of zeros near the central point of elliptic curve L-functions*, Experimental Mathematics **15** (2006), no. 3, 257–279. `http://arxiv.org/pdf/math/0508150`
- *Low lying zeros of L–functions with orthogonal symmetry* (with Christopher Hughes), Duke Mathematical Journal **136** (2007), no. 1, 115–172. `http://arxiv.org/abs/math/0507450`
- *Lower order terms in the 1-level density for families of holomorphic cuspidal newforms*, Acta Arithmetica **137** (2009), 51–98.
- *The effect of convolving families of L-functions on the underlying group symmetries* (with Eduardo Dueñez), Proceedings of the London Mathematical Society, 2009; doi: 10.1112/plms/pdp018. `http://arxiv.org/pdf/math/0607688.pdf`
- *The lowest eigenvalue of Jacobi Random Matrix Ensembles and Painlevé VI*, (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), Journal of Physics A: Mathematical and Theoretical **43** (2010) 405204 (27pp). `http://arxiv.org/pdf/1005.1298`
- *Models for zeros at the central point in families of elliptic curves* (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp).`http://arxiv.org/pdf/1107.4426`
- *The low-lying zeros of level 1 Maass forms* (with Levent Alpoge), preprint 2013.
- *Moment Formulas for Ensembles of Classical Compact Groups* (with Geoffrey Iyer and Nicholas Triantafillou), preprint 2013.

Excised Orthogonal Ensembles

Excised Orthogonal Ensemble: Preliminaries

Characteristic polynomial of $A \in \mathrm{SO}(2N)$ is

$$\Lambda_A(e^{i\theta}, N) := \det(I - Ae^{-i\theta}) = \prod_{k=1}^{N}(1 - e^{i(\theta_k - \theta)})(1 - e^{i(-\theta_k - \theta)}),$$

with $e^{\pm i\theta_1}, \dots, e^{\pm i\theta_N}$ the eigenvalues of A.

Motivated by the arithmetical size constraint on the central values of the L-functions, consider Excised Orthogonal Ensemble $T_{\mathcal{X}}$: $A \in \mathrm{SO}(2N)$ with $|\Lambda_A(1, N)| \geq \exp(\mathcal{X})$.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases. The one-level density excised orthogonal ensemble:

$$R_1^{T_{\mathcal{X}}}(\theta_1) := C_{\mathcal{X}} \cdot N \int_0^{\pi} \cdots \int_0^{\pi} H(\log|\Lambda_A(1, N)| - \mathcal{X}) \times \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N,$$

Here $H(x)$ denotes the Heaviside function

$$H(x) = \begin{cases} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0, \end{cases}$$

and $C_{\mathcal{X}}$ is a normalization constant.

One-Level Densities

One-level density $R_1^{G(N)}$ for a (circular) ensemble $G(N)$:

$$R_1^{G(N)}(\theta) = N \int \dots \int P(\theta, \theta_2, \dots, \theta_N) d\theta_2 \dots d\theta_N,$$

where $P(\theta, \theta_2, \dots, \theta_N)$ is the joint probability density function of eigenphases. The one-level density excised of the orthogonal ensemble is

$$R_1^{T_{\mathcal{X}}}(\theta_1) = \frac{C_{\mathcal{X}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{Nr} \frac{\exp(-r\mathcal{X})}{r} R_1^{J_N}(\theta_1; r - 1/2, -1/2) dr,$$

where $C_{\mathcal{X}}$ is a normalization constant and

$$R_1^{J_N}(\theta_1; r - 1/2, -1/2) = N \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} w^{(r-1/2,-1/2)}(\cos\theta_j) \times \prod_{j<k} (\cos\theta_j - \cos\theta_k)^2 d\theta_2 \cdots d\theta_N$$

is the one-level density for the Jacobi ensemble J_N with weight function

$$w^{(\alpha,\beta)}(\cos\theta) = (1 - \cos\theta)^{\alpha+1/2}(1 + \cos\theta)^{\beta+1/2}, \qquad \alpha = r - 1/2 \text{ and } \beta = -1/2.$$

Results

- With $C_{\mathcal{X}}$ normalization constant and $P(N,r,\theta)$ defined in terms of Jacobi polynomials,

$$R_1^{T_{\mathcal{X}}}(\theta) = \frac{C_{\mathcal{X}}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\exp(-r\mathcal{X})}{r}2^{N^2+2Nr-N}\times$$
$$\times\prod_{j=0}^{N-1}\frac{\Gamma(2+j)\Gamma(1/2+j)\Gamma(r+1/2+j)}{\Gamma(r+N+j)}\times$$
$$\times(1-\cos\theta)^r\frac{2^{1-r}}{2N+r-1}\frac{\Gamma(N+1)\Gamma(N+r)}{\Gamma(N+r-1/2)\Gamma(N-1/2)}P(N,r,\theta)\,dr.$$

- Residue calculus implies $R_1^{T_{\mathcal{X}}}(\theta)=0$ for $d(\theta,\mathcal{X})<0$ and

$$R_1^{T_{\mathcal{X}}}(\theta) = R_1^{\mathrm{SO}(2N)}(\theta) + C_{\mathcal{X}}\sum_{k=0}^{\infty} b_k\exp((k+1/2)\mathcal{X}) \quad \text{for } d(\theta,\mathcal{X})\geq 0,$$

where $d(\theta,\mathcal{X}) := (2N-1)\log 2+\log(1-\cos\theta)-\mathcal{X}$ and b_k are coefficients arising from the residues. As $\mathcal{X}\to-\infty$, θ fixed, $R_1^{T_{\mathcal{X}}}(\theta)\to R_1^{\mathrm{SO}(2N)}(\theta)$.

Numerical check

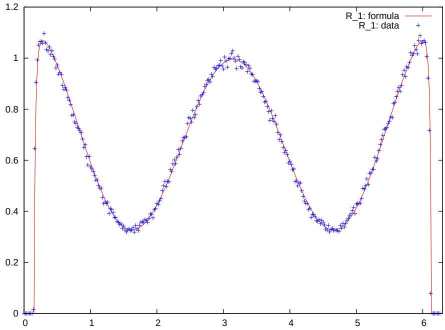


Figure: One-level density of excized $SO(2N), N = 2$ with cut-off $|\Lambda_A(1, N)| \geq 0.1$. The red curve uses our formula. The blue crosses give the empirical one-level density of 200,000 numerically generated matrices.

Theory vs Experiment

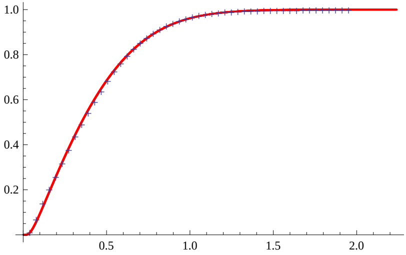


Figure: Cumulative probability density of the first eigenvalue from 3×10^6 numerically generated matrices $A \in SO(2N_{\text{std}})$ with $|\Lambda_A(1, N_{\text{std}})| \geq 2.188 \times \exp(-N_{\text{std}}/2)$ and $N_{\text{std}} = 12$ red dots compared with the first zero of even quadratic twists $L_{E_{11}}(s, \chi_d)$ with prime fundamental discriminants $0 < d \leq 400{,}000$ blue crosses. The random matrix data is scaled so that the means of the two distributions agree.