Zeros of Dirichlet L-Functions over the Rational Function Field

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L-Functions

Roughly, an L-function is a series

$$L(s,f) := \sum_{n=1}^{\infty} a_f(n) n^{-s}$$

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1. $a_f(n) \ll n^{\epsilon}$ for all $\epsilon > 0$.
2. Euler product $L(s,f) = \prod_p L_p(s,f)$.
3. Meromorphic continuation to $\mathbb{C}$.
4. Functional equation.

Interested in statistical behavior of zeros of L-functions.

Montgomery-Dyson conjectured that, as $T \to \infty$,

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$$\to \text{spacings of eigenangles in the Gaussian Unitary Ensemble}$$

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2. Should study zeros in the local regime near $s = 1/2$, in the limit $N \to \infty$. Also called the low-lying zeros.

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3. Can find classical compact matrix group G such that

$$\text{average low-lying zero distribution in } \mathscr{F}_N \text{ as } N \to \infty$$
$$= \text{distribution of eigenangles in } G \text{, as dimension} \to \infty$$

$(G = \text{U}, \text{USp}, \text{SO(even)}, \text{SO(odd)}.)$

Density Conjecture

Definition (1-Level Density)

Assume $L(s, f)$ has RH. Write its zeros in the form $1/2 + i\gamma$. For all integrable $\phi : \mathbb{R} \to \mathbb{C}$, let

$$W_f(\phi) := \lim_{T\to\infty} \sum_{0\le\gamma\le T} \phi\left(\gamma \cdot \frac{\log c_f}{2\pi}\right)$$

Above, $(\log c_f)/2\pi$ normalizes the average consecutive spacing to be 1 in the limit $T \to \infty$.

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Conjecture that

$$\mathbb{E}W_f(\phi) := \frac{1}{\#\mathscr{F}_N} \sum_{f\in\mathscr{F}_N} W_f(\phi) \to \int_{\mathbb{R}} \phi(t) W_G(t)\,\mathrm{d}t$$

as $N \to \infty$, where W_G depends only on G.

$$W_G(t) = \begin{cases} 1 & G = \mathrm{U} \\ 1 - \frac{\sin 2\pi t}{2\pi t} & G = \mathrm{USp} \\ 1 + \frac{\sin 2\pi t}{2\pi t} & G = \mathrm{SO(even)} \end{cases}$$

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Current results are limited to test functions ϕ such that $\widehat{\phi}$ is supported in some small interval $[-\sigma, +\sigma]$.

1. Iwaniec-Luo-Sarnak (2000): Holomorphic cuspidal newforms of weight k and level N, as $kN \to \infty$, should be orthogonal.
2. Rubinstein (2001), Gao (2013): Quadratic Dirichlet characters of discriminant $\leq D$, as $D \to \infty$, should be symplectic.
3. Hughes-Rudnick (2002): Dirichlet characters of prime conductor N, as $N \to \infty$, should be unitary.

Three Regimes

1-level density is a low-lying statistic: Due to the $(\log c_f)/2\pi \asymp \log N$ term, it "zooms in" on intervals that shrink as $N \to \infty$.

Can study intervals $\mathscr{I}$ in three regimes:

1. *Local or low-lying regime.* $|\mathscr{I}| \ll \log N$ as $N \to \infty$.
2. *Mesoscopic regime.* $|\mathscr{I}| \to 0$ but $|\mathscr{I}| \log N \to \infty$ as $N \to \infty$.
3. *Global regime.* $\mathscr{I}$ is fixed.

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Decreasing order of difficulty $\iff$ fewer restrictions on ϕ.

Recently, interest in mesoscopic and global regimes for L-functions over *function fields.*

L-Functions over $\mathbb{F}_q(T)$

Fix a prime power q. Then $\mathbb{F}_q[T]$ is a "cousin" of the integers $\mathbb{Z}$:

$\mathbb{Z} \subseteq \mathbb{Q}$	$\mathbb{F}_q[T] \subseteq \mathbb{F}_q(T)$
Positive n	Monic f
Prime p	Irreducible P
$\lvert n \rvert = \#\mathbb{Z}/n\mathbb{Z}$	$\lvert f \rvert = \#\mathbb{F}_q[T]/f\mathbb{F}_q[T] = q^{\deg f}$

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Prime p	Irreducible P
$\lvert n\rvert = \#\mathbb{Z}/n\mathbb{Z}$	$\lvert f\rvert = \#\mathbb{F}_q[T]/f\mathbb{F}_q[T] = q^{\deg f}$
$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$	$\zeta_{\mathbb{F}_q(T)}(s) = \sum_f' \lvert f\rvert^{-s}$
$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$	$L_{\mathbb{F}_q(T)}(s,\chi) = \sum_f' \chi(f) \lvert f\rvert^{-s}$

where $\sum'$ means a sum over monic polynomials.

For example, $\zeta_{\mathbb{F}_q(T)}(s) = (1-q^{1-s})^{-1}$. Its completed version is

$$\zeta(s, \mathbb{P}^1_{\mathbb{F}_q}) = \zeta_{\mathbb{F}_q(T)}(s)\zeta_{\mathbb{F}_q(T)}(1-s) = \frac{1}{(1-q^{1-s})(1-q^{-s})}$$

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Other L-functions $L(s, f)$ over $\mathbb{F}_q(T)$ occur as factors of $\zeta(s, V)/\zeta(s, \mathbb{P}^1_{\mathbb{F}_q})$ for some curve $V/\mathbb{F}_q$.

1. *Rationality.* $L(s, f)$ is a *polynomial* in q^{-s}.
2. *Riemann Hypothesis.* The zeros of $L(s, f)$ live on the line $s = 1/2$ (with period $2\pi/\log q$).

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Katz-Sarnak proved the (local) Density Conjecture in the limit $q, g \to \infty$. Interested in the limit where q is *fixed* and $g \to \infty$.

Motivation

Fix $Q \in \mathbb{F}_q[T]$ of degree $d > 0$. Let $L(s, \chi)$ be a primitive Dirichlet L-function over $\mathbb{F}_q(T)$ of conductor Q.

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1. Faifman-Rudnick (2010) showed there are $d|\mathscr{I}| + O(d/\log d)$ normalized zeros of $L(s,\chi)$ in a fixed interval $\mathscr{I} \subseteq [-1/2, +1/2]$, as $d \to \infty$.
2. Quadratic $L(s,\chi)$ are factors of $\zeta(s,C)$ for hyperelliptic $C : Y^2 = Q$. Averaging over all C of fixed genus, the higher moments in the distribution of zeros are Gaussian in the global and mesoscopic regimes.
3. Xiong (2010) extended this work to ℓ-fold covers $Y^\ell = Q$ such that $\ell \equiv 1 \pmod{q}$ and Q is ℓth-power-free.

Instead of averaging over a “geometric” family of zeta functions, we will average over the most naïve family of L-functions:

$$\mathscr{F}_Q = \text{family of primitive Dirichlet characters modulo } Q$$

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Our approach generalizes to test functions that are not indicator functions of intervals.

1. Refine bounds in some cases.
2. Facilitate comparison between different regimes.

Explicit Formula

Let $\mathscr{F}_Q$ be the family of primitive Dirichlet characters modulo Q, where $d := \deg Q \geq 2$.

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Theorem (A-M-P-T)

Let $\chi \in \mathscr{F}_Q$. Write the zeros of $L(s,\chi)$ in the form $1/2 + i\gamma_\chi$, and let $\mathrm{d}\gamma_\chi$ be the probability distribution of the normalized ordinates $\gamma_\chi \cdot \frac{\log q}{2\pi} \in [-1/2, +1/2]$. Let $\psi(s) = \sum_{n\in\mathbb{Z}} \widehat{\psi}(n) e^{-2\pi i n s}$ be a test function on $[-1/2, +1/2]$. Then

$$\begin{aligned} S_\chi(\psi) &:= \int_{-1/2}^{+1/2} \psi(s)\,\mathrm{d}\gamma_\chi(s) \\ &= \widehat{\psi}(0) - \frac{\lambda_\infty(\chi)}{d-1} \sum_{n\in\mathbb{Z}} \frac{\widehat{\psi}(n)}{q^{|n|/2}} - \frac{2}{d-1} \sum_{n=0}^{\infty} \sum_{\deg f = n}{}' \Lambda(f)\chi(f) \frac{\widehat{\psi}(n)}{q^{n/2}} \end{aligned}$$

where Λ is the von Mangoldt function and $\lambda_\infty(\chi)$ is 1 if χ is even and 0 otherwise.

1. Let $1/2 < c < 1$ and $T_q = 2\pi/\log q$. By Cauchy's Theorem,

$$\begin{aligned} S_\chi(\psi) &= \sum_{\gamma_\chi \in [0,T_q)} \frac{1}{d-1}\psi(\gamma_\chi/T_q) \\ &= \frac{1}{2\pi i}\left(\int_{\ell_c} - \int_{\ell_{1-c}}\right) \frac{L'}{L}(s,\chi)\frac{1}{d-1}\psi\left(-\frac{i(s-1/2)}{T_q}\right) \mathrm{d}s \\ &= S_\chi(\psi;c) - S_\chi(\psi;1-c) \end{aligned}$$

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$$= S_\chi(\psi;c) - S_\chi(\psi;1-c)$$

2. Replace the integral on the line $1-c$ using the functional equation of $L(s,\chi)$ and send $c \to 1/2$.

$$-S_\chi(\psi;1/2)$$
$$= \frac{1}{T_q}\int_{-T_q/2}^{+T_q/2}\psi\left(\frac{t}{T_q}\right)\mathrm{d}t + S_\chi(\psi;1/2)$$
$$+ \frac{\lambda_\infty(\chi)}{(d-1)\,T_q}\int_{-T_q/2}^{+T_q/2}\left(-1+\frac{1}{1-q^{1/2+it}}+\frac{1}{1-q^{1/2-it}}\right)\psi(t/T_q)\,\mathrm{d}t$$

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2. Simplification of the gamma-factor term:

$$\frac{1}{T_q}\int_{-T_q/2}^{+T_q/2}\left(-1+\frac{1}{1-q^{1/2+it}}+\frac{1}{1-q^{1/2-it}}\right)\psi(t/T_q)\,\mathrm{d}t$$

$$\frac{1}{T_q}\int_{-T_q/2}^{+T_q/2}\left(-1-\frac{q^{-1/2-it}}{1-q^{-1/2-it}}-\frac{q^{-1/2+it}}{1-q^{-1/2+it}}\right)\psi(t/T_q)\,\mathrm{d}t$$

$$= -\int_{-1/2}^{+1/2}\left(\sum_{n\in\mathbb{Z}} q^{-|n|/2} e^{-2\pi i n s}\right)\psi(u)\,\mathrm{d}u$$

$$= -\sum_{n\in\mathbb{Z}} \frac{\widehat{\psi}(n)}{q^{|n|/2}}$$

Specific to the function-field setting.

Global Regime

Theorem (A-M-P-T)

Suppose $Q \in \mathbb{F}_q[T]$ *is irreducible. Let* $\psi(s) = \sum_{n\in\mathbb{Z}} \widehat{\psi}(n) e^{-2\pi ins}$ *be a test function on* $[-1/2, +1/2]$ *such that, for some* $\epsilon > 0$*,*

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for all n *large enough.*

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for all n *large enough. Then*

$$\begin{aligned} \mathbb{E}S_\chi(\psi) &:= \frac{1}{\#\mathscr{F}_Q} \sum_{\chi \in \mathscr{F}_Q} S_\chi(\psi) \\ &= \widehat{\psi}(0) - \frac{1}{(d-1)(q-1)} \sum_{n \in \mathbb{Z}} \frac{\widehat{\psi}(n)}{q^{|n|/2}} + O\left(\frac{1}{\epsilon d^{1+\epsilon} q^d} \right) \end{aligned}$$

as $d \to \infty$.

1. Using Schur Orthogonality, reduce to estimating

$$\mathbb{E}S_\chi(\psi;1/2) = \frac{2}{d-1}\frac{|Q|-1}{|Q|-2}\sum_{n=0}^{\infty}\sum_{\substack{\deg f=n \\ f\equiv 1 \pmod{Q}}}{}' \Lambda(f)\frac{\widehat{\psi}(n)}{q^{n/2}}$$

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3. We "win" as long as $d^{-1}\sum_{n\geq d} q^{n/2-d}\widehat{\psi}(n) \to 0$ as $d \to \infty$. In particular, this happens when

$$\widehat{\psi}(n) \ll \frac{1}{n^{1+\epsilon}q^{n/2}}$$

for some $\epsilon > 0$.

Local Regime

Corollary (A-M-P-T)

Suppose $Q \in \mathbb{F}_q[T]$ *is irreducible. Let* $T_q = 2\pi/\log q$*. Let* ϕ *be a rapid-decay test function on* $\mathbb{R}$*, and let*

$$W_\chi(\phi) := \sum_{\gamma_\chi \in [0,T_q)} \sum_{n \in \mathbb{Z}} \phi\left((d-1)\left(\frac{\gamma_\chi}{T_q} + n\right)\right)$$

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1. *If* $\operatorname{supp}\widehat{\phi} \subseteq [-2,+2]$. *then* $\mathbb{E}W_\chi(\phi) = \widehat{\phi}(0) + O(1/d)$.
2. *Let*

$$\sigma(\phi)^2 = \int |t|\widehat{\phi}(t)^2 \,\mathrm{d}t + O(1/d)$$

If $\operatorname{supp}\widehat{\phi} \subseteq (-2/m, +2/m)$, *then*

$$\mathbb{E}(W_\chi(\phi) - \mathbb{E}W_\chi(\phi))^m = \begin{cases} \frac{m!}{2^{m/2}(m/2)!}\sigma(\phi)^m & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

Gaussian vs. Mock-Gaussian

The local-regime results seem to give Gaussian behavior for the centered moments of $W_\chi(\phi)$.

1. Cannot be the case, because if ϕ is an indicator function, then the limiting distribution is discrete.
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In the mesoscopic and global regimes, we do not zoom in as fast as $O(\deg Q)$.

1. Here, we can prove Gaussian behavior for test functions ψ on $[-1/2, +1/2]$ that are indicator functions, approximating them with Beurling-Selberg polynomials.
2. Compare to work of Faifman-Rudnick on hyperelliptic ensemble.

Conditional Improvements

Fiorilli-Miller (2013) showed that certain arithmetic hypotheses *beyond GRH* allow one to extend the support for $\widehat{\phi}$ in the local regime, on the number-field side.

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Their function-field analogues, in the global regime, again allow us to remove restrictions on ψ.

Conjecture (Montgomery-A-M-P-T)

Let $Q \in \mathbb{F}_q[T]$ *be of degree* $d \geq 2$*. Let* $s = 1/2 + i\gamma_\chi$ *run through zeros of* $L(s,\chi) \in \mathscr{F}_Q$*. Then there exists* $\delta > 0$ *such that*

$$\sum_{\chi \neq \chi_0} \sum_{\gamma_\chi} q^{in\gamma_\chi} \ll (d-1)(\#\mathscr{F}_Q)^{1-\delta}$$

for all n*.*

Theorem (A-M-P-T)

Let $Q \in \mathbb{F}_q[T]$ be of degree $d \geq 2$. Let $s = 1/2 + i\gamma_\chi$ run through zeros of $L(s,\chi) \in \mathscr{F}_Q$. Suppose that for some $0 \leq \epsilon_1, \epsilon_2 < 1$,

$$\sum_{\chi \neq \chi_0} \sum_{\gamma_\chi} q^{in\gamma_{\chi,j}} \ll (d-1)^{1-\epsilon_1} \# \mathscr{F}_Q^{1-\epsilon_2}$$

holds for all n. Then

$$\mathbb{E} S_\chi(\psi) = \widehat{\psi}(0) + O\left(\frac{\epsilon_2 d}{d^{\epsilon_1} (\# \mathscr{F}_Q)^{\epsilon_2}}\right)$$

for all test functions $\psi(s) = \sum_{n \in \mathbb{Z}} \widehat{\psi}(n) e^{-2\pi i n s}$.

Proof by Erdős-Turán.

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